

EXISTENCE OF SOLUTIONS FOR ELLIPTIC EQUATIONS HAVING NATURAL GROWTH TERMS IN ORLICZ SPACES

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Existence result for strongly nonlinear elliptic equation with a natural growth condition on the nonlinearity is proved.

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 2$) with the segment property.

Consider the nonlinear Dirichlet problem

$$A(u) + g(x, u, \nabla u) = f, \quad (1.1)$$

where $A(u) = -\operatorname{div} a(x, u, \nabla u)$ is a Leray-Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega) \rightarrow W^{-1} L_{\overline{M}}(\Omega)$ with M an N -function and where g is a nonlinearity with the "natural" growth condition

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + M(|\xi|)) \quad (1.2)$$

and which satisfies the classical sign condition $g(x, s, \xi)s \geq 0$. The right-hand side f is assumed to belong to $W^{-1} E_{\overline{M}}(\Omega)$.

It is well known that Gossez [12] solved (1.1) in the case where g depends only on x and u . If g depends also on ∇u , existence theorems have recently been proved by Benkirane and Elmahi in [3, 4] by making some restrictions.

In [3], g is supposed to satisfy a "nonnatural" growth condition of the form

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + P(|\xi|)) \quad \text{with } P \ll M, \quad (1.3)$$

and in [4], g is supposed to satisfy a natural growth of the form (1.2) but the result is restricted to N -functions M satisfying a Δ_2 -condition.

It is our purpose in this paper to extend the result of [4] to general N -functions (i.e., without assuming a Δ_2 -condition on M) and hence generalize the results of [3, 4, 7].

As an example of equations to which the present result can be applied, we give
(1)

$$\begin{aligned}
 -\operatorname{div}\left(\exp(m|u|)\frac{\exp(|\nabla u|)-1}{|\nabla u|^2}\nabla u\right)+u\sin^2 u\exp(|\nabla u|)=f, \quad m \geq 0, \\
 \text{with } f=f_0+\sum_{i=1}^N\frac{\partial f_i}{\partial x_i}, \int_{\Omega}f_i\log|f_i|dx<\infty,
 \end{aligned}
 \tag{1.4}$$

(2)

$$-\operatorname{div}\left(\frac{p(|\nabla u|)}{|\nabla u|}\nabla u\right)+ug(u)p(|\nabla u|)=f,
 \tag{1.5}$$

with suitable data f , where p is a given positive and continuous function which increases from 0 to $+\infty$ and where g is a positive function on \mathbb{R} .

For classical existence results for nonlinear elliptic equations in Orlicz-Sobolev spaces, see, for example, [2, 3, 4, 6, 8, 9, 10].

2. Preliminaries

2.1. Let $M:\mathbb{R}^+\rightarrow\mathbb{R}^+$ be an N -function, that is, M is continuous and convex, with $M(t)>0$ for $t>0$, $M(t)/t\rightarrow 0$ as $t\rightarrow 0$, and $M(t)/t\rightarrow\infty$ as $t\rightarrow\infty$.

Equivalently, M admits the following representation: $M(t)=\int_0^t m(\tau)d\tau$, where $m:\mathbb{R}^+\rightarrow\mathbb{R}^+$ is nondecreasing and right continuous, with $m(0)=0$, $m(t)>0$ for $t>0$, and $m(t)\rightarrow\infty$ as $t\rightarrow\infty$.

The N -function \overline{M} , conjugate to M , is defined by $\overline{M}(t)=\int_0^t \overline{m}(\tau)d\tau$, where $\overline{m}:\mathbb{R}^+\rightarrow\mathbb{R}^+$ is given by $\overline{m}(t)=\sup\{s:m(s)\leq t\}$ (see [1, 14, 15]).

The N -function M is said to satisfy the Δ_2 -condition if, for some $k>0$,

$$M(2t)\leq kM(t) \quad \forall t\geq 0.
 \tag{2.1}$$

When (2.1) holds only for $t\geq$ some $t_0>0$, then M is said to satisfy the Δ_2 -condition near infinity.

We will extend these N -functions into even functions on all \mathbb{R} .

