LOCAL EXACT CONTROLLABILITY OF THE AGE-DEPENDENT POPULATION DYNAMICS WITH DIFFUSION

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Received 15 June 2001

We investigate the local exact controllability of a linear age and space population dynamics model where the birth process is nonlocal. The methods we use combine the Carleman estimates for the backward adjoint system, some estimates in the theory of parabolic boundary value problems in L^k and the Banach fixed point theorem.

1. Introduction

We consider a linear model describing the dynamics of a single species population with age dependence and spatial structure. Let $p(a,t,x)$ be the distribution of individuals of age $a \geq 0$ at time $t \geq 0$ and location $x \in \Omega$, a bounded domain of $\mathbb{R}^N$, $N \in \{1,2,3\}$, with a suitably smooth boundary $\partial\Omega$. Let $a_\dagger$ be the life expectancy of an individual and T a positive constant. Let $\beta(a) \geq 0$ be the natural fertility-rate and $\mu(a) \geq 0$ the natural death-rate of individuals of age a. We assume that the flux of population takes the form $k\nabla p(a,t,x)$ with $k > 0$, where ∇ is the gradient vector with respect to the spatial variable. The evolution of the distribution p is governed by the system

$$\begin{cases} Dp(a,t,x)+\mu(a)p(a,t,x)-k\Delta p(a,t,x)=f(a,x)+m(x)u(a,t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a,t,x)\in Q_T, \\ \dfrac{\partial p}{\partial \nu}(a,t,x)=0, \qquad\qquad\qquad\qquad (a,t,x)\in \Sigma_T, \\ p(0,t,x)=\displaystyle\int_0^{a_\dagger}\beta(a)p(a,t,x)\,da, \qquad (t,x)\in (0,T)\times\Omega, \\ p(a,0,x)=p_0(a,x), \qquad\qquad\qquad (a,x)\in \big(0,a_\dagger\big)\times\Omega, \end{cases} \tag{1.1}$$

where u is a control function, m is the characteristic function of ω, f is a supply

Abstract and Applied Analysis 6:6 (2001) 357–368
2000 Mathematics Subject Classification: 93B05, 35K05, 46B70, 92D25
URL: http://aaa.hindawi.com/volume-6/S108533750100063X.html

of individuals and p_0 is the initial distribution. Here $\omega \subset \Omega$ is a nonempty open subset, $Q_T = (0, a_\dagger) \times (0, T) \times \Omega$, $\Sigma_T = (0, a_\dagger) \times (0, T) \times \partial\Omega$.

We have denoted by

$$Dp(a,t,x) = \lim_{\varepsilon \to 0} \frac{p(a+\varepsilon, t+\varepsilon, x) - p(a,t,x)}{\varepsilon} \tag{1.2}$$

the directional derivative of p with respect to the direction $(1,1,0)$. It is obvious that for p smooth enough,

$$Dp = \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a}. \tag{1.3}$$

Let p_s be a steady-state of (1.1), corresponding to $u \equiv 0$ and such that

$$p_s(a,x) \geq \rho_0 > 0 \quad \text{a.e. } (a,x) \in \big(0, a_1\big) \times \Omega, \tag{1.4}$$

where $\rho_0 > 0$ is constant and a_1 is a constant which will be later defined and belongs to $(0, a_\dagger)$.

The main goal of this paper is to prove the existence of a control u such that the solution of (1.1) satisfies

$$p(a,T,x) = p_s(a,x) \quad \text{a.e. } (a,x) \in \big(0, a_\dagger\big) \times \Omega, \tag{1.5}$$

$$p(a,t,x) \geq 0 \quad \text{a.e. } (a,t,x) \in Q_T. \tag{1.6}$$

Condition (1.6) is natural because p represents the density of a population.

We notice that if p is the solution to (1.1), then $p - p_s$ is the solution to

$$\begin{cases} Dp + \mu(a)p - k\Delta p = m(x)u(a,t,x), & (a,t,x) \in Q_T, \\ \dfrac{\partial p}{\partial \nu}(a,t,x) = 0, & (a,t,x) \in \Sigma_T, \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a)p(a,t,x)\,da, & (t,x) \in (0,T) \times \Omega, \\ p(a,0,x) = \bar{p}_0(a,x), & (a,x) \in \big(0, a_\dagger\big) \times \Omega, \end{cases} \tag{1.7}$$

where $\bar{p}_0 = p_0 - p_s$.

