ON THE STABILITY OF THE LINEAR DELAY DIFFERENTIAL AND DIFFERENCE EQUATIONS

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We consider the initial-value problem for linear delay partial differential equations of the parabolic type. We give a sufficient condition for the stability of the solution of this initial-value problem. We present the stability estimates for the solutions of the first and second order accuracy difference schemes for approximately solving this initial-value problem. We obtain the stability estimates in Hölder norms for the solutions of the initial-value problem of the delay differential and difference equations of the parabolic type.

1. Introduction

Methods for numerical solutions of the delay ordinary differential equations have been studied extensively by many researchers (cf. [1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16] and the references therein) and developed over the last two decades. In this literature, the linear delay equations was considering under the condition, such that in the case of the constant coefficients follows obviously from characteristic equations as the necessary condition. The stability of the solution of this initial-value problem and of the difference schemes for approximately solving the initial-value problem were obtained.

However, the theory and numerical solution of the delay partial differential equations have received less attention than the delay ordinary differential equations. In the present paper, we consider the initial-value problem for linear delay differential equations

$$\frac{du(t)}{dt} + Au(t) = B(t)u(t - \omega), \quad t \geq 0,$$
$$u(t) = g(t), \quad -\omega \leq t \leq 0, \tag{1.1}$$

in an arbitrary Banach space E with unbounded linear operators A and $B(t)$ in E with dense domain $D(A) \subseteq D(B(t))$. Let A be a strongly positive operator,

that is, $-A$ is the generator of the analytic semigroup $\exp\{-tA\}$ ($t \geq 0$) of the linear bounded operators with exponentially decreasing norm when $t \to \infty$. This means the following estimates hold:

$$\left\| \exp\{-tA\} \right\|_{E \mapsto E} \leq M e^{-\delta t}, \quad \left\| tA \exp\{-tA\} \right\|_{E \mapsto E} \leq M e^{-\delta t}, \quad t > 0 \quad (1.2)$$

for some $M > 0$, $\delta > 0$ (see [11]).

A function $u(t)$ is called a solution of problem (1.1) if the following conditions are satisfied:

(i) The function $u(t)$ is continuously differentiable on the interval $[-\omega, \infty)$. The derivative at the endpoint $t = -\omega$ is understood as the appropriate unilateral derivative.

(ii) The function $u(t)$ belongs to $D(A)$ for all $t \in [-\omega, \infty)$, and the functions $Au(t)$ and $B(t)u(t)$ are continuous on the interval $[-\omega, \infty)$.

(iii) The function $u(t)$ satisfies the equation and the initial condition (1.1).

A solution $u(t)$ of the initial-value problem (1.1) is said to be stable if

$$\left\| u(t) \right\|_E \leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_E \tag{1.3}$$

for every t, $-\omega \leq t < \infty$. We are interested to study the stability of solutions of the initial-value problem under the assumption that

$$\left\| B(t)A^{-1} \right\|_{E \mapsto E} \leq 1 \tag{1.4}$$

holds for every $t \geq 0$. We have not been able to obtain the estimate (1.3) in the arbitrary Banach space E. Nevertheless, we can establish the analog of estimates (1.3) where the space E is replaced by the fractional spaces E_α ($0 < \alpha < 1$) under the more strong assumption than (1.4).

2. The delay differential equation

It is known that (cf. [17]) the strongly positive operator A defines the fractional spaces $E_\alpha = E_\alpha(A, E)$ ($0 < \alpha < 1$) consisting of all $v \in E$ for which the following norms are finite:

$$\|v\|_{E_\alpha} = \sup_{\lambda > 0} \left\| \lambda^{1-\alpha} A \exp\{-\lambda A\} v \right\|_E. \tag{2.1}$$

First, we consider problem (1.1) when

$$A^{-1}B(t)x = B(t)A^{-1}x, \quad x \in D(A). \tag{2.2}$$

THEOREM 2.1. *Assume that the condition*

$$\left\| B(t)A^{-1} \right\|_{E \mapsto E} \leq \frac{(1-\alpha)}{M2^{2-\alpha}} \tag{2.3}$$

holds for every $t \geq 0$. Then for every $t \geq 0$,

$$\left\| u(t) \right\|_{E_\alpha} \leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha}. \tag{2.4}$$

Proof. Using the formula

$$u(t) = \exp\{-tA\}g(0) + \int_0^t \exp\left\{ -(t-s)A \right\} B(s)g(s-\omega)ds, \tag{2.5}$$

the semigroup property, condition (2.3), and the estimates (1.2) we have

$$\lambda^{1-\alpha} \left\| A\exp\{-\lambda A\}u(t) \right\|_E$$

$$\leq \lambda^{1-\alpha} \left\| A\exp\left\{ -(\lambda+t)A \right\}g(0) \right\|_E$$

$$+ \lambda^{1-\alpha} \int_0^t \left\| A\exp\left\{ -\frac{\lambda+t-s}{2}A \right\} \right\|_{E \to E} \left\| B(s)A^{-1} \right\|_{E \to E}$$

$$\times \left\| A\exp\left\{ -\frac{\lambda+t-s}{2}A \right\}g(s-\omega) \right\|_E ds$$

$$\leq \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}} \left\| g(0) \right\|_{E_\alpha} + \frac{1-\alpha}{M2^{2-\alpha}} \int_0^t \frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}} ds \max_{0 \leq s \leq \omega} \left\| g(s-\omega) \right\|_{E_\alpha}$$

$$\leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha} \tag{2.6}$$

for every t, $0 \leq t \leq \omega$ and λ, $\lambda > 0$. This shows that

$$\left\| u(t) \right\|_{E_\alpha} \leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha} \tag{2.7}$$

for every $t, 0 \leq t \leq \omega$. Applying the mathematical induction, one can easily show that it is true for every t. Namely, assume that inequality (2.4) is true for t, $(n-1)\omega \leq t \leq n\omega$, $n = 1, 2, 3, \ldots$. Using the formula

$$u(t) = \exp\left\{ -(t-n\omega)A \right\}u(n\omega) + \int_{n\omega}^t \exp\left\{ -(t-s)A \right\} B(s)u(s-\omega)ds, \tag{2.8}$$

the semigroup property, condition (2.3), and the estimates (1.2) we have

$$
\begin{aligned}
\lambda^{1-\alpha} & \left\| A \exp\{-\lambda A\} u(t) \right\|_E \\
&\leq \lambda^{1-\alpha} \left\| A \exp\left\{ -(\lambda+t-n\omega)A \right\} u(n\omega) \right\|_E \\
&\quad + \lambda^{1-\alpha} \int_{n\omega}^{t} \left\| A \exp\left\{ -\frac{\lambda+t-s}{2}A \right\} \right\|_{E\to E} \left\| B(s)A^{-1} \right\|_{E\to E} \\
&\qquad \times \left\| A \exp\left\{ -\frac{\lambda+t-s}{2}A \right\} u(s-\omega) \right\|_E ds \\
&\leq \frac{\lambda^{1-\alpha}}{(\lambda+t-n\omega)^{1-\alpha}} \left\| u(n\omega) \right\|_{E_\alpha} + \frac{1-\alpha}{M2^{2-\alpha}} \int_{n\omega}^{t} \frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}} ds \\
&\qquad \times \max_{n\omega \leq s \leq (n+1)\omega} \left\| u(s-\omega) \right\|_{E_\alpha} \\
&\leq \max_{-(n-1)\omega \leq t \leq n\omega} \left\| u(t) \right\|_{E_\alpha}
\end{aligned}
\tag{2.9}
$$

for every t, $n\omega \leq t \leq (n+1)\omega$, $n = 1, 2, 3, \ldots$ and λ, $\lambda > 0$. This shows that

$$
\left\| u(t) \right\|_{E_\alpha} \leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha}
\tag{2.10}
$$

for every t, $n\omega \leq t \leq (n+1)\omega$, $n = 1, 2, 3, \ldots$. This result completes the proof of Theorem 2.1. □

Now, we consider problem (1.1) when

$$
A^{-1}B(t)x \neq B(t)A^{-1}x, \quad x \in D(A).
\tag{2.11}
$$

Note that (see [11]) A is a strongly positive operator in a Banach space E if and only if its spectrum $\sigma(A)$ lies in the interior sector of angle φ, $0 < 2\varphi < \pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1 = [z = \rho \exp(i\varphi) : 0 \leq \rho < \infty]$ and $S_2 = [z = \rho \exp(-i\varphi) : 0 \leq \rho < \infty]$, and outside it the resolvent $(z - A)^{-1}$ is subject to the bound

$$
\left\| (z-A)^{-1} \right\|_{E\to E} \leq \frac{M_1}{1+|z|}
\tag{2.12}
$$

for some $M_1 > 0$. Throughout, the operator $(zI - A)^{-1}$ will be written $(z - A)^{-1}$. First of all we prove a lemma that will be needed in the sequel.

LEMMA 2.2. *For any z on the edges of the sector, $S_1 = [z = \rho \exp(i\varphi) : 0 \leq \rho < \infty]$, and $S_2 = [z = \rho \exp(-i\varphi) : 0 \leq \rho < \infty]$, and outside it the estimate*

$$
\left\| A(z-A)^{-1}\varphi \right\|_E \leq \frac{M_1^\alpha M^\alpha \left(1+M_1\right)^{1-\alpha} 2^{(2-\alpha)\alpha}}{\alpha(1-\alpha)\left(1+|z|\right)^\alpha} \|\varphi\|_{E_\alpha}
\tag{2.13}
$$

holds. Here and in the future M and M_1 are some constants of the estimates (1.2) and (2.12).