Let P and Q be two N -functions. $P\ll Q$ means that P grows essentially less rapidly than Q , that is, for each $\varepsilon>0$,

$$\frac{P(t)}{Q(\varepsilon t)}\rightarrow 0 \quad \text{as } t\rightarrow\infty.
 \tag{2.2}$$

This is the case if and only if

$$\lim_{t\rightarrow\infty}\frac{Q^{-1}(t)}{P^{-1}(t)}=0.
 \tag{2.3}$$

2.2. Let Ω be an open subset of \mathbb{R}^N . The Orlicz class $\mathcal{L}_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions

u on Ω such that

$$\int_{\Omega} M(u(x)) dx < +\infty \quad \left(\text{resp., } \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0 \right). \tag{2.4}$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_M = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) dx \leq 1 \right\} \tag{2.5}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 -condition for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_{\Omega} u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M}}$.

The space $L_M(\Omega)$ is reflexive if and only if M and \overline{M} satisfy the Δ_2 -condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.3. We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_M, \tag{2.6}$$

thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M , we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$,

$$\int_{\Omega} M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) dx \rightarrow 0 \quad \forall |\alpha| \leq 1; \tag{2.7}$$

this implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If M satisfies the Δ_2 -condition on \mathbb{R}^+ (near infinity only if Ω has finite measure), then modular convergence coincides with norm convergence.

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp., $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order less than or equal to 1 of functions in $L_{\overline{M}}(\Omega)$ (resp., $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [9, 11]). Consequently, the action of a distribution S in $W^{-1}L_{\overline{M}}(\Omega)$ on an element u of $W_0^1L_M(\Omega)$ is well defined. It will be denoted by $\langle S, u \rangle$.

3. The main result

Let Ω be a bounded open subset of \mathbb{R}^N ($N \geq 2$) with the segment property. Let M and P be two N -functions such that $P \ll M$.

Let $A : D(A) \subset W_0^1 L_M(\Omega) \rightarrow W^{-1} L_{\overline{M}}(\Omega)$ be a mapping (not everywhere defined) given by

$$A(u) = -\operatorname{div} a(x, u, \nabla u), \tag{3.1}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e. $x \in \Omega$, and for all $s \in \mathbb{R}$ and all $\xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$,

$$|a(x, s, \xi)| \leq \beta [c(x) + \overline{P}^{-1} M(\gamma|s|) + \overline{M}^{-1} M(\gamma|\xi|)], \tag{3.2}$$

$$[a(x, s, \xi) - a(x, s, \xi^*)][\xi - \xi^*] > 0, \tag{3.3}$$

$$\alpha M(|\xi|) \leq a(x, s, \xi)\xi, \tag{3.4}$$

where $c(x)$ belongs to $E_{\overline{M}}(\Omega)$, $c \geq 0$, and $\alpha, \beta, \gamma > 0$.

Furthermore, let $g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$,

$$g(x, s, \xi)s \geq 0, \tag{3.5}$$

$$|g(x, s, \xi)| \leq b(|s|)(c'(x) + M(|\xi|)), \tag{3.6}$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and non decreasing function and $c'(x)$ is a given non-negative function in $L^1(\Omega)$. Finally, we assume that

$$f \in W^{-1} E_{\overline{M}}(\Omega). \tag{3.7}$$

Consider the following elliptic problem with Dirichlet boundary condition:

$$\begin{aligned} u \in W_0^1 L_M(\Omega), \quad g(x, u, \nabla u) \in L^1(\Omega), \quad g(x, u, \nabla u)u \in L^1(\Omega), \\ \langle A(u), v \rangle + \int_{\Omega} g(x, u, \nabla u)v \, dx = \langle f, v \rangle \\ \text{for all } v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega) \text{ and for } v = u. \end{aligned} \tag{3.8}$$

We will prove the following existence theorem.

THEOREM 3.1. *Assume that (3.2), (3.3), (3.4), (3.5), (3.6), and (3.7) hold true. Then there exists at least one solution u of (3.8).*

Remark 3.2. Note that conditions (3.4) and (3.6) can be replaced by the following ones:

$$\begin{aligned} \alpha M\left(\frac{|\xi|}{\lambda}\right) &\leq a(x, s, \xi)\xi, \\ |g(x, s, \xi)| &\leq b(|s|)\left(c'(x) + M\left(\frac{|\xi|}{\lambda'}\right)\right), \end{aligned} \tag{3.9}$$

with $\lambda' \geq \lambda > 0$.