The above formulated problem is equivalent with the exact null controllability problem for (1.7). If we denote now by p the solution to (1.7), then condition (1.6) becomes

$$p(a,t,x) \geq -p_s(a,x) \quad \text{a.e. } (a,t,x) \in Q_T. \tag{1.8}$$

The main result of this paper amounts to saying that system (1.7) is exactly null controllable for p_0 in a neighborhood of p_s.

We recall that the internal null controllability of the linear heat equation, when the control acts on a subset of the domain, was established by Lebeau and Robbiano [10] and was later extended to some semilinear equation by Fursikov and Imanuvilov [4], in the sublinear case and by Barbu [2] and Fernandez-Cara [3], in the superlinear case.

The paper is organized as follows. We first give the hypotheses and state the main result. The existence of a steady-state of (1.1) with $u \equiv 0$ is established in Section 3. The proof of the local exact null controllability is given in Section 4. The proof is based on Carleman's inequality for the backward adjoint system associated with (1.7).

2. Assumptions and the main result

Assume that the following hypotheses hold:

(A1) $\beta \in L^\infty(0,a_\dagger)$, $\beta(a) \geq 0$ a.e. $a \in (0,a_\dagger)$,

(A2) there exists $a_0, a_1 \in (0,a_\dagger)$ such that $\beta(a) = 0$ a.e. $a \in (0,a_0) \cup (a_1,a_\dagger)$, $\beta(a) > 0$ a.e. in (a_0,a_1),

(A3) $\mu \in L^1_{\text{loc}}([0,a_\dagger))$, $\mu(a) \geq 0$ a.e. $a \in (0,a_\dagger)$,

(A4) $\int_0^{a_\dagger} \mu(a)da = +\infty$,

(A5) $p_0 \in L^\infty((0,a_\dagger)\times\Omega)$, $p_0(a,t) \geq 0$ a.e. in $(0,a_\dagger)\times\Omega$, $f \in L^\infty((0,a_\dagger)\times\Omega)$, $f(a,x) \geq 0$ a.e. in $(0,a_\dagger)\times\Omega$.

For the biological significance of the hypotheses and the basic existence results for the solution to (1.1) we refer to [5, 6, 11].

Let p_s be a steady-state of (1.1), corresponding to $u \equiv 0$ and such that

$$p_s(a,x) \geq \rho_0 > 0 \quad \text{a.e. } (a,x) \in \big(0,a_1\big)\times\Omega, \tag{2.1}$$

where $\rho_0 > 0$ is constant.

Denote by $\bar{p}_0 = p_0 - p_s$. Then we have the following theorem.

THEOREM 2.1. *If $\|\bar{p}_0\|_{L^\infty((0,a_\dagger)\times\Omega)}$ is small enough, then there exists $u \in L^2(Q_T)$ such that the solution p of (1.7) satisfies*

$$\begin{aligned} p(a,T,x) &= 0 \quad \textit{a.e. } (a,x) \in \big(0,a_\dagger\big)\times\Omega, \\ p(a,t,x) &\geq -p_s(a,x) \quad \textit{a.e. } (a,t,x) \in Q_T. \end{aligned} \tag{2.2}$$

3. Existence of steady states to (1.1)

In this section, we will study the existence of p_s, a steady-state of (1.1), corresponding to $u \equiv 0$, which satisfies (1.4). The steady-state p_s should be a solution to

$$\begin{cases} \dfrac{\partial p_s}{\partial a} + \mu(a)p_s - k\Delta p_s = f(a,x), & (a,x) \in \big(0,a_\dagger\big)\times\Omega, \\ \dfrac{\partial p_s}{\partial \nu}(a,x) = 0, & (a,x) \in \big(0,a_\dagger\big)\times\partial\Omega, \\ p_s(0,x) = \displaystyle\int_0^{a_\dagger} \beta(a)p_s(a,x)da, & x \in \Omega. \end{cases} \tag{3.1}$$

Denote by $R = \int_0^{a_\dagger} \beta(a) e^{-\int_0^a \mu(s)ds} da$ the reproductive number and by f_0 a nonnegative constant.

THEOREM 3.1. *If $R < 1$ and $f(a,x) \geq f_0 > 0$ a.e. $(a,x) \in (0,a_\dagger) \times \Omega$, then there exists a unique solution to (3.1), which in addition satisfies (1.4).*

If $R = 1$ and $f \equiv 0$, then there exist infinitely many solutions to (3.1), which satisfy (1.4). If $R > 1$, then there is no nonnegative solution to (3.1), satisfying (1.4).

