BOUNDARY VALUE PROBLEMS FOR SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH OPERATOR COEFFICIENTS

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Let Ω_T be some bounded simply connected region in $\mathbb{R}^2$ with $\partial\Omega_T = \bar{\Gamma}_1 \cap \bar{\Gamma}_2$. We seek a function $u(x,t)$, $((x,t) \in \Omega_T)$ with values in a Hilbert space H which satisfies the equation $ALu(x,t) = Bu(x,t) + f(x,t,u,u_t)$, $(x,t) \in \Omega_T$, where $A(x,t)$, $B(x,t)$ are families of linear operators (possibly unbounded) with everywhere dense domain D (D does not depend on (x,t)) in H and $Lu(x,t) = u_{tt} + a_{11}u_{xx} + a_1u_t + a_2u_x$. The values $u(x,t)$; $\partial u(x,t)/\partial n$ are given in Γ_1. This problem is not in general well posed in the sense of Hadamard. We give theorems of uniqueness and stability of the solution of the above problem.

1. Introduction

Let $\varphi_i(t) \in C^1$ $(t \geq t_0,\ i = 1,2)$, $t \in [t_0, T]$, and $|\varphi_i'(t)|(t-t_0)^{1/2} < \mu$ $(i = 1,2)$, where μ is a constant. Let Ω_T be a bounded simply connected region in $\mathbb{R}^2$ defined as follows:

$$\Omega_T = \{(x,t) : 0 \leq t_0 < t < T,\ \varphi_1(t) < x < \varphi_2(t),\ \varphi_1(t_0) = \varphi_2(t_0)\}, \quad (1.1)$$

and $\partial\Omega_T = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, where

$$\begin{aligned} \Gamma_1 &= [(x,t) : x = \varphi_i(t)\ (i = 1,2),\ t_0 \leq t < T], \\ \Gamma_2 &= [(x,t) : t = T,\ \varphi_1(t) < x < \varphi_2(t)]. \end{aligned} \quad (1.2)$$

Let D (D does not depend on (x,t)) be an everywhere dense domain in H, and $A(x,t)$, $B(x,t)$ are families of linear operators (possibly unbounded) with domain D, let $u(x,t)$, $((x,t) \in \Omega_T)$ be a function with values in the space H. Let $u(x,t)$ satisfy the equation

$$A(x,t)Lu(x,t) = Bu(x,t) + f(x,t,u,u_t), \quad (x,t) \in \Omega, \quad (1.3)$$

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where $Lu(x,t) \equiv u_{tt} + a_{11}u_{xx} + a_2u_x + a_1u_t$,

$$a_{11} \in C^2(\Omega_T) \quad (a_{11} > 0), \qquad a_1(x,t),\ a_2(x,t) \in C^1(\bar{\Omega}_T), \tag{1.4}$$

and in the part Γ_1 of the bound $\partial\Omega_T$ are given

$$\left.\frac{\partial u(x,t)}{\partial n}\right|_{\Gamma_1} = g, \quad u(x,t)\big|_{\Gamma_1} = f_1, \tag{1.5}$$

where

$$f_1 \in C^1(\Gamma_1; H), \quad g \in C(\Gamma_1; H). \tag{1.6}$$

Definition 1.1. A solution of (1.3) is called a two times smooth differentiable function which belongs to the domain of operators A, B for every $(x,t) \in \Omega_T$ and satisfies (1.3).

The Cauchy problem is the problem to find the solution of (1.3) which satisfies condition (1.5) with $f, g \in D(A) \cap D(B)$.

The Cauchy problem (1.3), (1.4), and (1.5) is not in general well posed in the sense of Hadamard. This type of problems for differential-operator equations were studied by Kreĭn [3], Levine [5], Buchgeim [1], and others. We will prove theorems of uniqueness and stability of the solution of the Cauchy problem using Lavrent'ev's method [2, 4].

2. Uniqueness

THEOREM 2.1. *Let $A = 1$, $Lu \equiv \triangle u \equiv u_{xx} + u_{tt}$, and let B be a selfadjoint constant operator. Suppose $w(x,t)$ satisfies*

$$\triangle w = Bw + f(t,x,w,w_t), \tag{2.1}$$

$$v \in C^1(\Omega_T; H) \cap C^2(\Omega_T; H) \cap L_2(\Omega_T; D) \tag{2.2}$$

is such that

$$\triangle v = Bv + f(t,x,v,v_t) - \varepsilon(v) \tag{2.3}$$

is defined, and $w|_{\Gamma_1} = 0$, $v|_{\Gamma 1} = 0$. Let $u = w - v$, and

$$\|f(t,x,v,v_t) - f(t,x,w,w_t)\|^2 \leq c_1 \int_{t_0}^t \|u(\tau)\|^2 d\tau + c_2 \int_{t_0}^t \|u_\tau(\tau)\|^2 d\tau, \tag{2.4}$$

where c_1, c_2 are positive constants. Then there exist constants $\Theta_i \geq 0$, $i = 1,2$, such that with

$$\gamma = \Theta_1\Big(\max_{\Gamma_1}\big(\|u_t\|^2 + \|u_x\|^2\big)\Big) + \Theta_2 \int_{t_0}^t \|\varepsilon(v)\|^2 d\tau \tag{2.5}$$

the function

$$\Psi(t) = \ln\left(\iint_{\Omega_t} \|u\|^2 ds\, d\tau + \gamma\right) \tag{2.6}$$

satisfies

$$\Psi''(t) + p\Psi'(t) + q \geq 0 \tag{2.7}$$

for some nonnegative computable constants p and q which depend on T, c_i $(i = 1, 2)$.