Proof. Applying the formula (see [11])

$$\varphi = \int_0^\infty A \exp\{-\tau A\}\varphi \, d\tau, \tag{2.14}$$

and using the semigroup property we obtain

$$A(z-A)^{-1}\varphi = \int_0^N A(z-A)^{-1}A\exp\{-\tau A\}\varphi \, d\tau$$
$$+ \int_N^\infty (z-A)^{-1}A\exp\left\{-\frac{\tau}{2}A\right\}A\exp\left\{-\frac{\tau}{2}A\right\}\varphi \, d\tau. \tag{2.15}$$

Using the estimates (1.2), (2.12), and the definition of the spaces E_a, we obtain

$$\left\|A(z-A)^{-1}\varphi\right\|_E \le \int_0^N \left\|A(z-A)^{-1}\right\|_{E\to E}\left\|A\exp\{-\tau A\}\varphi\right\|_E d\tau$$
$$+ \int_N^\infty \left\|(z-A)^{-1}\right\|_{E\mapsto E}\left\|A\exp\left\{-\frac{\tau}{2}A\right\}\right\|_{E\mapsto E}$$
$$\times \left\|A\exp\left\{-\frac{\tau}{2}A\right\}\varphi\right\|_E d\tau$$
$$\le \left(\int_0^N (1+M_1)\tau^{\alpha-1}d\tau + \int_N^\infty \frac{MM_1 2^{2-\alpha}\tau^{\alpha-2}}{|z|+1}d\tau\right)\|\varphi\|_{E_\alpha}$$
$$= \psi(N)\|\varphi\|_{E_\alpha}. \tag{2.16}$$

Here

$$\psi(N) = (1+M_1)\frac{N^\alpha}{\alpha} + \frac{MM_1 2^{2-\alpha}N^{\alpha-1}}{(|z|+1)(1-\alpha)}. \tag{2.17}$$

Taking the minimum of the function $\psi(N)$ over all N, $0 < N < \infty$, we obtain Lemma 2.2. $\square$

LEMMA 2.3. *For all $\tau > 0$ the following estimate holds:*

$$\left\|A^{-1}\left[A\exp\{-\tau A\}B(s) - B(s)A\exp\{-\tau A\}\right]\varphi\right\|_E$$
$$\le \frac{e(\alpha+1)M^\alpha M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha}\|Q\|_{E\mapsto E}\|\varphi\|_{E_\alpha}}{\tau^{1-\alpha}\pi\alpha^2(1-\alpha)}. \tag{2.18}$$

Here $Q = \overline{A^{-1}(AB(s) - B(s)A)A^{-1}}$.

Proof. Applying the Cauchy-Riesz formula, we obtain the representation

$$
A \exp\{-\tau A\}\varphi = \frac{1}{2\pi i} \int_{S_1 \cup S_2} \exp\{-\tau z\} A(z - A)^{-1} dz\varphi
$$

$$
= -\frac{1}{2\pi i} \int_{S_1 \cup S_2} \exp\{-\tau z\} dz\varphi \tag{2.19}
$$

$$
+ \frac{1}{2\pi i} \int_{S_1 \cup S_2} \exp\{-\tau z\} z(z - A)^{-1} dz\varphi
$$

for every τ, $\tau > 0$. From that it follows that

$$
A^{-1} \big[A \exp\{-\tau A\} B(s) - B(s) A \exp\{-\tau A\} \big] \varphi
$$

$$
= \frac{1}{2\pi i} \int_{S_1 \cup S_2} \exp\{-\tau z\} z A^{-1} \big[(z - A)^{-1} B(s) - B(s)(z - A)^{-1} \big] dz\varphi \tag{2.20}
$$

$$
= \frac{1}{2\pi i} \int_{S_1 \cup S_2} \exp\{-\tau z\} z(z - A)^{-1} Q A(z - A)^{-1} dz\varphi.
$$

Making the substitution $z = -1/\tau + iy$, $-\infty < y < \infty$ and using integration by parts, we can write the formula

$$
A^{-1} \big[A \exp\{-\tau A\} B(s) - B(s) A \exp\{-\tau A\} \big] \varphi
$$

$$
= \frac{e}{2\pi} \int_{-\infty}^{\infty} \exp(-\tau iy) \left(-\frac{1}{\tau} + iy \right) \left(-\frac{1}{\tau} + iy - A \right)^{-1} Q A
$$

$$
\times \left(-\frac{1}{\tau} + iy - A \right)^{-1} dy\varphi
$$

$$
= \frac{e}{2\pi\tau} \int_{-\infty}^{\infty} \exp(-\tau iy) \left(-\frac{1}{\tau} + iy - A \right)^{-1} \tag{2.21}
$$

$$
\times \left[Q - \left(-\frac{1}{\tau} + iy \right) \left(-\frac{1}{\tau} + iy - A \right)^{-1} Q \right.
$$

$$
\left. - \left(-\frac{1}{\tau} + iy \right) Q \left(-\frac{1}{\tau} + iy - A \right)^{-1} \right]
$$

$$
\times A \left(-\frac{1}{\tau} + iy - A \right)^{-1} dy\varphi.
$$

By Lemma 2.2 and performing the change of variable $\tau y = x$, and using the estimate (2.12), we obtain

$$
\left\| A^{-1} \big[A \exp\{-\tau A\} B(s) - B(s) A \exp\{-\tau A\} \big] \varphi \right\|_E
$$

$$
\leq \frac{e}{2\pi\tau} \int_{-\infty}^{\infty} \left| \exp(-\tau i y) \right| \left\| \left(-\frac{1}{\tau} + i y - A \right)^{-1} \right\|_{E \mapsto E}
$$

$$
\times (1 + 2M_1) \| Q \|_{E \mapsto E} \left\| A \left(-\frac{1}{\tau} + i y - A \right)^{-1} dy\varphi \right\|_E
$$

$$
\leq \frac{e}{2\pi\tau} \int_{-\infty}^{\infty} \frac{dy}{\left(1/\tau^2 + y^2 \right)^{(1+\alpha)/2}}
$$

$$
\times \frac{(1 + 2M_1) \| Q \|_{E \mapsto E} M_1^{1+\alpha} M^\alpha (1 + M_1)^{1-\alpha} 2^{(2-\alpha)\alpha} \| \varphi \|_{E_\alpha}}{\alpha(1-\alpha)}
$$

$$
= \frac{e}{\pi\tau^{1-\alpha}} \int_0^{\infty} \frac{dx}{\left(1 + x^2 \right)^{(1+\alpha)/2}}
$$

$$
\times \frac{(1 + 2M_1) \| Q \|_{E \mapsto E} M_1^{1+\alpha} M^\alpha (1 + M_1)^{1-\alpha} 2^{(2-\alpha)\alpha} \| \varphi \|_{E_\alpha}}{\alpha(1-\alpha)}
$$

$$
\leq \frac{e(1+\alpha)(1 + 2M_1) \| Q \|_{E \mapsto E} M_1^{1+\alpha} M^\alpha (1 + M_1)^{1-\alpha} 2^{(2-\alpha)\alpha} \| \varphi \|_{E_\alpha}}{\pi\tau^{1-\alpha} \alpha^2 (1-\alpha)}.
$$

$$
(2.22)
$$

Lemma 2.3 is thus proved. $\square$

Suppose that

$$
\left\| \overline{A^{-1} \big(AB(t) - B(t)A \big) A^{-1}} \right\|_{E \mapsto E}
$$

$$
\leq \frac{\pi(1-\alpha)^2 \alpha^2 \epsilon}{e M^{1+\alpha} M_1^{1+\alpha} (1 + 2M_1)(1 + M_1)^{1-\alpha} 2^{2+\alpha-\alpha^2}(1+\alpha)}
$$

$$
(2.23)
$$

holds for every $t \geq 0$. Here and in the future ϵ is some constant, $0 \leq \epsilon \leq 1$.

THEOREM 2.4. *Assume that the condition*

$$
\left\| \overline{A^{-1} B(t)} \right\|_{E \mapsto E} \leq \frac{(1-\alpha)(1-\epsilon)}{M 2^{2-\alpha}}
$$

$$
(2.24)
$$

holds for every $t \geq 0$. Then for every $t \geq 0$ the estimate (2.4) holds.