Proof. If $R < 1$, then there exists a unique (and nonnegative) solution to (3.1) via Banach fixed point theorem. If, in addition, $f(a,x) \geq f_0 > 0$ a.e. $(a,x) \in (0,a_\dagger) \times \Omega$, then by the comparison result in [5] we get that

$$p_s(a,x) \geq p_i(a,t,x) \quad \text{a.e. } (a,t,x) \in Q = \big(0,a_\dagger\big) \times (0,+\infty) \times \Omega, \tag{3.2}$$

where p_i is the solution to

$$\begin{aligned}
&Dp_i + \mu p_i - k\Delta p_i = f_0, && (a,t,x) \in Q,\\
&\frac{\partial p_i}{\partial \nu} = 0, && (a,t,x) \in \Sigma,\\
&p_i(0,t,x) = \int_0^{a_\dagger} \beta(a) p_i(a,t,x)da, && (t,x) \in (0,+\infty) \times \Omega,\\
&p_i(a,0,x) = 0, && (a,x) \in \big(0,a_\dagger\big) \times \Omega
\end{aligned} \tag{3.3}$$

$(\Sigma = (0,a_\dagger) \times (0,+\infty) \times \partial\Omega)$; p_i does not explicitly depend on x. So, we will write $p_i(a,t)$ instead of $p_i(a,t,x)$ and

$$p_s(a,x) \geq p_i(a,t) \quad \forall t \in [0,+\infty), \quad \text{a.e. } (a,x) \in \big(0,a_\dagger\big) \times \Omega, \tag{3.4}$$

where p_i is the solution of

$$\begin{aligned}
&Dp_i + \mu p_i = f_0, && (a,t) \in \big(0,a_\dagger\big) \times (0,+\infty),\\
&p_i(0,t) = \int_0^{a_\dagger} \beta(a) p_i(a,t)da, && t \in (0,+\infty),\\
&p_i(a,0) = 0, && a \in \big(0,a_\dagger\big).
\end{aligned} \tag{3.5}$$

For $t > a_\dagger$ we have

$$p_i(0,t) > 0, \quad p_i(0,t) \text{ is continuous with respect to } t \tag{3.6}$$

(see [7]). As a consequence we obtain that there exists $\rho_0 > 0$ such that, for t large enough, and for any $a \in (0,a_1)$,

$$p_i(a,t) > \rho_0, \tag{3.7}$$

and in conclusion we get that p_s satisfies (1.4).

If $R = 1$ and $f \equiv 0$, then any function defined by

$$p(a,x) = ce^{-\int_0^a \mu(s)ds} da, \quad (a,x) \in \left(0, a_\dagger\right) \times \Omega \tag{3.8}$$

is a solution to (3.1) (for any $c \in \mathbb{R}$). In fact these are all the solutions to (3.1) in this case. It is now obvious that there exist infinitely many solutions to (3.1), which satisfy (1.4).

If $R > 1$ and if it would exist a nonnegative solution p_s to (3.1) satisfying (1.4), then $p(a,t,x) = p_s(a,x)$, $(a,t,x) \in Q$ is the solution to

$$\begin{aligned}
&Dp + \mu p - k\Delta p = f(a,x), && (a,t,x) \in Q,\\
&\frac{\partial p}{\partial \nu} = 0, && (a,t,x) \in \Sigma,\\
&p(0,t,x) = \int_0^{a_\dagger} \beta(a) p(a,t,x), && (t,x) \in (0,+\infty) \times \Omega,\\
&p(a,0,x) = p_s(a,x), && (a,x) \in \left(0, a_\dagger\right) \times \Omega
\end{aligned} \tag{3.9}$$

and for $t \to +\infty$ we have (see [9])

$$\lim_{t \to +\infty} \left\| p(t) \right\|_{L^2((0,a_\dagger) \times \Omega)} = +\infty. \tag{3.10}$$

On the other hand,

$$\left\| p(t) \right\|_{L^2((0,a_\dagger) \times \Omega)} = \left\| p_s \right\|_{L^2((0,a_\dagger) \times \Omega)}, \tag{3.11}$$

and so $\| p_s \|_{L^2((0,a_\dagger) \times \Omega)} = +\infty$, which is absurd.

4. Proof of Theorem 2.1

In what follows we will use the general Carleman inequality for linear parabolic equations given in [4]. Namely, let $\tilde{\omega} \subset\subset \omega$ be a nonempty bounded set, $T_0 \in (0,+\infty)$ and $\psi \in C^2(\bar{\Omega})$ be such that

$$\begin{aligned}
\psi(x) &> 0, && \forall x \in \Omega,\\
\psi(x) &= 0, && \forall x \in \partial\Omega,\\
\left|\nabla \psi(x)\right| &> 0, && \forall x \in \bar{\Omega} \backslash \tilde{\omega}
\end{aligned} \tag{4.1}$$

and set

$$\alpha(t,x) = \frac{e^{\lambda \psi(x)} - e^{2\lambda \|\psi\|_{C(\bar{\Omega})}}}{t\left(T_0 - t\right)}, \tag{4.2}$$

where λ is an appropriate positive constant.