Proof. From (2.1) and (2.3) we find

$$\triangle u = Bu + \alpha\big(t, x, u, u_t\big) + \varepsilon(u). \tag{2.8}$$

We denote the estimating integral by

$$\varphi(t) = \iint_{\Omega_t} \big\|u(s, \tau)\big\|^2 ds\, d\tau. \tag{2.9}$$

We differentiate $\varphi(t)$, and take into account the condition $u|_{\Gamma_1} = 0$. We get

$$\begin{aligned} \varphi'(t) &= 2\iint_{\Omega_t} \operatorname{Re}\big(u_\tau, u\big)\, ds\, d\tau, \\ \varphi''(t) &= 2\iint_{\Omega_t} \operatorname{Re}\big(u_{\tau\tau}, u\big)\, ds\, d\tau + 2\iint_{\Omega_t} \big(u_\tau, u_\tau\big)\, ds\, d\tau. \end{aligned} \tag{2.10}$$

Taking into account (2.1) we receive

$$\begin{aligned} \varphi''(t) = &-2\iint_{\Omega_t} \operatorname{Re}\big(u_{ss}, u\big)\, ds\, d\tau + 2\iint_{\Omega_t} (Bu, u)\, ds\, d\tau \\ &+ 2\iint_{\Omega_t} \big(\varepsilon(v), u\big)\, ds\, d\tau + 2\iint_{\Omega_t} \operatorname{Re}\big(\alpha\big(\tau, s, u, u_\tau\big), u\big)\, ds\, d\tau \\ &+ 2\iint_{\Omega_t} \big(u_\tau, u_\tau\big)\, ds\, d\tau, \end{aligned} \tag{2.11}$$

where $\alpha(t, s, u, u_t) = f(t, s, w, w_t) - f(t, s, v, v_t)$.

Using the integration by part and taking into account the condition $u|_{\Gamma_1} = 0$, (2.8) we get from (2.11)

$$\begin{aligned} \varphi''(t) = &-2\iint_{\Omega_t} \big[\big(u_s, u_s\big) + \big(u_\tau, u_\tau\big) + (Bu, u)\big]\, ds\, d\tau \\ &+ 2\iint_{\Omega_t} \operatorname{Re}\big(\alpha\big(\tau, s, u, u_\tau\big) + \varepsilon(v), u\big)\, ds\, d\tau. \end{aligned} \tag{2.12}$$

We differentiate

$$p(t)=\int_{\varphi_1(t)}^{\varphi_2(t)}\left\|u_s(s,t)\right\|^2ds \tag{2.13}$$

and obtain

$$\begin{aligned}
p'(t)&=\int_{\varphi_1(t)}^{\varphi_2(t)}\frac{\partial}{\partial t}\left(u_s,u_s\right)ds+\sum_{i=1}^{2}(-1)^i\varphi_i'(t)\left\|u_s\left(\varphi_i(t),t\right)\right\|^2\\
&=-\int_{\varphi_1(t)}^{\varphi_2(t)}2\operatorname{Re}\left(u_t,u_{ss}\right)ds+2\sum_{i=1}^{2}(-1)^i\operatorname{Re}\left(u_t\left(\varphi_i(t),t\right)u_s\left(\varphi_i(t),t\right)\right)\\
&\quad+\sum_{i=1}^{2}(-1)^i\varphi_i'(t)\left\|u_s\left(\varphi_i(t),t\right)\right\|^2\\
&=\frac{d}{dt}\int_{\varphi_1(t)}^{\varphi_2(t)}\left[\left(u_t,u_t\right)+(Bu,u)\right]ds-2\int_{\varphi_1(t)}^{\varphi_2(t)}\operatorname{Re}\left(\alpha\left(t,s,u,u_t\right)+\varepsilon(v),u_t\right)ds\\
&\quad+\sum_{i=1}^{2}(-1)^i\Big\{\varphi_i'(t)\Big[\left\|u_s\left(\varphi_i(t),t\right)\right\|^2-\left\|u_t\left(\varphi_i(t),t\right)\right\|^2\Big]\\
&\qquad\qquad+2\operatorname{Re}\left(u_t\left(\varphi_i(t),t\right),u_s\left(\varphi_i(t),t\right)\right)\Big\}.
\end{aligned} \tag{2.14}$$