Proof. Using formula (2.5) we have

$$\lambda^{1-\alpha} A \exp\{-\lambda A\} u(t)$$

$$= \lambda^{1-\alpha} A \exp\left\{-(\lambda+t)A\right\} g(0)$$

$$+ \lambda^{1-\alpha} \int_0^t \exp\left\{-\frac{\lambda+t-s}{2}A\right\} B(s)$$

$$\times A \exp\left\{-\frac{\lambda+t-s}{2}A\right\} g(s-\omega) ds$$

$$+ \lambda^{1-\alpha} \int_0^t \exp\left\{-\frac{\lambda+t-s}{2}A\right\}$$

$$\times \left[A \exp\left\{-\frac{\lambda+t-s}{2}A\right\} B(s) \right.$$

$$\left. - B(s) A \exp\left\{-\frac{\lambda+t-s}{2}A\right\} \right] g(s-\omega) ds$$

$$= I_1 + I_2 + I_3.$$

$$(2.25)$$

Using the estimates (1.2), (2.12), and condition (2.24) we have

$$\|I_1\|_E = \lambda^{1-\alpha} \|A \exp\{-(\lambda+t)A\} g(0)\|_E$$

$$\leq \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}} \|g(0)\|_{E_\alpha} \leq \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}} \max_{-\omega \leq t \leq 0} \|g(t)\|_{E_\alpha},$$

$$\|I_2\|_E \leq \lambda^{1-\alpha} \int_0^t \left\| A \exp\left\{-\frac{\lambda+t-s}{2}A\right\} \right\|_{E \mapsto E} \left\| \overline{A^{-1}B(s)} \right\|_{E \mapsto E}$$

$$\times \left\| A \exp\left\{-\frac{\lambda+t-s}{2}A\right\} g(s-\omega) \right\|_E ds$$

$$\leq \max_{0 \leq t \leq \omega} \left\| \overline{A^{-1}B(t)} \right\|_{E \mapsto E} \int_0^t \frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}} ds \max_{0 \leq s \leq \omega} \|g(s-\omega)\|_{E_\alpha}$$

$$\leq \max_{-\omega \leq t \leq 0} \|g(t)\|_{E_\alpha} \left(1 - \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}} \right) (1-\epsilon)$$

$$(2.26)$$

for every t, $0 \leq t \leq \omega$ and λ, $\lambda > 0$. Now we estimate I_3. By Lemma 2.3 and using the estimate (2.23), we obtain

$$
\begin{aligned}
\left\| I_3 \right\|_E \leq \lambda^{1-\alpha} \int_0^t & \left\| A \exp\left\{ -\frac{\lambda + t - s}{2} A \right\} \right\|_{E \mapsto E} \\
& \times \left\| A^{-1} \left[A \exp\left\{ -\frac{\lambda + t - s}{2} A \right\} B(s) - B(s) \right. \right. \\
& \left. \left. \times A \exp\left\{ -\frac{\lambda + t - s}{2} A \right\} \right] g(s - \omega) \right\|_E ds \\
\leq \lambda^{1-\alpha} e(1+\alpha) & M^{1+\alpha} M_1^{1+\alpha} (1 + 2M_1)(1 + M_1)^{1-\alpha} 2^{(2-\alpha)\alpha} \\
& \times \int_0^t \frac{\overline{\left\| A^{-1}(AB(s) - B(s)A)A^{-1} \right\|}_{E \mapsto E} 2^{2-\alpha} \left\| g(s - \omega) \right\|_{E_\alpha} ds}{(\lambda + t - s)^{2-\alpha} \pi \alpha^2 (1 - \alpha)} \\
\leq \max_{0 \leq s \leq \omega} & \overline{\left\| A^{-1}(AB(s) - B(s)A)A^{-1} \right\|}_{E \mapsto E} \\
& \times \int_0^t \frac{e(1+\alpha) M^{1+\alpha} M_1^{1+\alpha} (1 + 2M_1)(1 + M_1)^{1-\alpha} 2^{(2-\alpha)\alpha} 2^{2-\alpha} ds}{(\lambda + t - s)^{2-\alpha} \pi \alpha^2 (1 - \alpha)} \\
& \times \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha} \\
\leq \max_{-\omega \leq t \leq 0} & \left\| g(t) \right\|_{E_\alpha} \left(1 - \frac{\lambda^{1-\alpha}}{(\lambda + t)^{1-\alpha}} \right) \epsilon
\end{aligned}
\tag{2.27}
$$

for every t, $0 \leq t \leq \omega$ and λ, $\lambda > 0$. Using the triangle inequality we obtain

$$
\lambda^{1-\alpha} \left\| A \exp\{ -\lambda A \} u(t) \right\|_E \leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha}
\tag{2.28}
$$

for every t, $0 \leq t \leq \omega$ and λ, $\lambda > 0$. This shows that

$$
\left\| u(t) \right\|_{E_\alpha} \leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha}
\tag{2.29}
$$

for every t, $0 \leq t \leq \omega$. Applying the mathematical induction, one can easily show that it is true for every t. Namely, assume that inequality (2.4) is true for every t, $(n-1)\omega \leq t \leq n\omega$, $n = 1, 2, 3, \ldots$. Therefore, using formula (2.24)

we have

$$\lambda^{1-\alpha} A \exp\{-\lambda A\} u(t)$$

$$= \lambda^{1-\alpha} A \exp\left\{-(\lambda + t - n\omega)A\right\} u(n\omega)$$

$$+ \lambda^{1-\alpha} \int_{n\omega}^{t} \exp\left\{-\frac{\lambda + t - s}{2}A\right\} B(s) A \exp\left\{-\frac{\lambda + t - s}{2}A\right\} u(s - \omega) ds$$

$$+ \lambda^{1-\alpha} \int_{n\omega}^{t} \exp\left\{-\frac{\lambda + t - s}{2}A\right\} \left[A \exp\left\{-\frac{\lambda + t - s}{2}A\right\} B(s)\right.$$

$$\left. - B(s) A \exp\left\{-\frac{\lambda + t - s}{2}A\right\}\right]$$

$$\times u(s - \omega) ds$$

$$= I_{1n} + I_{2n} + I_{3n}.$$

$$(2.30)$$

Using the estimates (1.2), (2.12), and condition (2.24), we have

$$\|I_{1n}\|_E = \lambda^{1-\alpha} \|A \exp\left\{-(\lambda + t - n\omega)A\right\} u(n\omega)\|_E$$

$$\leq \frac{\lambda^{1-\alpha}}{(\lambda + t - n\omega)^{1-\alpha}} \|u(n\omega)\|_{E_\alpha} \leq \frac{\lambda^{1-\alpha}}{(\lambda + t - n\omega)^{1-\alpha}} \max_{-\omega \leq t \leq 0} \|g(t)\|_{E_\alpha},$$

$$\|I_{2n}\|_E \leq \lambda^{1-\alpha} \int_{n\omega}^{t} \left\|A \exp\left\{-\frac{\lambda + t - s}{2}A\right\}\right\|_{E \mapsto E} \|\overline{A^{-1} B(s)}\|_{E \mapsto E}$$

$$\times \left\|A \exp\left\{-\frac{\lambda + t - s}{2}A\right\} u(s - \omega)\right\|_E ds$$

$$\leq \max_{n\omega \leq t \leq (n+1)\omega} \|\overline{A^{-1} B(t)}\|_{E \mapsto E} \int_{n\omega}^{t} \frac{M \lambda^{1-\alpha} 2^{2-\alpha}}{(\lambda + t - s)^{2-\alpha}} ds$$

$$\times \max_{n\omega \leq s \leq (n+1)\omega} \|u(s - \omega)\|_{E_\alpha}$$

$$\leq \max_{-\omega \leq t \leq 0} \|g(t)\|_{E_\alpha} \left(1 - \frac{\lambda^{1-\alpha}}{(\lambda + t)^{1-\alpha}}\right)(1 - \epsilon)$$

$$(2.31)$$

for every t, $n\omega \leq t \leq (n+1)\omega$ and λ, $\lambda > 0$. Now we estimate I_{3n}. By Lemma 2.3 and using the estimate (2.23), we obtain

$$
\begin{aligned}
\left\| I_{3n} \right\|_E &\leq \lambda^{1-\alpha} \int_{n\omega}^t \left\| A \exp\left\{ -\frac{\lambda+t-s}{2} A \right\} \right\|_{E \mapsto E} \\
&\quad \times \left\| A^{-1} \left[A \exp\left\{ -\frac{\lambda+t-s}{2} A \right\} B(s) \right.\right. \\
&\qquad\qquad \left.\left. - B(s) A \exp\left\{ -\frac{\lambda+t-s}{2} A \right\} \right] u(s-\omega) \right\|_E ds \\
&\leq \lambda^{1-\alpha} e(1+\alpha) M^{1+\alpha} M_1^{1+\alpha} \left(1+2M_1\right)\left(1+M_1\right)^{1-\alpha} 2^{(2-\alpha)\alpha} \\
&\quad \times \int_{n\omega}^t \frac{\overline{\left\| A^{-1}\left(AB(s)-B(s)A\right)A^{-1} \right\|}_{E \mapsto E} 2^{2-\alpha} \left\| u(s-\omega) \right\|_{E_\alpha} ds}{(\lambda+t-s)^{2-\alpha} \pi \alpha^2 (1-\alpha)} \\
&\leq \max_{n\omega \leq s \leq (n+1)\omega} \overline{\left\| A^{-1}\left(AB(s)-B(s)A\right)A^{-1} \right\|}_{E \mapsto E} \\
&\quad \times \int_{n\omega}^t \frac{e(1+\alpha) M^{1+\alpha} M_1^{1+\alpha} \left(1+2M_1\right)\left(1+M_1\right)^{1-\alpha} 2^{(2-\alpha)\alpha} 2^{2-\alpha} ds}{(\lambda+t-s)^{2-\alpha} \pi \alpha^2 (1-\alpha)} \\
&\quad \times \max_{n\omega \leq t \leq (n+1)\omega} \left\| u(s-\omega) \right\|_{E_\alpha} \\
&\leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha} \left(1 - \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}} \right) \epsilon
\end{aligned}
$$