Denote by $D_{T_0} = (0, T_0) \times \Omega$. □

Lemma 4.1. *There exist positive constants C_1, s_1 such that*

$$\frac{1}{s}\int_{D_{T_0}} t\big(T_0-t\big)e^{2s\alpha}\big(\left|w_t\right|^2+|\Delta w|^2\big)dx\,dt+s\int_{D_{T_0}}\frac{e^{2s\alpha}}{t\big(T_0-t\big)}|\nabla w|^2dx\,dt$$
$$+s^3\int_{D_{T_0}}\frac{e^{2s\alpha}}{t^3\big(T_0-t\big)^3}|w|^2dx\,dt$$
$$\leq C_1\bigg[\int_{D_{T_0}} e^{2s\alpha}\left|w_t+\Delta w\right|^2dx\,dt+s^3\int_{(0,T_0)\times\omega}\frac{e^{2s\alpha}}{t^3\big(T_0-t\big)^3}|w|^2dx\,dt\bigg], \tag{4.3}$$

for all $w\in C^2(\bar{D}_{T_0})$, $(\partial w/\partial\nu)(t,x)=0$, $\forall(t,x)\in(0,T_0)\times\partial\Omega$ and $s\geq s_1$.

For the proof of this result we refer to [4]. Let $T_0\in(0,\min\{a_0,T/2,a_\dagger-a_1\})$. Define

$$K=L^\infty\big(\big(0,2T_0\big)\times\Omega\big). \tag{4.4}$$

In what follows we will denote by the same symbol C, several constants independent of $\bar{p}_0$ and all other variables.

For $b\in K$ arbitrary but fixed and for any $\varepsilon>0$, consider the following optimal control problem:

$$\text{Minimize}\bigg\{\int_G\int_\Omega\varphi(a,t,x)\left|u(a,t,x)\right|^2dx\,dt\,da+\frac{1}{\varepsilon}\int_{\Gamma_0}\int_\Omega\left|p(a,t,x)\right|^2dx\,dl\bigg\}, \tag{4.5}$$

subject to (4.7) ($u\in L^2(Q_{2T_0})$ and p is the solution of (4.7) corresponding to u). Here

$$G=\big(0,a_\dagger\big)\times\big(0,T_0\big)\cup\big(0,T_0\big)\times\big(T_0,2T_0\big) \tag{4.6}$$

(see Figure 4.1)

$$\Gamma_0=\big\{T_0\big\}\times\big(T_0,2T_0\big)\cup\big(T_0,a_\dagger-T_0\big)\times\big\{T_0\big\},$$
$$\varphi(a,t,x)=\begin{cases} e^{-2s\alpha(t,x)}t^3\big(T_0-t\big)^3, & \text{if } t<a,\ (a,t)\in G,\\ e^{-2s\alpha(a,x)}a^3\big(T_0-a\big)^3, & \text{if } a<t,\ (a,t)\in G,\end{cases}$$

$$\begin{aligned}
&Dp+\mu p-k\Delta p = m(x)\tilde{m}(a,t)u(a,t,x), && (a,t,x)\in Q_{2T_0},\\
&\frac{\partial p}{\partial \nu}=0, && (a,t,x)\in \Sigma_{2T_0},\\
&p(0,t,x)=b(t,x), && (t,x)\in \big(0,2T_0\big)\times\Omega,\\
&p(a,0,x)=p_0(a,x), && (a,x)\in \big(0,a_\dagger\big)\times\Omega.
\end{aligned} \tag{4.7}$$

Here $\tilde{m}$ is the characteristic function of G.

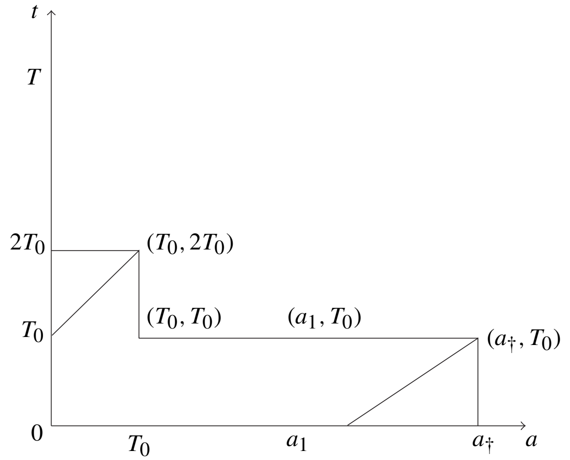

Figure 4.1. Case where $T_0 = a_0$.