By deducing this formula we use (2.8) and integration by parts. Integrating (2.14) from t_0 till t we get the following:

$$\begin{aligned}
&\int_{\varphi_1(t)}^{\varphi_2(t)}\left(\left\|u_s\right\|^2+(Bu,u)\right)ds\\
&\quad=\int_{\varphi_1(t)}^{\varphi_2(t)}\left\|u_t\right\|^2ds-2\iint_{\Omega_t}\operatorname{Re}\left(\alpha\left(\tau,s,u,u_\tau\right)+\varepsilon(v),u_\tau\right)ds\,d\tau+b_0(t),
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
b_0(t)=\sum_{i=1}^{2}(-1)^i\int_{t_0}^{t}\varphi_i'(\tau)\Big[\left\|u_s\left(\varphi_i(t),t\right)\right\|^2-\left\|u_t\left(\varphi_i(\tau),\tau\right)\right\|^2\Big]\\
+2\operatorname{Re}\left(u_\tau\left(\varphi_i(\tau),\tau\right),u_s\left(\varphi_i(\tau),\tau\right)\right)d\tau.
\end{aligned} \tag{2.16}$$

Substituting (2.15) in (2.12) we get

$$\begin{aligned}
\varphi''(t)=4\iint_{\Omega_t}\left\|u_\tau\right\|^2ds\,d\tau-4\int_{t_0}^{t}\iint_{\Omega_t}\operatorname{Re}\left(\alpha+\varepsilon(v),u_\tau\right)ds\,d\tau_1\,d\tau\\
+2\iint_{\Omega_t}\operatorname{Re}\left(\alpha+\varepsilon(v),u\right)ds\,d\tau+\int_{t_0}^{t}b_0(\tau)d\tau.
\end{aligned} \tag{2.17}$$

We notice that

$$\left|\int_{t_0}^{t} b_0(\tau)dt\right| \le \Theta_0 \sum_{i=1}^{2} \int_{t_0}^{t} (t-\tau)\left|\varphi_i'(\tau)\right|\Big[\left\|u_\tau\right\|^2 + \left\|u_s\right\|^2\Big]_{s=\varphi_i(t)} d\tau \le \gamma, \tag{2.18}$$

where γ is defined by (2.3). From (2.17) we can write

$$\begin{aligned}
4\int_{t_0}^{t}&\iint_{\Omega_t} \left\|u_{\tau_1}(\tau_1,s)\right\|^2 ds\,d\tau_1\,d\tau \\
&\le \varphi'(t) + 4\left|\int_{t_0}^{t}\iint_{\Omega_t}\big(\alpha+\varepsilon(v),u_{\tau_2}\big)\,ds\,d\tau_2\,d\tau_1\,d\tau\right| \\
&\qquad + 2\left|\int_{t_0}^{t}\iint_{\Omega_t}\operatorname{Re}\big(\alpha+\varepsilon(v),u\big)\,ds\,d\tau_1\,d\tau\right| + \left|\int_{t_0}^{t}\int_{t_0}^{\tau} b_0(\tau_1)d\tau_1\,d\tau\right|.
\end{aligned} \tag{2.19}$$

Using the Cauchy and Bellman inequalities and the inequality

$$|ab| \le \frac{a^2}{2}\beta + \frac{b^2}{2\beta}, \quad \beta > 0, \tag{2.20}$$

from (2.19) we receive the following inequality:

$$\begin{aligned}
\int_{t_0}^{t}&\iint_{\Omega_t} \left\|u_{\tau_1}(\tau_1,s)\right\|^2 ds\,d\tau_1\,d\tau \\
&\le k_1\varphi'(t) + k_2\varphi(t) + k_3\int_{t_0}^{t}\iint_{\Omega_t}\left\|\varepsilon(v)\right\|^2 ds\,d\tau_1\,d\tau + T\gamma,
\end{aligned} \tag{2.21}$$

where k_1, k_2, and k_3 are nonnegative constants which depend on T and the constants c_1 and c_2. Then for $\varphi''(t)$ we get the following:

$$\begin{aligned}
\varphi''(t) \ge 4&\iint_{\Omega_t}\left\|u_\tau(s,\tau)\right\|^2 ds\,d\tau \\
&- k_4\varphi'(t) - k_5\varphi(t) - k_6\iint_{\Omega_t}\left\|\varepsilon(v)\right\|^2 d\tau\; ds - k_7\gamma.
\end{aligned} \tag{2.22}$$

We consider now the function $\psi(t) = \ln[\varphi(t)+\gamma]$, using the Cauchy inequality,

$$|ab| \le \frac{a^2}{2}\beta + \frac{b^2}{2\beta}, \quad \beta > 0, \tag{2.23}$$