(2.32)

for every t, $n\omega \leq t \leq (n+1)\omega$ and λ, $\lambda > 0$. Using the triangle inequality we obtain

$$
\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} u(t) \right\|_E \leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha}
\tag{2.33}
$$

for every t, $n\omega \leq t \leq (n+1)\omega$, $n = 1, 2, 3, \ldots$, and λ, $\lambda > 0$. This shows that

$$
\left\| u(t) \right\|_{E_\alpha} \leq \max_{-\omega \leq t \leq 0} \left\| g(t) \right\|_{E_\alpha}
\tag{2.34}
$$

for every t, $n\omega \leq t \leq (n+1)\omega$, $n = 1, 2, 3, \ldots$. This result completes the proof of Theorem 2.4. $\qquad\square$

3. The delay difference equation

Using the first and second order accuracy implicit difference schemes for differential equations without delay (cf. [5]), we have the following approximate

solution for the initial-value problem (1.1)

$$\frac{1}{\tau}(u_k - u_{k-1}) + Au_k = B_k u_{k-N},$$

$$B_k = B(t_k), \quad t_k = k\tau, \ 1 \le k, \qquad N\tau = \omega, \tag{3.1}$$

$$u_k = g(t_k), \quad t_k = k\tau, \ -N \le k \le 0;$$

$$\frac{1}{\tau}(u_k - u_{k-1}) + \left(A + \frac{1}{2}\tau A^2\right)u_k = \frac{1}{2}\left(I + \frac{\tau}{2}A\right)B_k(u_{k-N} + u_{k-N-1}),$$

$$B_k = B\left(t_k - \frac{\tau}{2}\right), \quad t_k = k\tau, \ 1 \le k, \tag{3.2}$$

$$u_k = g(t_k), \quad t_k = k\tau, \ -N \le k \le 0.$$

THEOREM 3.1. *Assume that all the conditions of Theorem 2.1 are satisfied. Then for the solution of difference scheme (3.1) the estimate*

$$\|u_k\|_{E_\alpha} \le \max_{-N \le k \le 0} \|g(t_k)\|_{E_\alpha} \tag{3.3}$$

holds for any $k \ge 1$.

Proof. Consider $1 \le k \le N$. In this case

$$u_k = R^k g(0) + \sum_{j=1}^{k} R^{k-j+1} g(t_{j-N})\tau, \tag{3.4}$$

where $R = (1 + \tau A)^{-1}$. Using the formula (see [5, 11])

$$(I + \tau A)^{-k} = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \exp(-\tau t A) dt, \quad k \ge 1, \tag{3.5}$$

and condition (2.3), we have

$$\lambda^{1-\alpha} \|A \exp\{-\lambda A\} u_k\|_E$$

$$\le \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \|A \exp\{-(\tau t + \lambda)A\} g(0)\|_E$$

$$+ \lambda^{1-\alpha} \sum_{j=1}^{k} \tau \frac{1}{(k-j)!}$$

$$\times \int_0^\infty t^{k-j} \exp(-t) \left\|A \exp\left\{-\frac{(\tau t + \lambda)A}{2}\right\}\right\|_{E \mapsto E} \|B_j A^{-1}\|_{E \mapsto E}$$

$$\times \left\|A \exp\left\{-\frac{(\tau t + \lambda)A}{2}\right\} g(t_{j-N})\right\|_E$$

$$\leq \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}} \left\| g(0) \right\|_{E_\alpha}$$

$$+\lambda^{1-\alpha} \frac{1-\alpha}{M2^{2-\alpha}} \sum_{j=1}^k \tau \frac{1}{(k-j)!}$$

$$\times \int_0^\infty t^{k-j} \exp(-t) \frac{M2^{2-\alpha}\,dt}{(t\tau+\lambda)^{2-\alpha}} \left\| g(t_{j-N}) \right\|_{E_\alpha}$$

$$\leq J \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha}, \tag{3.6}$$

where

$$J = \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}}$$

$$+\lambda^{1-\alpha}(1-\alpha) \sum_{j=1}^k \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{dt}{(t\tau+\lambda)^{2-\alpha}}. \tag{3.7}$$

We will calculate J. Making the substitution $k-j=m$ and using Taylor's formula and integration by parts, we obtain

$$J = \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}}$$

$$+\lambda^{1-\alpha}(1-\alpha) \sum_{m=0}^{k-1} \tau \frac{1}{m!} \int_0^\infty t^m \exp(-t) \frac{dt}{(t\tau+\lambda)^{2-\alpha}}$$

$$= \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}}$$

$$+\lambda^{1-\alpha}(1-\alpha) \int_0^\infty \exp(t) \left[1 - \int_0^t \frac{\xi^{k-1}\exp(-\xi)d\xi}{(k-1)!} \right] \exp(-t) \frac{dt}{(t\tau+\lambda)^{2-\alpha}}$$

$$= \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}} + 1$$

$$-\lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}}$$

$$= 1. \tag{3.8}$$

Therefore,

$$\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} u_k \right\|_E \leq \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{3.9}$$

for every k, $1 \le k \le N$ and λ, $\lambda > 0$. This shows that

$$\left\| u_k \right\|_{E_\alpha} \le \max_{-N \le k \le 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{3.10}$$

for every k, $1 \le k \le N$. Assume that the estimate (3.2) is true for k, where $(n-1)N \le k \le nN$, $n = 1, 2, 3, \ldots$. Then for any k, with $nN \le k \le (n+1)N$ using formula (3.5) and

$$u_k = R^{k-nN} u_{nN} + \sum_{j=nN+1}^{k} \tau R^{k-j+1} B_j u_{j-N} \tag{3.11}$$

and the estimates (1.2) and condition (2.3), we have

$$\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} u_k \right\|_E$$

$$\le \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \left\| A \exp\left(-(\tau t + \lambda)A \right) u_{nN} \right\|_E$$

$$+ \lambda^{1-\alpha} \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \left\| A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} \right\|_{E \to E}$$

$$\times \left\| B_j A^{-1} \right\|_{E \to E} \times \left\| A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} u_{j-N} \right\|_E$$

$$\le \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \frac{dt}{(t\tau + \lambda)^{1-\alpha}} \left\| u_{nN} \right\|_{E_\alpha}$$

$$+ \lambda^{1-\alpha} \frac{1-\alpha}{M 2^{2-\alpha}} \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{M 2^{2-\alpha} dt}{(t\tau + \lambda)^{2-\alpha}} \left\| u_{j-N} \right\|_{E_\alpha}$$

$$\le J_n \max_{(n-1)N \le k \le nN} \left\| u_k \right\|_{E_\alpha}, \tag{3.12}$$

where

$$J_n = \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \frac{dt}{(t\tau + \lambda)^{1-\alpha}}$$

$$+ \lambda^{1-\alpha}(1-\alpha) \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{dt}{(t\tau + \lambda)^{2-\alpha}}. \tag{3.13}$$

We will calculate J_n. Making the substitution $k - j = m$ and using Taylor's formula and integration by parts, we obtain

$$
\begin{aligned}
J_n &= \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}} \\
&\quad + \lambda^{1-\alpha}(1-\alpha) \sum_{m=0}^{k-nN-1} \tau \frac{1}{m!} \int_0^\infty t^m \exp(-t) \frac{dt}{(t\tau+\lambda)^{2-\alpha}} \\
&= \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}} \\
&\quad + \lambda^{1-\alpha}(1-\alpha) \int_0^\infty \exp(t) \\
&\qquad\qquad \times \left[1 - \int_0^t \frac{\xi^{k-nN-1}\exp(-\xi)d\xi}{(k-nN-1)!} \right] \exp(-t) \frac{dt}{(t\tau+\lambda)^{2-\alpha}} \\
&= \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}} \\
&\quad + 1 - \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}} \\
&= 1.
\end{aligned}
\tag{3.14}
$$

Therefore

$$
\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} u_k \right\|_E \le \max_{-N \le k \le 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{3.15}
$$

for every k, $nN \le k \le (n+1)N$, $n = 1, 2, 3, \ldots$, and λ, $\lambda > 0$. This shows that

$$
\left\| u_k \right\|_{E_\alpha} \le \max_{-\omega \le k \le 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{3.16}
$$

for every k, $nN \le k \le (n+1)N$, $n = 1, 2, 3, \ldots$. This result completes the proof of Theorem 3.1. □

THEOREM 3.2. *Assume that all the conditions of Theorem 2.4 are satisfied. Then for the solution of the difference scheme (3.1), the estimate (3.3) holds for any* $k \ge 1$.