Denote by $\Psi_\varepsilon(u)$ the value of the cost function in u. Since the cost function $\Psi_\varepsilon : L^2(Q_{2T_0}) \to \mathbb{R}^+$ is convex, continuous and

$$\lim_{\|u\|_{L^2(Q_{2T_0})}\to+\infty} \Psi_\varepsilon(u) = +\infty, \tag{4.8}$$

then it follows that there exists at least one minimum point for Ψ_ε and consequently an optimal pair $(u_\varepsilon, p_\varepsilon)$ for (4.5). By standard arguments we have

$$u_\varepsilon(a,t,x) = m(x)\tilde{m}(a,t)q_\varepsilon(a,t,x)\varphi^{-1}(a,t,x) \quad \text{a.e. } (a,t,x)\in Q_{2T_0}, \tag{4.9}$$

where q_ε is the solution of

$$\begin{aligned}
&Dq-\mu q+k\Delta q=0, && (a,t,x)\in G\times\Omega,\\
&\frac{\partial q}{\partial \nu}=0, && (a,t,x)\in G\times\partial\Omega,\\
&q(a,t,x)=0, && (a,t,x)\in \big(\Gamma\setminus\Gamma_0\big)\times\Omega,\\
&q(a,t,x)=-\frac{1}{\varepsilon}p_\varepsilon(a,t,x), && (a,t,x)\in \Gamma_0\times\Omega.
\end{aligned} \tag{4.10}$$

Here $\Gamma = (0,T_0)\times\{2T_0\}\cup\{a_\dagger\}\times(0,T_0)\cup\Gamma_0\cup(a_\dagger - T_0, a_\dagger)\times\{T_0\}$.

Multiplying the first equation in (4.10) by p_ε and integrating on Q_{2T_0}, we obtain after some calculation (and using (4.7) and (4.9)) that

$$\begin{aligned}\int_G\int_\omega \varphi(a,t,x)\big|u_\varepsilon(a,t,x)\big|^2 dx\,da\,dt+\frac{1}{\varepsilon}\int_{\Gamma_0}\int_\Omega \big|p_\epsilon(a,t,x)\big|^2 dx\,dl \\ =-\int_0^{T_0}\int_\Omega b(t,x)q_\varepsilon(0,t,x)dx\,dt \\ -\int_0^{a_\dagger-T_0}\int_\Omega \bar{p}_0(a,x)q_\varepsilon(a,0,x)dx\,da.\end{aligned} \tag{4.11}$$

Let S be an arbitrary characteristic line of equation

$$S=\big\{(\gamma+t,\theta+t);t\in\big(0,T_0\big)\big\},\quad (\gamma,\theta)\in\big(0,a_\dagger-T_0\big)\times\{0\}\cup\{0\}\times\big(0,T_0\big). \tag{4.12}$$

Define

$$\begin{aligned}\tilde{u}(t,x)&=u(\gamma+t,\theta+t,x), && (t,x)\in\big(0,T_0\big)\times\Omega,\\ \tilde{p}_\varepsilon(t,x)&=p_\varepsilon(\gamma+t,\theta+t,x), && (t,x)\in\big(0,T_0\big)\times\Omega,\\ \tilde{q}_\varepsilon(t,x)&=q_\varepsilon(\gamma+t,\theta+t,x), && (t,x)\in\big(0,T_0\big)\times\Omega,\\ \tilde{\mu}(t)&=\mu(\gamma+t), && t\in\big(0,T_0\big).\end{aligned} \tag{4.13}$$

$(\tilde{u}_\varepsilon,\tilde{p}_\varepsilon,\tilde{q}_\varepsilon)$ satisfies

$$\begin{aligned}\big(\tilde{p}_\varepsilon\big)_t+\tilde{\mu}\tilde{p}_\varepsilon-k\Delta\tilde{p}_\varepsilon=m(x)\tilde{u}_\varepsilon(t,x), &\quad (t,x)\in\big(0,T_0\big)\times\Omega,\\ \frac{\partial\tilde{p}_\varepsilon}{\partial\nu}=0, &\quad (t,x)\in\big(0,T_0\big)\times\partial\Omega,\\ \tilde{p}_\varepsilon(0,x)=\begin{cases}b(\theta,x), & \gamma=0\\ \bar{p}_0(\gamma,x), & \theta=0\end{cases} &\quad x\in\Omega.\end{aligned} \tag{4.14}$$