and (2.22) we transform the second derivative

$$
\begin{aligned}
\psi''(t) &= \frac{\varphi''(t)\cdot\bigl(\varphi(t)+\gamma\bigr)-\bigl(\varphi'(t)\bigr)^2}{\bigl(\varphi(t)+\gamma\bigr)^2} \\
&\geq \frac{1}{\bigl(\varphi(t)+\gamma\bigr)^2}\Biggl\{\biggl(4\iint_{\Omega_t}\bigl\|u_\tau(s,\tau)\bigr\|^2 ds\,d\tau - k_4\varphi'(t) \\
&\qquad\qquad - k_5\varphi(t) - k_6\iint_{\Omega_t}\bigl\|\varepsilon(u)\bigr\|^2 ds\,d\tau - k_7\gamma\biggr) \\
&\qquad\qquad \times\biggl(\iint_{\Omega_t}\|u\|^2 ds\,d\tau+\gamma\biggr) - 4\biggl(\iint_{\Omega t}\operatorname{Re}\bigl(u_\tau,u\bigr)\,ds\,d\tau\biggr)^2\Biggr\} \\
&\geq -k_8\frac{\varphi'}{\varphi(t)+\gamma} - k_9,
\end{aligned}
\tag{2.24}
$$

or

$$
\psi'' + p\psi' + q \geq 0, \tag{2.25}
$$

where p, q are nonnegative constants that depend on T and the constants c_1, c_2. Theorem 2.1 is proved. □

Remark 2.2. It is known from the theory of ordinary differential equations if the function $\psi(t)$ satisfies the inequality (2.7), then it satisfies the following inequality:

$$
\psi(t) \leq \psi_0(t), \tag{2.26}
$$

where $\psi_0(t)$ is a solution of the differential equation

$$
\psi_0''(t) + p\psi_0'(t) + q = 0 \tag{2.27}
$$

with boundary conditions $\psi_0(t_0) = \psi(t_0)$, $\psi_0(T) = \psi(T)$. It is not difficult to see that

$$
\psi_0(t) = \ell_1 + \ell_2\exp(-pt) - q\frac{t}{p};
$$

$$
\ell_1 = \frac{\psi(T)\exp\bigl(-pt_0\bigr) - \psi\bigl(t_0\bigr)\exp(-pT) + (q/p)\bigl(T\exp\bigl(-pt_0\bigr) - t_0\exp(-pT)\bigr)}{\exp\bigl(-pt_0\bigr) - \exp(-pT)};
$$

$$
\ell_2 = \psi(0) - \psi(T) + \frac{q}{p}\bigl(t_0 - T\bigr)\bigl(\exp\bigl(-pt_0\bigr) - \exp(-pT)\bigr);
$$

$$
\psi(t) \leq \bigl(1-\omega(t)\bigr)\psi\bigl(t_0\bigr) + \omega(t)\psi(T) + q\frac{\omega(t)\bigl(T-t_0\bigr) + \bigl(t_0 - t\bigr)}{p}, \tag{2.28}
$$

where

$$\omega(t) = \frac{\exp\left(-pt_0\right) - \exp(-pt)}{\exp\left(-pt_0\right) - \exp(-pT)}. \tag{2.29}$$

Further, it is not difficult to see, from (2.7), that

$$\iint_{\Omega_t} \|u\|^2 ds\, d\tau \le c(t)\gamma^{1-\omega(t)} \left\{ \iint_{\Omega_t} \|u\|^2 ds\, d\tau + \gamma \right\}^{\omega(t)} - \gamma. \tag{2.30}$$

Corollary 2.3. *The solution of the Cauchy problem for (1.3) is unique in the space*

$$C^1\left(\bar{\Omega}_T; H\right) \cap C^2\left(\Omega_T; H\right) \cap L_2\left(\Omega_T; D\right). \tag{2.31}$$

Proof. Let v, w be solutions of the Cauchy problem for (2.1) and (2.3), respectively. Then $u = w - v$ is the solution of the homogeneous Cauchy problem for the equation

$$\triangle u = Bu + \alpha\left(t, u, u_t\right) \tag{2.32}$$

and $\gamma = 0$ and from (2.30) follows $u \equiv 0$ or $w \equiv v$. Corollary 2.3 is proved. □

From inequality (2.30) it is easy to see that the following corollary follows.

Corollary 2.4. *The solution of the Cauchy problem for (1.3) is stable in the space*

$$\begin{gathered} C^1\left(\bar{\Omega}_T; H\right) \cap C^2\left(\Omega_T; H\right) \cap L_2\left(\Omega_T; D\right), \\ u \in \left\{ u : \iint_{\Omega_t} \|u\|^2 ds\, d\tau \le M \right\}. \end{gathered} \tag{2.33}$$

3. Stability

Let A be a constant selfadjoint operator and $(Au, u) > 0$ for all $u \neq 0$, $(Au, u) = 0$ if and only if $u = 0$. Let $B(x, t)$ be a selfadjoint operator for every $(x, y) \in D$ and with

$$\begin{aligned} c = \max\Bigg[\max_{(x,t)\in[\Omega_T]} & \frac{\left(\left|a_{11_r}\right| + \left|a_1/2\right| + \left|a_{11_s}/2\right|\right)}{a_{11}}; \\ & \max\left(\left|\frac{a_{11_s}}{2}\right| + \left|\frac{\left(a_1\right)}{2}\right| + \left|\left(a_2\right)\right| + 2c_1 + \beta_9\right)\Bigg] \end{aligned} \tag{3.1}$$