Remark 3.3. The Euler equation of the integral

$$\int_{\Omega} \left(a(u) \int_0^{|\nabla u|} \frac{M(t)}{t} dt \right) dx - \langle f, u \rangle \tag{3.10}$$

is

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(a(u) \frac{M(|\nabla u|)}{|\nabla u|^2} \frac{\partial u}{\partial x_i} \right) + a'(u) \int_0^{|\nabla u|} \frac{M(t)}{t} dt = f, \tag{3.11}$$

where $a(s)$ is a smooth function satisfying $a'(s)s \geq 0$. Note that

$$a'(u) \int_0^{|\nabla u|} \frac{M(t)}{t} dt \tag{3.12}$$

satisfies the growth condition (3.6) and then Theorem 3.1 can be applied to Dirichlet problems related to (3.11).

Proof of Theorem 3.1

Step 1 (a priori estimates). Consider the sequence of approximate problems

$$\begin{aligned} u_n &\in W_0^1 L_M(\Omega), \\ \langle A(u_n), v \rangle + \int_{\Omega} g_n(x, u_n, \nabla u_n) v dx &= \langle f, v \rangle \quad \forall v \in W_0^1 L_M(\Omega), \end{aligned} \tag{3.13}$$

where

$$g_n(x, s, \xi) = T_n(g(x, s, \xi)) \tag{3.14}$$

and where for $k > 0$, T_k is the usual truncation at height k defined by $T_k(s) = \max(-k, \min(k, s))$ for all $s \in \mathbb{R}$.

Note that $g_n(x, s, \xi)s \geq 0$, $|g_n(x, s, \xi)| \leq |g(x, s, \xi)|$, and $|g_n(x, s, \xi)| \leq n$. Since g_n is bounded for any fixed $n > 0$, there exists at least one solution u_n of (3.13) (see [13, Propositions 1 and 5]).

Using in (3.13) the test function u_n , we get

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f, u_n \rangle. \tag{3.15}$$

Consequently, one has that (u_n) is bounded in $W_0^1 L_M(\Omega)$. By [13, Proposition 5] (see [13, Remark 8]), $(a(x, u_n, \nabla u_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$,

$$\int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx \leq C, \tag{3.16}$$

where C is a real constant which does not depend on n .

Passing to a subsequence, if necessary, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \text{ strongly in } E_M(\Omega), \text{ and a.e. in } \Omega; \\ a(x, u_n, \nabla u_n) &\rightharpoonup h \text{ and } a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } (L_{\overline{M}}(\Omega))^N \\ &\text{for } \sigma(\Pi L_{\overline{M}}, \Pi E_M) \text{ for some } h \text{ and } h_k \in (L_{\overline{M}}(\Omega))^N. \end{aligned} \tag{3.17}$$

Step 2 (almost everywhere convergence of the gradients). Fix $k > 0$ and let $\varphi(t) = te^{\sigma t^2}$, $\sigma > 0$. It is well known that when $\sigma \geq (b(k)/2\alpha)^2$, one has

$$\varphi'(t) - \frac{b(k)}{\alpha} |\varphi(t)| \geq \frac{1}{2} \quad \forall t \in \mathbb{R}. \tag{3.18}$$

Take a sequence $(v_j) \subset \mathcal{D}(\Omega)$ which converges to u for the modular convergence in $W_0^1 L_M(\Omega)$ (cf. [11]) and set $\theta_n^j = T_k(u_n) - T_k(v_j)$, $\theta^j = T_k(u) - T_k(v_j)$, and $z_n^j = \varphi(\theta_n^j)$.

Using in (3.13) the test function z_n^j , we get

$$\langle A(u_n), z_n^j \rangle + \int_{\Omega} g_n(x, u_n, \nabla u_n) z_n^j dx = \langle f, z_n^j \rangle. \tag{3.19}$$

Denote by $\varepsilon_i(n, j)$ ($i = 0, 1, 2, \dots$) various sequences of real numbers which tend to 0 when n and $j \rightarrow \infty$, that is,

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \varepsilon_i(n, j) = 0. \tag{3.20}$$