This yields

$$\tilde{u}_\varepsilon(t,x)=m(x)\tilde{q}_\varepsilon(t,x)\cdot\frac{e^{2s\alpha(t,x)}}{t^3\big(T_0-t\big)^3} \tag{4.15}$$

a.e. $(t,x)\in(0,T_0)\times\Omega$,

$$\begin{aligned}\big(\tilde{q}_\varepsilon\big)_t+k\Delta\tilde{q}_\varepsilon=\tilde{\mu}\tilde{q}_\varepsilon, &\quad (t,x)\in\big(0,T_0\big)\times\Omega,\\ \frac{\partial\tilde{q}_\varepsilon}{\partial\nu}=0, &\quad (t,x)\in\big(0,T_0\big)\times\partial\Omega,\\ \tilde{q}_\varepsilon\big(T_0,x\big)=-\frac{1}{\varepsilon}\tilde{p}_\varepsilon\big(T_0,x\big) &\quad x\in\Omega.\end{aligned} \tag{4.16}$$

Multiplying the first equation in (4.16) by $\tilde{p}_\varepsilon$ and integrating on D_{T_0}, we obtain that

$$\begin{aligned}\int_0^{T_0}\int_\omega e^{-2s\alpha(t,x)}t^3\big(T_0-t\big)^3\big|\tilde{u}_\varepsilon(t,x)\big|^2dx\,dt+\frac{1}{\varepsilon}\int_\Omega\big|\tilde{p}_\varepsilon\big(T_0,x\big)\big|^2dx\\ =-\int_\Omega\tilde{p}_\varepsilon(0,x)\tilde{q}_\varepsilon(0,x)dx.\end{aligned}\tag{4.17}$$

By Carleman's inequality (4.3) we infer that

$$\begin{aligned}\int_0^{T_0}\int_\Omega e^{2s\alpha}\Big[\frac{t\big(T_0-t\big)}{s}\big(\big|\big(\tilde{q}_\varepsilon\big)_t\big|^2+\big|\Delta\tilde{q}_\varepsilon\big|^2\big)\\ +\frac{s}{t\big(T_0-t\big)}\big|\nabla\tilde{q}_\varepsilon\big|^2+\frac{s^3}{t^3\big(T_0-t\big)^3}\big|\tilde{q}_\varepsilon\big|^2\Big]dx\,dt\\ \le C_1\Big[\int_0^{T_0}\int_\Omega e^{2s\alpha}\|\tilde{\mu}\|^2_{C([0,T_0])}\cdot\big|\tilde{q}_\varepsilon\big|^2dx\,dt\\ +s^3\int_{(0,T_0)\times\omega}\frac{e^{2s\alpha}}{t^3\big(T_0-t\big)^3}\big|\tilde{q}_\varepsilon\big|^2dx\,dt\Big]\end{aligned}\tag{4.18}$$

and consequently

$$\begin{aligned}\int_0^{T_0}\int_\Omega e^{2s\alpha}\Big[\frac{t\big(T_0-t\big)}{s}\big(\big|\big(\tilde{q}_\varepsilon\big)_t\big|^2+\big|\Delta\tilde{q}_\varepsilon\big|^2\big)\\ +\frac{s}{t\big(T_0-t\big)}\big|\nabla\tilde{q}_\varepsilon\big|^2+\frac{s^3}{t^3\big(T_0-t\big)^3}\big|\tilde{q}_\varepsilon\big|^2\Big]dx\,dt\\ \le C\int_0^{T_0}\int_\omega e^{2s\alpha}\frac{s^3}{t^3\big(T_0-t\big)^3}\big|\tilde{q}_\varepsilon\big|^2dx\,dt,\end{aligned}\tag{4.19}$$

for $s\ge\max(s_1, C\|\mu\|^{2/3}_{C([0,a_\dagger-T_0])})$.

Multiplying the first equation in (4.16) by $\tilde{q}_\varepsilon$ we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega\big|\tilde{q}_\varepsilon(t,x)\big|^2dx-k\int_\Omega\big|\nabla\tilde{q}_\varepsilon(t,x)\big|^2dx-\int_\Omega\tilde{\mu}(t)\big|\tilde{q}_\varepsilon(t,x)\big|^2dx=0,\\ \frac{d}{dt}\int_\Omega\big|\tilde{q}_\varepsilon(t,x)\big|^2dx\ge 0\qquad\text{a.e. } t\in\big(0,T_0\big).\end{aligned}\tag{4.20}$$