satisfies

$$\left(B_t u, u\right) \ge -c(Bu, u). \tag{3.2}$$

Let w satisfy

$$ALw = Bw + F\left(x, t, w, w_t\right) \tag{3.3}$$

and v satisfy

$$ALv = Bv + f\big(x,t,v,v_t\big) - \varepsilon(v). \tag{3.4}$$

Let $u = w - v$ and $\alpha = f(x,t,w,w_t) - f(x,t,v,v_t)$ and

$$\|\alpha\|^2 \le c_1\|u\|^2 + c_2\big\|u_t\big\|^2, \tag{3.5}$$

then

$$ALu = Bu + \alpha\big(x,t,u,u_t\big) + \varepsilon(v), \tag{3.6}$$

where c_1, c_2 are constants.

THEOREM 3.1. *Let the coefficients a_{11}, a_1, and a_2 satisfy condition (1.4). If the solution of (3.6) is equal to zero on Γ_1 and satisfies the inequality*

$$\begin{gathered}\iint_{\Omega_T}(u,Au)\,ds\,d\tau \le M,\\ \gamma = \Theta\max_{\Gamma_1}\big[\big(u_t,Au_t\big)+\big(u_x,Au_x\big)\big] + \iint_{\Omega_T}\big\|\varepsilon(v)\big\|^2 ds\,d\tau,\end{gathered} \tag{3.7}$$

then for $u \in C^1(\bar{\Omega}_T;H)\cap C^2(\Omega_T;H)\cap L_2(\Omega_T;D)$ the following inequality is true

$$\iint_{\Omega_t}(u,Au)\,ds\,d\tau \le \gamma^{1-\omega(t)}(M+\gamma)^{\omega(t)}c(t) - \gamma, \tag{3.8}$$

where

$$\begin{aligned} c(t) &= \exp\frac{q\big(\gamma(t)T - t\big)}{p},\\ \omega(t) &= \frac{\exp\big(-pt_0\big)-\exp(-pt)}{\exp\big(-pt_0\big)-\exp(-pT)},\end{aligned} \tag{3.9}$$

and Θ, p, q are constants that depend on the coefficients T and c_i, $i = 1,2$.

Proof. Let

$$F(t) = \iint_{\Omega_t}(u,Au)\,ds\,d\tau, \tag{3.10}$$

then

$$\begin{aligned} F'(t) &= \iint_{\Omega_t} 2\operatorname{Re}\big(u_\tau, Au\big)\,ds\,d\tau,\\ F''(t) &= \iint_{\Omega_t} 2\operatorname{Re}\big(u_{\tau\tau}, Au\big)\,ds\,d\tau + F(t)\\ &= \iint_{\Omega_t} 2\big(u_\tau, Au_\tau\big)\,ds\,d\tau,\end{aligned} \tag{3.11}$$

because $u|_{\Gamma_1} = 0$. We transform the first term in the expression for $F''(t)$ using (3.6) and integration by parts

$$\begin{aligned}
&\iint_{\Omega_t} 2\operatorname{Re}\big(u_{\tau\tau}, Au\big)\,ds\,d\tau \\
&\quad = \iint_{\Omega_t} 2(Bu,u)\,ds\,d\tau \iint_{\Omega_t} 2a_{11}\big(u_s, Au_s\big)\,ds\,d\tau \\
&\qquad + \iint_{\Omega_t} 2\operatorname{Re}\big(u, \alpha\big(s,\tau,u,u_\tau\big)\big)\,ds\,d\tau + \iint_{\Omega_t} 2\operatorname{Re}\big(u, \varepsilon(v)\big)\,ds\,d\tau \\
&\qquad - \iint_{\Omega_t} 2\operatorname{Re}\big(-a_{11_s} + a_2\big)\big(u_s, Au\big)\,ds\,d\tau - \iint_{\Omega_t} 2a_1\operatorname{Re}\big(u_\tau, Au\big)\,ds\,d\tau.
\end{aligned} \tag{3.12}$$

After the transform we get

$$\begin{aligned}
F''(t) = {} & \iint_{\Omega_1} 2\big(\big(u_\tau, Au_\tau\big) + a_{11}\big(u_s, Au_s\big) + (u, Bu)\big)\,ds\,d\tau \\
& + \iint_{\Omega_1} 2\big[\big(a_{11_s} - a_2\big)\big(u_s, Au\big) + \operatorname{Re}\big(u, \alpha\big(s,\tau,u,u_\tau\big)\big) \\
& \qquad - \operatorname{Re}\big(u, \varepsilon(v)\big) - a_1\operatorname{Re}\big(u_\tau, Au\big)\big]\,ds\,d\tau.
\end{aligned} \tag{3.13}$$