Proof. Consider $1 \le k \le N$. Using formula (3.5) we have

$$\lambda^{1-\alpha} A \exp\{-\lambda A\} u_k = \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) A \exp\left(-(\tau t + \lambda)A\right) g(0)$$

$$+ \lambda^{1-\alpha} \sum_{j=1}^k \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \exp\left\{-\frac{\tau t + \lambda}{2}A\right\}$$

$$\times B_j A \exp\left\{-\frac{\tau t + \lambda}{2}A\right\} g(t_{j-N})$$

$$+ \lambda^{1-\alpha} \sum_{j=1}^k \tau \frac{1}{(k-j)!}$$

$$\times \int_0^\infty t^{k-j} \exp(-t) A \exp\left\{-\frac{\tau t + \lambda}{2}A\right\} A^{-1}$$

$$\times \left[\exp\left\{-\frac{\tau t + \lambda}{2}A\right\} A B_j - B_j A \exp\left\{-\frac{\tau t + \lambda}{2}A\right\}\right] g(t_{j-N})$$

$$= T_1 + T_2 + T_3.$$

$$(3.17)$$

Using the estimates (1.2), (2.12) and assumption (2.24), we have

$$\|T_1\|_E \leq \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \|A \exp\left\{-(\tau t + \lambda)A\right\} g(0)\|_E$$

$$\leq \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \frac{dt}{(t\tau + \lambda)^{1-\alpha}} \max_{-N \leq k \leq 0} \|g(t_k)\|_{E_\alpha},$$

$$\|T_2\|_E \leq \lambda^{1-\alpha} \sum_{j=1}^k \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \left\|A \exp\left\{-\frac{\tau t + \lambda}{2}A\right\}\right\|_{E \to E}$$

$$\times \|B_j A^{-1}\|_{E \to E} \times \left\|A \exp\left\{-\frac{\tau t + \lambda}{2}A\right\} g(t_{j-N})\right\|_E$$

$$\leq \frac{\lambda^{1-\alpha}(1-\alpha)(1-\epsilon)}{M2^{2-\alpha}}$$

$$\times \sum_{j=1}^k \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{M2^{2-\alpha} dt}{(t\tau + \lambda)^{2-\alpha}} \|g(t_{j-N})\|_{E_\alpha}$$

$$\leq \lambda^{1-\alpha}(1-\alpha)(1-\epsilon) \sum_{j=1}^k \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{dt}{(t\tau + \lambda)^{2-\alpha}}$$

$$\times \max_{-N \leq k \leq 0} \|g(t_k)\|_{E_\alpha}$$

$$(3.18)$$

for every k, $1 \le k \le N$ and λ, $\lambda > 0$. Now we estimate T_3. By Lemma 2.3 and using the estimate (2.23), we obtain

$$
\|T_3\|_E \le \lambda^{1-\alpha} \sum_{j=1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \left\| A \exp\left\{ -\frac{\tau t + \lambda}{2} A \right\} \right\|_{E \to E}
$$

$$
\times \left\| A^{-1} \left[A \exp\left\{ -\frac{\tau t + \lambda}{2} A \right\} B_j - B_j A \exp\left\{ -\frac{\tau t + \lambda}{2} A \right\} \right] g(t_{j-N}) \right\|_E
$$

$$
\le \lambda^{1-\alpha} \max_{1 \le k \le N} \left\| \overline{A^{-1}(AB_j - B_j A)A^{-1}} \right\|_{E \to E} \sum_{j=1}^{k} \tau \frac{1}{(k-j)!}
$$

$$
\times \int_0^\infty t^{k-j} \exp(-t) \frac{e(1+\alpha) M^\alpha M_1^{1+\alpha} (1+2M_1)(1+M_1)^{1-\alpha} 2^{(2-\alpha)\alpha}}{\pi \alpha^2 (1-\alpha)}
$$

$$
\times \frac{M 2^{2-\alpha} \, dt}{(t\tau + \lambda)^{2-\alpha}} \left\| g(t_{j-N}) \right\|_{E_\alpha}
$$

$$
\le \lambda^{1-\alpha}(1-\alpha)\epsilon \sum_{j=1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{dt}{(t\tau + \lambda)^{2-\alpha}}
$$

$$
\times \max_{-N \le k \le 0} \left\| g(t_k) \right\|_{E_\alpha}
$$

$$
\tag{3.19}
$$

for every k, $1 \le k \le N$ and λ, $\lambda > 0$. Using the triangle inequality we obtain

$$
\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} u_k \right\|_E \le \max_{-N \le k \le 0} J \left\| g(t_k) \right\|_{E_\alpha} \tag{3.20}
$$

for every k, $1 \le k \le N$ and λ, $\lambda > 0$. This shows that

$$
\left\| u_k \right\|_{E_\alpha} \le \max_{-N \le k \le 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{3.21}
$$

for every k, $1 \le k \le N$. Applying the mathematical induction, one can easily show that it is true for every k. Namely, assume that inequality (3.2) is true for every k, $(n-1)N \le k \le nN, n = 1, 2, 3, \ldots$. Then for any $k, nN \le k \le (n+1)N$ using formula (2.8) we have

$$\lambda^{1-\alpha} A \exp\{-\lambda A\} u_k$$

$$= \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) A \exp\{-(\tau t + \lambda)A\} u_{nN}$$

$$+ \lambda^{1-\alpha} \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \exp\left\{-\frac{\tau t + \lambda}{2} A\right\} B_j$$

$$\times A \exp\left\{-\frac{\tau t + \lambda}{2} A\right\} u_{j-N}$$

$$+ \lambda^{1-\alpha} \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) A \exp\left\{-\frac{\tau t + \lambda}{2} A\right\}$$

$$\times A^{-1}\left[\exp\left\{-\frac{\tau t + \lambda}{2} A\right\} A B_j - B_j A \exp\left\{-\frac{\tau t + \lambda}{2} A\right\}\right] u_{j-N}$$

$$= T_{1n} + T_{2n} + T_{3n}.$$

$$(3.22)$$

Using the estimates (1.2), (2.12), and assumption (2.24) we have

$$\left\| T_{1n} \right\|_E \leq \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \left\| A \exp\{-(\tau t + \lambda)A\} u_{nN} \right\|_E$$

$$\leq \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \frac{dt}{(t\tau+\lambda)^{1-\alpha}} \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha},$$

$$\left\| T_{2n} \right\|_E \leq \lambda^{1-\alpha} \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \left\| A \exp\left\{-\frac{\tau t + \lambda}{2} A\right\} \right\|_{E \to E}$$

$$\times \left\| \overline{A^{-1} B_j} \right\|_{E \to E} \times \left\| A \exp\left\{-\frac{\tau t + \lambda}{2} A\right\} u_{j-N} \right\|_E$$

$$\leq \frac{\lambda^{1-\alpha}(1-\alpha)(1-\epsilon)}{M2^{2-\alpha}}$$

$$\times \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{M2^{2-\alpha} dt}{(t\tau+\lambda)^{2-\alpha}} \left\| u_{j-N} \right\|_{E_\alpha}$$

$$\leq \lambda^{1-\alpha}(1-\alpha)(1-\epsilon) \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{dt}{(t\tau+\lambda)^{2-\alpha}}$$

$$\times \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha}$$

$$(3.23)$$

for every $k, nN \le k \le (n+1)N, n = 1, 2, 3, \dots$, and $\lambda, \lambda > 0$. Now we estimate T_{3n}. By Lemma 2.3 and using the estimate (2.23), we obtain

$$
\begin{aligned}
\left\| T_{3n} \right\|_E &\le \lambda^{1-\alpha} \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \left\| A \exp\left\{ -\frac{\tau t + \lambda}{2} A \right\} \right\|_{E \to E} \\
&\quad \times \left\| A^{-1} \left[A \exp\left\{ -\frac{\tau t + \lambda}{2} A \right\} B_j - B_j A \exp\left\{ -\frac{\tau t + \lambda}{2} A \right\} \right] u_{j-N} \right\|_E \\
&\le \lambda^{1-\alpha} \max_{nN+1 \le k \le (n+1)N} \left\| \overline{A^{-1} (AB_j - B_j A) A^{-1}} \right\|_{E \to E} \sum_{j=1}^{k} \tau \frac{1}{(k-j)!} \\
&\quad \times \int_0^\infty t^{k-j} \exp(-t) \frac{e(1+\alpha) M^\alpha M_1^{1+\alpha} (1+2M_1)(1+M_1)^{1-\alpha} 2^{(2-\alpha)\alpha}}{\pi \alpha^2 (1-\alpha)} \\
&\quad \times \frac{M 2^{2-\alpha} \, dt}{(t\tau + \lambda)^{2-\alpha}} \left\| u_{j-N} \right\|_{E_\alpha} \\
&\le \lambda^{1-\alpha} (1-\alpha) \epsilon \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{dt}{(t\tau + \lambda)^{2-\alpha}} \\
&\quad \times \max_{-N \le k \le 0} \left\| g(t_k) \right\|_{E_\alpha}
\end{aligned}
\tag{3.24}
$$

for every $k, nN \le k \le (n+1)N, n = 1, 2, 3, \dots$, and $\lambda, \lambda > 0$. Using the triangle inequality we obtain

$$
\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} u_k \right\|_E \le \max_{-N \le k \le 0} J_n \left\| g(t_k) \right\|_{E_\alpha}
\tag{3.25}
$$

for every $k, nN \le k \le (n+1)N, n = 1, 2, 3, \dots$, and $\lambda, \lambda > 0$. This shows that

$$
\left\| u_k \right\|_{E_\alpha} \le \max_{-\omega \le k \le 0} \left\| g(t_k) \right\|_{E_\alpha}
\tag{3.26}
$$

for every $k, nN \le (n+1)N, n = 1, 2, 3, \dots$. This result completes the proof of Theorem 3.2. □

Now we consider the difference scheme (3.2). We have not been able to obtain the same result for solution of the difference scheme (3.2) in the spaces E_α under the assumption for the difference scheme (3.1). But, we have the asymptotical stability of the solution of the difference scheme under the supplementary restriction of the operator A that is considered in the following section.