Integrating the last inequality we get that

$$\int_\Omega\big|\tilde{q}_\varepsilon(0,x)\big|^2dx\le C\int_0^{T_0}\int_\Omega\big|\tilde{q}_\varepsilon(t,x)\big|^2\frac{e^{2s\alpha(x,t)}}{t^3\big(T_0-t\big)^3}dx\tag{4.21}$$

and by Carleman's inequality we have

$$\int_\Omega\big|\tilde{q}_\varepsilon(0,x)\big|^2dx\le C\int_0^{T_0}\int_\omega\big|\tilde{q}_\varepsilon(t,x)\big|^2\cdot\frac{e^{2s\alpha(x,t)}}{t^3\big(T_0-t\big)^3}dx\,dt.\tag{4.22}$$

By Young's inequality (4.16), (4.22), and (4.15) we obtain

$$\int_{(0,T_0)\times\omega} e^{-2s\alpha} t^3 \big(T_0 - t\big)^3 \big|\tilde{u}_\varepsilon(t,x)\big|^2 dx\,dt + \frac{1}{\varepsilon}\int_\Omega \big|\tilde{p}_\varepsilon\big(T_0,x\big)\big|^2 dx \\ \leq C\big\|\tilde{p}_\varepsilon(0)\big\|^2_{L^2(\Omega)}, \tag{4.23}$$

for $s \geq \max(s_1, C\|\mu\|^{2/3}_{C([0,a_\dagger - T_0])})$.

Using now (4.19) we get

$$\int_0^{T_0}\int_\Omega e^{2s\alpha}\bigg[\frac{t\big(T_0-t\big)}{s}\Big(\big|\big(\tilde{q}_\varepsilon\big)_t\big|^2 + \big|\Delta\tilde{q}_\varepsilon\big|^2\Big) + \frac{s}{t\big(T_0-t\big)}\big|\nabla\tilde{q}_\varepsilon\big|^2 \\ + \frac{s^3}{t^3\big(T_0-t\big)^3}\big|\tilde{q}_\varepsilon\big|^2\bigg] dx\,dt \\ \leq C\big\|\tilde{p}_\varepsilon(0)\big\|^2_{L^2(\Omega)}, \tag{4.24}$$

for any $\varepsilon > 0$ and consequently

$$\big\|\tilde{v}_\varepsilon\big\|^2_{W_2^{1,2}((0,T_0)\times\Omega)} \leq C\big\|\tilde{p}_\varepsilon(0)\big\|^2_{L^2(\Omega)}, \tag{4.25}$$

where $\tilde{v}_\varepsilon(t,x) = (e^{2s\alpha(t,x)}/t^3(T_0-t)^3)\tilde{q}_\varepsilon$, $(t,x) \in (0,T_0)\times\Omega$. As

$$W_2^{1,2}\big(\big(0,T_0\big)\times\Omega\big) \subset L^l\big(\big(0,T_0\big)\times\Omega\big) \tag{4.26}$$

(where $l = +\infty$ for $N = 1,2$ and $l = 10$ for $N = 3$), we may infer that

$$\big\|\tilde{u}_\varepsilon\big\|^2_{L^{10}((0,T_0)\times\Omega)} = \big\|m\tilde{v}_\varepsilon\big\|^2_{L^{10}((0,T_0)\times\Omega)} \leq C\big\|\tilde{p}_\varepsilon(0)\big\|^2_{L^2(\Omega)}, \tag{4.27}$$

for any $\varepsilon > 0$ and $s \geq \max(s_1, C\|\mu\|^{2/3}_{C([0,a_\dagger - T_0])})$.

The last estimate and the existence theory of parabolic boundary value problems in L^r (see [8]) imply that on a subsequence we have that

$$\begin{aligned} \tilde{u}_\varepsilon &\longrightarrow \tilde{u} \quad \text{weakly in } L^{10}\big(\big(0,T_0\big)\times\Omega\big) \\ \tilde{p}_\varepsilon &\longrightarrow \tilde{p}^{\tilde{u}} \quad \text{weakly in } W_{10}^{1,2}\big(\big(0,T_0\big)\times\Omega\big), \end{aligned} \tag{4.28}$$

where $(\tilde{u}, \tilde{p}^{\tilde{u}})$ satisfies (4.14) and

$$\tilde{p}^{\tilde{u}}\big(T_0,x\big) = 0 \quad \text{a.e. } x \in \Omega. \tag{4.29}$$

By (4.14) we get

$$\big\|\tilde{p}^{\tilde{u}}\big\|^2_{L^\infty((0,T_0)\times\Omega)} \leq C\Big(\big\|\tilde{p}_\varepsilon(0)\big\|^2_{L^\infty(\Omega)} + \big\|m\tilde{u}_\varepsilon\big\|^2_{L^3((0,T_0)\times\Omega)}\Big) \tag{4.30}$$