From here and from (1.4) using the conditions for the coefficients and Hölder's inequality, we get

$$\begin{aligned}
F''(t) \geq {} & \iint_{\Omega t} 2\big(\big(u_\tau, Au_\tau\big) + a_{11}\big(u_s, Au_s\big) + (u, Bu)\big)\,ds\,d\tau \\
& - 2\beta_1\Big[\iint_{\Omega t}\big(u_s, Au_s\big)\,ds\,d\tau\, F(t)\Big]^{1/2} \\
& - 2\beta_2\Big[\iint_{\Omega t}\big(u_\tau, Au_\tau\big)\,ds\,d\tau\, F(t)\Big]^{1/2} \\
& - 2\beta_3 F(t) - \beta_4\Big[\iint_{\Omega_1}\big(u_\tau, Au_\tau\big)\,ds\,d\tau \iint_{\Omega_1}(u, Au)\,ds\,d\tau\Big]^{1/2} \\
& - \beta_5\Big[\iint_{\Omega_t}\|\varepsilon(v)\|\,ds\,d\tau \iint_{\Omega_t}(u, Au)\,ds\,d\tau\Big]^{1/2} \\
\geq {} & \iint_{\Omega_t} 2\big(\big(u_\tau, Au_\tau\big) + a_{11}\big(u_s, Au_s\big) + (u, Bu)\big)\,ds\,d\tau \\
& - \bigg(\frac{\beta_1^2}{a_{11}} + \bigg(\frac{(2\beta_2 + \beta_4)}{2}\bigg)^2 - 2\beta_3 + \frac{\beta_5^2}{2}\bigg)F(t) \\
& - 0.5\iint_{\Omega_t}\|\varepsilon(v)\|^2\,ds\,d\tau.
\end{aligned} \tag{3.14}$$

From the last inequality it follows that

$$\begin{aligned}\int_{t_0}^{t}\iint_{\Omega_t}\left(\left(u_{\tau_1}, Au_{\tau_1}\right)+a_{11}\left(u_s, Au_s\right)+(u, Bu)\right)ds\,d\tau_1\,d\tau\\ \le F'(t)+\beta_6 F(t)+0.5\iint_{\Omega_t}\left\|\varepsilon(v)\right\|^2 ds\,d\tau.\end{aligned}\tag{3.15}$$

Differentiating the first integral on the right-hand side of (3.12), we get

$$\begin{aligned}&\frac{d}{dt}\iint_{\Omega_t} a_{11}\left(u_s, Au_s\right)ds\,d\tau\\
&\quad=\frac{d}{dt}\iint_{\Omega_t}\left\{\left(u_\tau, Au_\tau\right)-(Bu,u)\right\}ds\,d\tau\\
&\qquad+\int_{t_0}^{t}\sum_{i=1}^{2}\Big\{(-1)^i\varphi_i'(t)\Big[a_{11}\left(u_s\left(\varphi_i(\tau),\tau\right), Au_s\left(\varphi_i(\tau),\tau\right)\right)\\
&\qquad\qquad-\left(u_\tau\left(\varphi_i(\tau),\tau\right), Au_\tau\left(\varphi_i(\tau),\tau\right)\right)\Big]\\
&\qquad\qquad+a_{11}\left(u_\tau\left(\varphi_i(\tau),\tau\right), Au_s\left(\varphi_i(\tau),\tau\right)\right)\Big\}d\tau\\
&\qquad+\iint_{\Omega_t}\left[a_{11_\tau}\left(u_s, Au_s\right)-a_{11_s}\left(u_\tau, Au_s\right)\right]ds\,d\tau\\
&\qquad+\iint_{\Omega_t}\Big[\left(u, Bu_\tau\right)+a_2\left(u_\tau, Au_\tau\right)+a_1\left(u_\tau, Au_s\right)\\
&\qquad\qquad-\left(u_\tau, \alpha+\varepsilon(v)\right)\Big]ds\,d\tau.\end{aligned}\tag{3.16}$$

After this transformation $F''(t)$ takes the form

$$\begin{aligned}F''(t)&=4\iint_{\Omega_t}\left(u_\tau, Au_\tau\right)ds\,d\tau\\
&\quad+2\int_{t_0}^{t}\iint_{\Omega_t}\Big[a_{11_\tau}\left(u_s, Au_s\right)-a_{11_s}\operatorname{Re}\left(u_\tau, Au_s\right)\\
&\qquad+\operatorname{Re}\left(u, Bu_\tau\right)+a_2\left(u_\tau, Au_\tau\right)+a_1\operatorname{Re}\left(u_\tau, Au_s\right)\Big]ds\,d\tau\\
&\quad+2\int_{t_0}^{t}\iint_{\Omega_t}\operatorname{Re}\left(u_\tau, \alpha+\varepsilon(v)\right)ds\,d\tau\\
&\quad+\iint_{\Omega t}\Big\{\left(a_{11_s}-a_2\right)\operatorname{Re}\left(u_s, Au\right)+2\operatorname{Re}\left(u_\tau, \alpha+\varepsilon(v)\right)\\
&\qquad-2a_1\operatorname{Re}\left(u_\tau, Au_\tau\right)\Big\}ds\,d\tau+b_0(\tau),\end{aligned}\tag{3.17}$$

where

$$b_0(\tau)=\int_{t_0}^{t}(t-\tau)\sum_{i=1}^{2}\Big\{(-1)^i\varphi_i'(t)\Big[a_{11}\big(u_x\big(\varphi_i(\tau),\tau\big),Au_x\big(\varphi_i(\tau),\tau\big)\big) - \big(u_\tau\big(\varphi_i(\tau),\tau\big),Au_\tau\big(\varphi_i(\tau),\tau\big)\big)\Big] + a_{11}\operatorname{Re}\big(u_\tau\big(\varphi_i(\tau),\tau\big),Au_x\big(\varphi_i(\tau),\tau\big)\big)\Big\}d\tau. \tag{3.18}$$