4. The difference scheme (3.2)

THEOREM 4.1. *Let condition (1.2) be satisfied. Suppose that the following estimate holds:*

$$\left\| (I + \tau A)\left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right\|_{E \mapsto E} \leq 1,$$

$$\left\| \left(I + \frac{\tau A}{2} \right)(I + \tau A)\left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right\|_{E \mapsto E} \leq \frac{1 + \sqrt{2}}{2},$$

(4.1)

and

$$\| B(t)A^{-1} \|_{E \mapsto E} \leq \frac{(1 - \alpha)}{M 2^{1-\alpha}(1 + \sqrt{2})} \tag{4.2}$$

holds for every $t \geq 0$.

Then for the solution of difference scheme (3.2) the estimate

$$\| u_k \|_{E_\alpha} \leq \max_{-N \leq k \leq 0} \| g(t_k) \|_{E_\alpha} \tag{4.3}$$

holds for any $k \geq 1$.

Proof. Consider $1 \leq k \leq N$. In this case

$$u_k = R^k g(0) + \sum_{j=1}^{k} R^{k-j+1}\left(I + \frac{\tau A}{2} \right)\left(g(t_{j-N}) + g(t_{j-N-1}) \right)\tau, \tag{4.4}$$

where $R = (I + \tau A + (\tau A)^2/2)^{-1}$. Using formula (3.5), the estimates (1.2), and assumption (4.1) we have

$$\lambda^{1-\alpha} \| A \exp\{-\lambda A\} u_k \|_E \leq \left\| \left((I + \tau A)\left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right)^k \right\|_{E \mapsto E}$$

$$\times \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \| A \exp\{ -(\tau t + \lambda)A \} g(0) \|_E$$

$$+ \lambda^{1-\alpha} \sum_{j=1}^{k} \tau \left\| \left((I + \tau A)\left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right)^{k-j} \right\|_{E \mapsto E}$$

$$\times \left\| \left(I + \frac{\tau A}{2} \right)(I + \tau A)\left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right\|_{E \mapsto E}$$

$$\times \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \left\| A \exp\left\{ -\frac{\tau t + \lambda}{2} A \right\} \right\|_{E \to E} \left\| B_j A^{-1} \right\|_{E \to E}$$

$$\times \left\| A \exp\left\{ -\frac{\tau t + \lambda}{2} A \right\} \frac{1}{2} \left(g(t_{j-N}) + g(t_{j-N-1}) \right) \right\|_E$$

$$\leq \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) \frac{dt}{(t\tau + \lambda)^{1-\alpha}} \left\| g(0) \right\|_{E_\alpha}$$

$$+ \frac{(1+\sqrt{2})}{2} \frac{1-\alpha}{M 2^{1-\alpha}(1+\sqrt{2})} \sum_{j=1}^k \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{M 2^{2-\alpha} dt}{(t\tau + \lambda)^{2-\alpha}}$$

$$\times \left\| \frac{1}{2} \left(g(t_{j-N}) + g(t_{j-N-1}) \right) \right\|_{E_\alpha}$$

$$\leq J \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha}. \tag{4.5}$$

Therefore

$$\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} u_k \right\|_E \leq \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{4.6}$$

for every k, $1 \leq k \leq N$ and λ, $\lambda > 0$. This shows that

$$\left\| u_k \right\|_{E_\alpha} \leq \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{4.7}$$

for every k, $1 \leq k \leq N$. Assume that the estimate (4.3) is true for k, with $(n-1)N \leq k \leq nN$, $n = 1, 2, 3, \ldots$. Then for any k, $nN \leq k \leq (n+1)N$ using formula (3.5) and

$$u_k = R^{k-nN} u_{nN} + \sum_{j=nN+1}^k \tau R^{k-j} \left(I + \frac{\tau A}{2} \right) B_j \frac{1}{2} \left(u_{j-N} + u_{j-N-1} \right), \tag{4.8}$$

the estimates (1.2), and assumption (4.1) we have that

$$\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} u_k \right\|_E$$

$$\leq \left\| \left((I + \tau A) \left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right)^{k-nN} \right\|_{E \mapsto E}$$

$$\times \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) \left\| A \exp\left\{ -(\tau t + \lambda) A \right\} u_{nN} \right\|_E$$

$$+\lambda^{1-\alpha}\sum_{j=nN+1}^{k}\tau\left\|\left((I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1}\right)^{k-j}\right\|_{E\mapsto E}$$

$$\times\left\|\left(I+\frac{\tau A}{2}\right)(I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1}\right\|_{E\mapsto E}$$

$$\times\frac{1}{(k-j)!}\int_0^\infty t^{k-j}\exp(-t)\left\|A\exp\left\{-\frac{(\tau t+\lambda)A}{2}\right\}\right\|_{E\to E}$$

$$\times\left\|B_jA^{-1}\right\|_{E\to E}\left\|A\exp\left\{-\frac{(\tau t+\lambda)A}{2}\right\}\frac{1}{2}\left(u_{j-N}+u_{j-N-1}\right)\right\|_E$$

$$\le\lambda^{1-\alpha}\frac{1}{(k-nN-1)!}\int_0^\infty t^{k-nN-1}\exp(-t)\frac{dt}{(t\tau+\lambda)^{1-\alpha}}\left\|u_{nN}\right\|_{E_\alpha}$$

$$+\frac{(1+\sqrt2)}{2}\frac{1-\alpha}{M2^{1-\alpha}(1+\sqrt2)}\sum_{j=nN+1}^{k}\tau\frac{1}{(k-j)!}$$

$$\times\int_0^\infty t^{k-j}\exp(-t)\frac{M2^{2-\alpha}dt}{(t\tau+\lambda)^{2-\alpha}}\left\|\frac{1}{2}\left(u_{j-N}+u_{j-N-1}\right)\right\|_{E_\alpha}$$

$$\le J_n\max_{(n-1)N\le k\le nN}\left\|u_k\right\|_{E_\alpha}. \tag{4.9}$$

Therefore

$$\lambda^{1-\alpha}\left\|A\exp\{-\lambda A\}u_k\right\|_E\le\max_{-N\le k\le 0}\left\|g(t_k)\right\|_{E_\alpha} \tag{4.10}$$

for every k, $nN\le k\le(n+1)N$, $n=1,2,3,\dots$, and λ, $\lambda>0$. This shows that

$$\left\|u_k\right\|_{E_\alpha}\le\max_{-N\le k\le 0}\left\|g(t_k)\right\|_{E_\alpha} \tag{4.11}$$

for every k, $nN\le(n+1)N$, $n=1,2,3,\dots$. This result completes the proof of Theorem 4.1. □

Now, we consider the difference scheme (3.2) when

$$A^{-1}B(t)x\ne B(t)A^{-1}x,\quad x\in D(A). \tag{4.12}$$

Suppose that

$$\left\|\overline{A^{-1}\left(AB(t)-B(t)A\right)A^{-1}}\right\|_{E\mapsto E}$$
$$\le\frac{\pi(1-\alpha)^2\alpha^2(1+\alpha)^{-1}\left(1+\sqrt2\right)^{-1}\epsilon}{eM^{1+\alpha}M_1^{1+\alpha}(1+2M_1)\left(1+M_1\right)^{1-\alpha}2^{2+\alpha-\alpha^2}} \tag{4.13}$$

holds for every $t\ge0$.

THEOREM 4.2. *Let the assumptions (1.2) and (4.13) be satisfied. Suppose that the following estimate holds:*

$$\left\|\overline{A^{-1}B(t)}\right\|_{E\mapsto E} \le \frac{(1-\alpha)(1-\epsilon)}{M2^{2-\alpha}\left(1+\sqrt{2}\right)} \tag{4.14}$$

for every $t \ge 0$. Then for the solution of difference scheme (3.2) the estimate (4.3) holds for any $k \ge 1$.

Proof. Consider $1 \le k \le N$. Using formula (3.5) we obtain

$$
\begin{aligned}
&\lambda^{1-\alpha} A \exp\{-\lambda A\} u_k \\
&= \left((I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1} \right)^k \\
&\quad \times \lambda^{1-\alpha} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \exp(-t) A \exp\left\{-(\tau t+\lambda)A\right\} g(0) \\
&\quad + \lambda^{1-\alpha} \sum_{j=1}^k \tau \left((I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1} \right)^{k-j} \left(I+\frac{\tau A}{2} \right) \\
&\quad \times (I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1} \\
&\quad \times \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \exp\left\{ -\frac{(\tau t+\lambda)A}{2} \right\} \\
&\quad \times B_j A \exp\left\{ -\frac{(\tau t+\lambda)A}{2} \right\} \frac{1}{2} \left(g(t_{j-N}) + g(t_{j-N-1}) \right) \\
&\quad + \lambda^{1-\alpha} \sum_{j=1}^k \tau \left((I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1} \right)^{k-j} \left(I+\frac{\tau A}{2} \right) \\
&\quad \times (I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1} \\
&\quad \times \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) A \exp\left\{ -\frac{(\tau t+\lambda)A}{2} \right\} A^{-1} \\
&\quad \times \left[\exp\left\{ -\frac{(\tau t+\lambda)A}{2} \right\} A B_j - B_j A \exp\left\{ -\frac{(\tau t+\lambda)A}{2} \right\} \right] \\
&\quad \times \frac{1}{2} \left(g(t_{j-N}) + g(t_{j-N-1}) \right) \\
&= P_1 + P_2 + P_3.
\end{aligned} \tag{4.15}
$$