(we recall that $W_3^{1,2}((0,T_0)\times\Omega)\subset L^\infty((0,T_0)\times\Omega)$ for $N\in\{1,2,3\}$; see [1, 8]). So by (4.27) we have

$$\left\|\tilde{p}^{\tilde{u}}\right\|^2_{L^\infty((0,T_0)\times\Omega)}\le C\left\|\tilde{p}_\varepsilon(0)\right\|^2_{L^\infty(\Omega)}. \tag{4.31}$$

For (u,p^u) given by $(\tilde{u},\tilde{p}^{\tilde{u}})$ on each characteristic line we have that $u\in L^2(Q_T)$, p^u is the solution of (4.7) and $p(a,t,x)=0$ a.e. $(a,t,x)\in\Gamma_0\times\Omega$. Moreover,

$$\left\|p^u\right\|_{L^\infty(Q_{2T_0})}\le C\left\|\bar{p}_0\right\|_{L^\infty((0,a_\dagger)\times\Omega)}+\|b\|_{L^\infty((0,2T_0)\times\Omega)}. \tag{4.32}$$

We are now ready to prove our null exact controllability result. For any $b\in K$, we denote by $\Phi(b)\subset L^2((0,2T_0)\times\Omega)$ the set of all $\int_0^{a_\dagger}\beta(a)p^u(a,t,x)da$, such that $u\in L^2(Q_{2T_0})$, p^u satisfies (4.32) and

$$p^u(a,t,x)=0,\qquad \text{a.e. } (a,t,x)\in\Gamma_0\times\Omega. \tag{4.33}$$

There exists an element in $\Phi(b)$ which does not depend on b:

• If $t>T_0$ then $\int_0^{a_\dagger}\beta(a)p^u(a,t,x)da=0$. This is because $\beta(a)=0$ a.e. $a\in(0,T_0)$ and $p^u(a,T_0,x)da=0$ for $a\geq T_0$.

• If $t\in(0,T_0)$ then $\int_0^{a_\dagger}\beta(a)p^u(a,t,x)da=\int_{T_0}^{a_\dagger-T_0}\beta(a)p^u(a,t,x)da$, and this depends only on $\bar{p}_0$ and not on b.

We also have that $p^u(a,2T_0,x)=0$ a.e. $(a,x)\in(0,a_\dagger)\times\Omega$ and

$$\int_{T_0}^{a_\dagger-T_0}\beta(a)p^u(a,t,x)da\le C\|\beta\|_{L^\infty(0,a_\dagger)}\cdot\left\|\bar{p}_0\right\|_{L^\infty((0,a_\dagger)\times\Omega)}. \tag{4.34}$$

So, for any u as above we can take

$$b(t,x)=\begin{cases}0 & \text{a.e. } (t,x)\in\big(T_0,2T_0\big)\times\Omega,\\ \int_0^{a_\dagger}\beta(a)p^u(a,t,x)da & \text{a.e. } (t,x)\in\big(0,T_0\big)\times\Omega\end{cases} \tag{4.35}$$

a fixed point of the multivalued function Φ. In addition by (4.32) and (4.34) we have

$$\left\|p^u\right\|_{L^\infty(Q_{2T_0})}\le C\left\|\bar{p}_0\right\|_{L^\infty((0,a_\dagger)\times\Omega)}. \tag{4.36}$$

So, if $\|\bar{p}_0\|_{L^\infty((0,a_\dagger)\times\Omega)}$ is small enough, there exists $u\in L^2(Q_{2T_0})$ and p, the solution of (1.7) satisfies

$$\begin{gathered}p\big(a,2T_0,x\big)=0\qquad \text{a.e. } (a,x)\in\big(0,a_\dagger\big)\times\Omega,\\ \|p\|_{L^\infty(Q_{2T_0})}\le C\left\|\bar{p}_0\right\|_{L^\infty((0,a_\dagger)\times\Omega)}\le\rho_0\end{gathered} \tag{4.37}$$

and in conclusion $p(a,t,x)\geq-\rho_0$ a.e. $(a,t,x)\in Q_{2T_0}$. On the other hand, $p(a,t,x)$ does not depend on the control for $(a,t,x)\in(a_\dagger-T_0,a_\dagger)\times(0,2T_0)\times\Omega$, so

$$p(a,t,x)\geq-p_s(a,x)\qquad \text{a.e. in } Q_{2T_0}. \tag{4.38}$$

Now if we extend this u by 0 outside $G \times \Omega$, we conclude the null controllability for (1.7) and the controllability for (1.1).

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