We remark that

$$\begin{aligned}\big|b_0(\tau)\big| &\le \sum_{i=1}^{2}\int_{t_0}^{t}(t-\tau)\Big\{a_{11}\Big[\big|\varphi_i'\big|\big(u_x\big(\varphi_i(\tau),\tau\big),Au_x\big(\varphi_i(\tau),\tau\big)\big) \\ &\qquad + \big|2\operatorname{Re}\big(u_x\big(\varphi_i(\tau),\tau\big),Au_\tau\big(\varphi_i(\tau),\tau\big)\big)\big|\Big] \\ &\qquad + \big|\varphi_i'\big|\big(u_\tau\big(\varphi_i(\tau),\tau\big),Au_\tau\big(\varphi_i(\tau),\tau\big)\big)\Big\}d\tau \\ &\le \gamma_1.\end{aligned} \tag{3.19}$$

We denote

$$s^2=\iint_{\Omega_t}\big(u_\tau,Au_\tau\big)\,ds\,d\tau\iint_{\Omega_t}(u,Au)\,ds\,d\tau-\left(\iint_{\Omega_t}\big(u_\tau,Au_\tau\big)\,ds\,d\tau\right)^2. \tag{3.20}$$

Then using Hölder's inequality, (1.4), (3.15) we obtain from (3.17)

$$\begin{aligned}F''(t) \ge{}& 4\iint_{\Omega_t}\big(u_\tau,Au_\tau\big)\,ds\,d\tau \\ &-2\int_{t_0}^{t}\iint_{\Omega_t}\bigg\{\bigg[\big|a_{11_r}\big|+\frac{\big|a_{1_1}\big|}{2}+\frac{\big|a_{11_s}\big|}{2}\bigg]\big(u_s,Au_s\big) \\ &\qquad +\bigg[\frac{\big|a_{11_s}\big|}{2}+\frac{\big|a_1\big|}{2}+\frac{\big|a_2\big|}{2}\bigg]\big(u_\tau,Au_\tau\big) \\ &\qquad +\big|\operatorname{Re}\big(u,B_\tau u\big)\big|\bigg\}\,ds\,d\tau\,d\tau_1 \\ &-2c_2\int_{t_0}^{t}\iint_{\Omega_{\tau_1}}\big(u_{\tau_1},Au_{\tau_1}\big)\,ds\,d\tau\,d\tau_1 \\ &-2c_1\bigg\{\int_{t_0}^{t}\iint_{\Omega_{\tau_1}}\big(u_{\tau_1},Au_{\tau_1}\big)\,ds\,d\tau\,d\tau_1F(t)\bigg\}^{1/2} \\ &-2\bigg\{\int_{t_0}^{t}\iint_{\Omega_{\tau_1}}\big(u_{\tau_1},Au_{\tau_1}\big)\,ds\,d\tau\,d\tau_1\int_{t_0}^{t}\iint_{\Omega_t}\big\|\varepsilon(v)\big\|^2ds\,d\tau\,d\tau_1\bigg\}^{1/2}\end{aligned}$$

$$
\begin{aligned}
&-\iint_{\Omega_t} \left|a_{11_s}-a_2\right| \operatorname{Re}(u, Au)\, ds\, d\tau - 2c_1 \iint_{\Omega_t} \operatorname{Re}(u, Au)\, ds\, d\tau \\
&-c_2\left\{\iint_{\Omega_t} \operatorname{Re}\left(u_\tau, Au_\tau\right) ds\, dt\right\}^{1/2} \\
&-2\beta\left\{\iint_{\Omega_t} \operatorname{Re}\left(u_\tau, Au_\tau\right) ds\, d\tau\, F(t)\right\}^{1/2} - \gamma \\
\geq{}& 4\iint_{\Omega_t} \left(u_\tau, Au_\tau\right) ds\, d\tau \\
&-\int_{t_0}^{t}\iint_{\Omega_t} \left\{ca_{11}\left(u_s, Au_s\right) + c\left(u_\tau, Au_\tau au\right) + c(u, Bu)\right\} ds\, d\tau\, d\tau_1 \\
&-\beta_6 F'(t) - \beta_7 F(t) - \beta_8 s - \gamma \\
\geq{}& 4\iint_{\Omega_t} \left(u_\tau, Au_\tau\right) ds\, d\tau - \beta_6 F'(t) - \beta_7 F(t) - \beta_s - \gamma, \\
&\left(c = \max\left\{\frac{2\left|a_{11_r}\right| + \left|a_1\right| + \left|a_{11_s}\right|}{2\left|a_{11}\right|}, \left[\frac{\left|a_{11_s}\right|}{2} + \frac{\left|a_1\right|}{2} + \left|a_2\right| + 2c_1 + \beta_9\right]\right\}\right).
\end{aligned}
\tag{3.21}
$$