Using the estimates (1.2), (2.12), and assumption (2.24), we have

$$\|P_1\|_E \leq \left\|\left((I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1}\right)^k\right\|_{E\mapsto E}$$

$$\times \lambda^{1-\alpha}\frac{1}{(k-1)!}\int_0^\infty t^{k-1}\exp(-t)\left\|A\exp\left(-(\tau t+\lambda)A\right)g(0)\right\|_E$$

$$\leq \lambda^{1-\alpha}\frac{1}{(k-1)!}\int_0^\infty t^{k-1}\exp(-t)\frac{dt}{(t\tau+\lambda)^{1-\alpha}}\max_{-N\leq k\leq 0}\|g(t_k)\|_{E_\alpha},$$

$$\|P_2\|_E \leq \lambda^{1-\alpha}\sum_{j=1}^k \tau\left\|\left((I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1}\right)^{k-j}\right\|_{E\mapsto E}$$

$$\times \left\|\left(I+\frac{\tau A}{2}\right)(I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1}\right\|_{E\mapsto E}$$

$$\times \frac{1}{(k-j)!}\int_0^\infty t^{k-j}\exp(-t)\left\|A\exp\left\{-\frac{(\tau t+\lambda)A}{2}\right\}\right\|_{E\to E}$$

$$\times \left\|\overline{A^{-1}B_j}\right\|_{E\to E}\left\|A\exp\left\{-\frac{(\tau t+\lambda)A}{2}\right\}\frac{1}{2}\left(g(t_{j-N})+g(t_{j-N-1})\right)\right\|_E$$

$$\leq \frac{(1+\sqrt{2})}{2}\frac{\lambda^{1-\alpha}(1-\alpha)(1-\epsilon)}{M2^{1-\alpha}(1+\sqrt{2})}$$

$$\times \sum_{j=1}^k \tau\frac{1}{(k-j)!}\int_0^\infty t^{k-j}\exp(-t)\frac{M2^{2-\alpha}\,dt}{(t\tau+\lambda)^{2-\alpha}}$$

$$\times \left\|\frac{1}{2}\left(g(t_{j-N})+g(t_{j-N-1})\right)\right\|_{E_\alpha}$$

$$\leq (1-\alpha)(1-\epsilon)\sum_{j=1}^k \tau\frac{1}{(k-j)!}\int_0^\infty t^{k-j}\exp(-t)\frac{\lambda^{1-\alpha}\,dt}{(t\tau+\lambda)^{2-\alpha}}$$

$$\times \max_{-N\leq k\leq 0}\|g(t_k)\|_{E_\alpha}$$

$$(4.16)$$

for every k, $1\leq k\leq N$ and λ, $\lambda>0$. Now we estimate P_3. By Lemma 2.3 and using the estimate (2.23), we obtain

$$\|P_3\|_E \leq \lambda^{1-\alpha}\sum_{j=1}^k \tau\left\|\left((I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1}\right)^{k-j}\right\|_{E\mapsto E}$$

$$\times \left\|\left(I+\frac{\tau A}{2}\right)(I+\tau A)\left(I+\tau A+\frac{(\tau A)^2}{2}\right)^{-1}\right\|_{E\mapsto E}$$

$$\times \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \left\| A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} \right\|_{E \to E}$$

$$\times \left\| A^{-1} \left[A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} B_j - B_j A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} \right] \right]$$

$$\times \frac{1}{2} \left(g(t_{j-N}) + g(t_{j-N-1}) \right) \Big\|_E$$

$$\leq \frac{\lambda^{1-\alpha}(1+\sqrt{2})}{2} \max_{1 \leq j \leq N} \left\| \overline{A^{-1}(AB_j - B_j A)A^{-1}} \right\|_{E \to E} \sum_{j=1}^k \tau \frac{1}{(k-j)!}$$

$$\times \int_0^\infty t^{k-j} \exp(-t) \frac{e(1+\alpha)M^\alpha M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha} 2^{(2-\alpha)\alpha}}{\pi \alpha^2 (1-\alpha)}$$

$$\times \frac{M 2^{2-\alpha} \, dt}{(t\tau + \lambda)^{2-\alpha}} \frac{1}{2} \left\| \left(g(t_{j-N}) + g(t_{j-N-1}) \right) \right\|_{E_\alpha}$$

$$\leq \lambda^{1-\alpha}(1-\alpha)\epsilon \sum_{j=1}^k \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{dt}{(t\tau + \lambda)^{2-\alpha}}$$

$$\times \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{4.17}$$

for every k, $1 \leq k \leq N$ and λ, $\lambda > 0$. Using the triangle inequality we obtain

$$\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} u_k \right\|_E \leq J \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{4.18}$$

for every k, $1 \leq k \leq N$ and λ, $\lambda > 0$. This shows that

$$\left\| u_k \right\|_{E_\alpha} \leq \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{4.19}$$

for every k, $1 \leq k \leq N$. Assume that inequality (3.3) is true for k, with $(n-1)N \leq k \leq nN$, $n = 1, 2, 3, \ldots$. Then for any k, $nN \leq k \leq (n+1)N$ using formula (2.8) we have that

$$\lambda^{1-\alpha} A \exp\{-\lambda A\} u_k$$

$$= \left((I + \tau A)\left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right)^{k-nN}$$

$$\times \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-nN-1} \exp(-t) A \exp\left\{ -(\tau t + \lambda)A \right\} u_{nN}$$

$$+ \lambda^{1-\alpha} \sum_{j=nN+1}^{k} \tau \left((I+\tau A) \left(I+\tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right)^{k-j}$$

$$\times \left(I + \frac{\tau A}{2} \right) (I+\tau A) \left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1}$$

$$\times \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} B_j A$$

$$\times \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} \frac{1}{2} (u_{j-N} + u_{j-N-1})$$

$$+ \lambda^{1-\alpha} \sum_{j=nN+1}^{k} \tau \left((I+\tau A) \left(I+\tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right)^{k-j}$$

$$\times \left(I + \frac{\tau A}{2} \right) (I+\tau A) \left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1}$$

$$\times \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\}$$

$$\times A^{-1} \left[\exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} A B_j - B_j A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} \right]$$

$$\times \frac{1}{2} (u_{j-N} + u_{j-N-1})$$

$$= P_{1n} + P_{2n} + P_{3n}. \tag{4.20}$$

Using the estimates (1.2), (2.12), and assumption (2.24) we have

$$\| P_{1n} \|_E \leq \left\| \left((I+\tau A) \left(I+\tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right)^{k-nN} \right\|_{E \mapsto E}$$

$$\times \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-1} \exp(-t) \| A \exp\{ -(\tau t + \lambda)A \} u_{nN} \|_E$$

$$\leq \lambda^{1-\alpha} \frac{1}{(k-nN-1)!} \int_0^\infty t^{k-1} \exp(-t) \frac{dt}{(t\tau + \lambda)^{1-\alpha}} \max_{-N \leq k \leq 0} \| g(t_k) \|_{E_\alpha},$$

$$\| P_{2n} \|_E \leq \lambda^{1-\alpha} \sum_{j=nN+1}^{k} \tau \left\| \left((I+\tau A) \left(I+\tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right)^{k-j} \right\|_{E \mapsto E}$$

$$\times \left\| \left(I + \frac{\tau A}{2} \right) (I+\tau A) \left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right\|_{E \mapsto E}$$

$$\times \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \left\| A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} \right\|_{E \to E}$$

$$\times \left\| \overline{A^{-1} B_j} \right\|_{E \to E} \left\| A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} \frac{1}{2} (u_{j-N} + u_{j-N-1}) \right\|_E$$

$$\leq \frac{(1+\sqrt{2})}{2} \frac{\lambda^{1-\alpha}(1-\alpha)(1-\epsilon)}{M 2^{1-\alpha} (1+\sqrt{2})}$$

$$\times \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{M 2^{2-\alpha} \, dt}{(t\tau + \lambda)^{2-\alpha}}$$

$$\times \left\| \frac{1}{2} (u_{j-N} + u_{j-N-1}) \right\|_{E_\alpha}$$

$$\leq (1-\alpha)(1-\epsilon) \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \frac{\lambda^{1-\alpha} \, dt}{(t\tau + \lambda)^{2-\alpha}}$$

$$\times \max_{-N \leq k \leq 0} \left\| g(t_k) \right\|_{E_\alpha} \tag{4.21}$$

for every k, $nN \leq k \leq (n+1)N$, $n = 1, 2, 3, \ldots$, and λ, $\lambda > 0$. Now we estimate P_{3n}. By Lemma 2.3 and using the estimate (2.23), we obtain

$$\| P_{3n} \|_E \leq \lambda^{1-\alpha} \sum_{j=nN+1}^{k} \tau \left\| \left((I + \tau A)\left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right)^{k-j} \right\|_{E \mapsto E}$$

$$\times \left\| \left(I + \frac{\tau A}{2} \right)(I + \tau A)\left(I + \tau A + \frac{(\tau A)^2}{2} \right)^{-1} \right\|_{E \mapsto E}$$

$$\times \frac{1}{(k-j)!} \int_0^\infty t^{k-j} \exp(-t) \left\| A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} \right\|_{E \to E}$$

$$\times \left\| A^{-1} \left[A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} B_j \right. \right.$$

$$\left. \left. - B_j A \exp\left\{ -\frac{(\tau t + \lambda)A}{2} \right\} \right] \frac{1}{2} (u_{j-N} + u_{j-N-1}) \right\|_E$$

$$\leq \frac{\lambda^{1-\alpha}(1+\sqrt{2})}{2} \max_{nN+1 \leq j \leq (n+1)N} \left\| \overline{A^{-1}(AB_j - B_j A)A^{-1}} \right\|_{E \to E}$$

$$\times \sum_{j=nN+1}^{k} \tau \frac{1}{(k-j)!}$$

$$\times \int_0^\infty t^{k-j} \exp(-t) \frac{e(1+\alpha)M^\alpha M_1^{1+\alpha}\left(1+2M_1\right)\left(1+M_1\right)^{1-\alpha}2^{(2-\alpha)\alpha}}{\pi\alpha^2(1-\alpha)}$$

$$\times \frac{M2^{2-\alpha}\,dt}{(t\tau+\lambda)^{2-\alpha}}\frac{1}{2}\left\|u_{j-N}+u_{j-N-1}\right\|_{E_\alpha}$$

$$\leq \lambda^{1-\alpha}(1-\alpha)\epsilon \sum_{j=nN+1}^{k} \tau\frac{1}{(k-j)!}$$

$$\times \int_0^\infty t^{k-j}\exp(-t)\frac{dt}{(t\tau+\lambda)^{2-\alpha}}\max_{-N\leq k\leq 0}\left\|g\left(t_k\right)\right\|_{E_\alpha} \tag{4.22}$$

for every k, $nN \leq k \leq (n+1)N$ and λ, $\lambda > 0$. Using the triangle inequality we obtain

$$\lambda^{1-\alpha}\left\|A\exp\{-\lambda A\}u_k\right\|_E \leq J_n \max_{-N\leq k\leq 0}\left\|g\left(t_k\right)\right\|_{E_\alpha} \tag{4.23}$$

for every k, $nN \leq k \leq (n+1)N$, $n = 1, 2, 3, \ldots$, and λ, $\lambda > 0$. This shows that

$$\left\|u_k\right\|_{E_\alpha} \leq \max_{-\omega\leq k\leq 0}\left\|g\left(t_k\right)\right\|_{E_\alpha} \tag{4.24}$$

for every k, $nN \leq (n+1)N$, $n = 1, 2, 3, \ldots$. This result completes the proof of Theorem 4.2. $\qquad\square$

5. Applications

We consider the initial-value problem on the range $\{0 \leq t \leq 1,\ x \in \mathbb{R}^n\}$ for $2m$-order multidimensional delay differential equations of parabolic type

$$\frac{\partial u(t,x)}{\partial t} + \sum_{|\tau|=2m} a_\tau(x)\frac{\partial^{|\tau|}u(t,x)}{\partial x_1^{\tau_1}\cdots\partial x_n^{\tau_n}} + \delta u(t,x)$$

$$= b(t)\left(\sum_{|\tau|=2m} a_\tau(x)\frac{\partial^{|\tau|}u(t-\omega,x)}{\partial x_1^{\tau_1}\cdots\partial x_n^{\tau_n}} + \delta u(t-\omega,x)\right), \quad 0 < t < \infty,\ x \in \mathbb{R}^n, \tag{5.1}$$

$$u(t,x) = g(t,x), \quad -\omega \leq t \leq 0,\ x \in \mathbb{R}^n, \qquad |\tau| = \tau_1 + \cdots + \tau_n, \tag{5.2}$$

where $a_r(x)$, $b(t)$, $g(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number. We will assume that the symbol

$$A_1^x(\xi) = \sum_{|\tau|=2m} a_r(x)(i\xi)^{r_1}\cdots(i\xi)^{r_n} \tag{5.3}$$

of the differential operator of the form

$$A_1^x = \sum_{|r=2m|} a_r(x)\frac{\partial^{|r|}}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}} \tag{5.4}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1|\xi|^{2m} \leq \left|A_1^x(\xi)\right| \leq M_2|\xi|^{2m} < \infty \quad \text{for } \xi \neq 0. \tag{5.5}$$

Problem (5.1) has a unique smooth solution. This allows us to reduce the initial-value problem (5.1) to the initial-value problem (1.1) in the Banach space E with a strongly positive operator $A^x = A_1^x + \delta I$ defined by (5.4). We give a number of corollaries of the abstract Theorems 2.1 and 3.1.

THEOREM 5.1. *Assume that*

$$\sup_{0 \leq t < \infty} \left|b(t)\right| \leq \frac{1-\alpha}{M2^{2-\alpha}}. \tag{5.6}$$

Then the solutions of the initial-value problem (5.1) satisfy the following stability estimates:

$$\sup_{0 \leq t < \infty} \left\|v(t, \cdot)\right\|_{C^{2m\alpha}(\mathbb{R}^n)} \leq M_2(\alpha) \max_{-\omega \leq t \leq 0} \left\|g(t, \cdot)\right\|_{C^{2m\alpha}(\mathbb{R}^n)}, \quad 0 < \alpha < \frac{1}{2m}, \tag{5.7}$$

where $M_2(\alpha)$ does not depend on $g(t, x)$. Here $C^\varepsilon(\mathbb{R}^n)$ is the space of functions satisfying a Hölder condition with the indicator $\varepsilon \in (0, 1)$.

The proof of Theorem 5.1 is based on the estimate

$$\left\|\exp\left\{-tA^x\right\}\right\|_{C(\mathbb{R}^n) \to C(\mathbb{R}^n)} \leq M, \quad t \geq 0, \tag{5.8}$$

and on the abstract Theorem 2.1, on the strong positivity of the operator A^x in $C^\varepsilon(\mathbb{R}^n)$, and on the fact of the equivalence of the norms in the spaces $E_\alpha = E_\alpha(A, C(\mathbb{R}^n))$ and $C^{2m\alpha}(\mathbb{R}^n)$ when $0 < \alpha < 1/2m$ (see [5]).

We define the grid space $\mathbb{R}_h^n$ ($0 < h \leq h_0$) as the set of all points of the Euclidean space $\mathbb{R}^n$ whose coordinates are given by

$$x_i = s_i h, \quad s_i = 0, \pm 1, \pm 2, \ldots, \quad i = 1, \ldots, n. \tag{5.9}$$

To the differential operator $A^x = A_1^x + \delta I$ defined by (5.4) we assign the difference operator A_h^x by the formula

$$A_h^x u^h(x) = \sum_{2m \leq |r| \leq S} b_r^h(x) D_h^r u^h(x) \tag{5.10}$$

of an arbitrary high order of accuracy that approximates this multidimensional elliptic operator A^x which acts on functions u_x^h defined on the entire space $\mathbb{R}_h^n$. We will assume that for $|\xi_k h| \leq \pi$, the symbol $A^x(\xi h, h)$ of the operator A_h^x satisfies the inequalities

$$\left|A^x(\xi h, h)\right| \geq M_3|\xi|^{2m}, \quad \left|\arg A^x(\xi h, h)\right| \leq \phi < \pi. \tag{5.11}$$

With the help of A_h^x we arrive at the initial-value problem

$$\begin{aligned}
\left(u^h(t,x)\right)' + A_h^x u^h(t,x) &= b(t) A_h^x u^h(t-\omega, x), \quad t \geq 0, \ x \in \mathbb{R}^n, \\
u^h(t,x) &= g^h(t,x) = g(t,x) \quad (-\omega \leq t \leq 0), \ x \in \mathbb{R}^n,
\end{aligned} \tag{5.12}$$

for an infinite system of ordinary differential equations. Now, we replace problem (5.12) by the first order accuracy in t difference scheme

$$\begin{aligned}
\frac{1}{\tau}\left(u_k^h(x) - u_{k-1}^h(x)\right) + A_h^x u_k^h(x) &= b(t_k) A_h^x u_{k-N}^h(x), \\
t_k = k\tau, \ 1 \leq k, \ N\tau &= \omega, \ x \in \mathbb{R}_h^n, \\
u_k^h(x) = g^h(t_k, x), \quad t_k = k\tau, \ -N &\leq k \leq 0, \ x \in \mathbb{R}_h^n.
\end{aligned} \tag{5.13}$$

THEOREM 5.2. *Assume that all the conditions of Theorem 5.1 are satisfied. Then for the solution of difference scheme (5.13) the estimate*

$$\sup_{1 \leq k < \infty} \left\| u_k^h \right\|_{C^{2m\alpha}(\mathbb{R}_h^n)} \leq M_2(\alpha) \max_{-N \leq t \leq 0} \left\| g_k^h \right\|_{C^{2m\alpha}(\mathbb{R}_h^n)}, \quad 0 < \alpha < \frac{1}{2m}, \quad (5.14)$$

holds.

The proof of Theorem 5.2 is based on the estimate

$$\left\| \exp\left\{ -t_k A_h^x \right\} \right\|_{C(\mathbb{R}_h^n) \to C(\mathbb{R}_h^n)} \leq M, \quad k \geq 0, \tag{5.15}$$

and on the abstract Theorem 3.1, the positivity of the operator A_h^x in $C^\varepsilon(\mathbb{R}_h^n)$, and on the fact that the $E_\alpha = E_\alpha(A_h^x, C(\mathbb{R}_h^n))$-norms are equivalent to the norms $C^{2m\alpha}(\mathbb{R}_h^n)$ uniformly in h for $0 < \alpha < 1/2m$ (see [5]).

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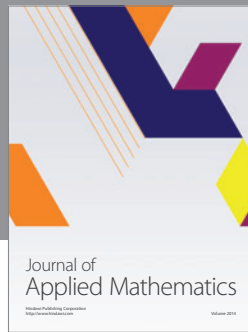
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