Let $\phi(t) = \ln[F(t)+\gamma]$, then

$$
\begin{aligned}
\Phi''(t) ={}& \frac{\left(F''(t)\left(F(t)+\gamma\right) - \left(F'(t)\right)^2\right)}{\left(F(t)+\gamma\right)^2} \\
\geq{}& \frac{4\iint_{\Omega_t}\left(u_\tau, Au_\tau\right) ds\, d\tau \iint_{\Omega_t}(u, Au)\, ds\, d\tau - \left(2\iint_{\Omega_t}\left(u_\tau, Au\right) ds\, d\tau\right)^2}{\left(F(t)+\gamma\right)^2} \\
&+4\frac{\iint_{\Omega t}\left(u_\tau, Au_\tau\right) ds\, d\tau\gamma}{\left(F(t)+\gamma\right)^2} - \frac{\beta_9 F'(t) + \beta_{10} F(t) + \beta_{11} S + \gamma}{F(t)+\gamma} \\
\geq{}& -p\frac{F'(t)}{F(t)+\gamma} - q.
\end{aligned}
\tag{3.22}
$$

Inequality (3.8) follows from (3.22), and the theorem is proved. □

Remark 3.2. We can obtain similar results for arbitrary second order elliptic operators L, using the methods of [4].

Remark 3.3. For

$$B(x,t) = B_1(x,t) + iB_2(x,t), \tag{3.23}$$

where $B_1(x,t)$ and $B_2(x,t)$ are selfadjoint operators for all $(x,t) \in \Omega_T$ and λ, μ, c are constants such that $(Au,u) \geq \lambda(u,u) > 0$, $\|B_2u\|^2 \leq \mu(Au,u)$, and $-(B_1)_t \leq cB_t$, a similar result is valid.

Example 3.4. We consider the equation

$$\operatorname{sgn}(y)\big(u_{tt}(t,x,y)+u_{xx}(t,x,y)\big)=u_{yy}(t,x,y) \tag{3.24}$$

in the region $Q=(-1,1)\times\Omega_T$ (Ω_T is defined as above). This equation is a mixed type equation. We will consider the problem of finding the solution of this equation in Q ($y\neq 0$) which satisfies the following boundary conditions:

$$u(t,x,y)|_{\Gamma_i'}=f, \qquad \frac{\partial u(t,x,y)}{\partial n}|_{\Gamma_1'}=g, \tag{3.25}$$

where $\Gamma_1'=\Gamma_1\times(-1,1)$, $\Gamma_2'=\Gamma_2\times(-1,1)$;

$$u(t,x,-1)=0, \quad u(t,x,1)=0, \quad (t,x)\in\Omega_T; \tag{3.26}$$

$$\begin{gathered} u(t,x,-0)=u(t,x,+0),\\ \frac{\partial u(t,x,-0)}{\partial y}=\frac{\partial u(t,x,+0)}{\partial y}. \end{gathered} \tag{3.27}$$

Here B is a selfadjoint positive definite in $L_2(-1,1)$ operator which is generated by the differential expression

$$Bu=-\frac{\partial^2 u}{\partial y^2} \tag{3.28}$$

and with boundary conditions $u|_{y=-1}=u|_{y=1}=0$. We define the operator A as the operator of multiplication with the function $\operatorname{sgn}(y)$. This problem is an ill-posed problem in the sense of Hadamard, since continuous dependence of the solution from the data is absent in it.

Using the above described method we can prove the following result.

THEOREM 3.5. *If a solution of this problem becomes zero on the surface Γ_1' and satisfies*

$$\int_0^T\int_{\varphi_1(t)}^{\varphi_2(t)}\int_{-1}^1\big(u_y(t,x,y)\big)^2dy\,ds\,dt\le M,$$

$$\Theta\max_{\Gamma_1}\left(\int_{-1}^1\big(u_{yx}\big)^2dy+\int_{-1}^1\big(u_{yt}\big)^2dy+\int_{-1}^1\big|\operatorname{sign}(x)\big(u_{yy}\big)^2dy\big|\right)=\gamma, \tag{3.29}$$

(Θ is constant that depends on T, Γ_1), then the inequality

$$\int_0^t\int_{\varphi_1(t)}^{\varphi_2(t)}\int_{-1}^1\big(u_y(t,x,y)\big)^2dy\,ds\,dt\le\gamma^{1-\omega_1(t)}(M+\gamma)^{\omega_1(t)}c_1(t)-\gamma \tag{3.30}$$

is valid, where $c_1(t)=\exp(t(T-t)/2)$, $\omega_1(t)=(1-t/T)$.

From this theorem one can easily see that the uniqueness and the conditional stability of the solution of this problem follows.

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