In view of (3.17), we have $z_n^j \rightarrow \varphi(\theta^j)$ weakly in $W_0^1 L_M(\Omega)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ as $n \rightarrow \infty$ and then $\langle f, z_n^j \rangle \rightarrow \langle f, \varphi(\theta^j) \rangle$ as $n \rightarrow \infty$. Using, now, the modular convergence of (v_j) , we get $\langle f, \varphi(\theta^j) \rangle \rightarrow 0$ as $j \rightarrow \infty$ so that

$$\langle f, z_n^j \rangle = \varepsilon_0(n, j). \tag{3.21}$$

Since $g_n(x, u_n, \nabla u_n) z_n^j \geq 0$ on the subset $\{x \in \Omega : |u_n| > k\}$, we have

$$\langle A(u_n), z_n^j \rangle + \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) z_n^j dx \leq \varepsilon_0(n, j). \tag{3.22}$$

The first term on the left-hand side of (3.22) reads as

$$\begin{aligned}
 \langle A(u_n), z_n^j \rangle &= \int_{\{|u_n| \leq k\}} a(x, u_n, \nabla u_n) [\nabla T_k(u_n) - \nabla T_k(v_j)] \varphi'(\theta_n^j) dx \\
 &\quad - \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \varphi'(\theta_n^j) dx \\
 &= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(u_n) - \nabla T_k(v_j)] \varphi'(\theta_n^j) dx \\
 &\quad - \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \varphi'(\theta_n^j) dx
 \end{aligned} \tag{3.23}$$

and then

$$\begin{aligned}
 \langle A(u_n), z_n^j \rangle &= \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)] \\
 &\quad \times [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s] \varphi'(\theta_n^j) dx \\
 &+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s] \varphi'(\theta_n^j) dx \\
 &- \int_{\Omega \setminus \Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \varphi'(\theta_n^j) dx \\
 &- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \varphi'(\theta_n^j) dx,
 \end{aligned} \tag{3.24}$$

where χ_j^s denotes the characteristic function of the subset

$$\Omega_j^s = \{x \in \Omega : |\nabla T_k(v_j)| \leq s\}. \tag{3.25}$$

We will pass to the limit in n and in j for s fixed in the last three terms of the right-hand side of (3.24).

Starting with the fourth term, observe that, since

$$|\nabla T_k(v_j) \chi_{\{|u_n| > k\}} \varphi'(\theta_n^j)| \leq \varphi'(2k) |\nabla T_k(v_j)| \leq \varphi'(2k) \|\nabla v_j\|_{\infty} = a_j \in \mathbb{R}, \tag{3.26}$$

we have

$$\nabla T_k(v_j) \chi_{\{|u_n| > k\}} \varphi'(\theta_n^j) \longrightarrow \nabla T_k(v_j) \chi_{\{|u| \geq k\}} \varphi'(\theta^j) \text{ strongly in } (E_M(\Omega))^N \text{ as } n \longrightarrow \infty, \tag{3.27}$$

and hence

$$\int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \varphi'(\theta_n^j) dx \longrightarrow \int_{\{|u| \geq k\}} h \nabla T_k(v_j) \varphi'(\theta^j) dx \text{ as } n \longrightarrow \infty. \tag{3.28}$$

Observe that

$$|\nabla T_k(v_j) \chi_{\{|u| \geq k\}} \varphi'(\theta^j)| \leq \varphi'(2k) |\nabla T_k(v_j)| \leq \varphi'(2k) |\nabla v_j|; \tag{3.29}$$

then, by using the modular convergence of $|\nabla v_j|$ in $L_M(\Omega)$ and Vitali's theorem, we get

$$\nabla T_k(v_j)\chi_{\{|u|\geq k\}}\varphi'(\theta^j) \rightarrow 0 \tag{3.30}$$

for the modular convergence in $(L_M(\Omega))^N$, and thus

$$\int_{\{|u|\geq k\}} h\nabla T_k(v_j)\varphi'(\theta^j)dx \rightarrow 0 \text{ as } j \rightarrow \infty. \tag{3.31}$$

We have then proved that

$$\int_{\{|u_n|>k\}} a(x, u_n, \nabla u_n)\nabla T_k(v_j)\varphi'(\theta_n^j)dx = \varepsilon_1(n, j). \tag{3.32}$$

The second term on the right-hand side of (3.24) tends to (by letting $n \rightarrow \infty$)

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j)\chi_j^s)[\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s]\varphi'(\theta^j)dx \tag{3.33}$$

since $a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)\varphi'(\theta_n^j) \rightarrow a(x, T_k(u), \nabla T_k(v_j)\chi_j^s)\varphi'(\theta^j)$ strongly in $(E_{\overline{M}}(\Omega))^N$ as $n \rightarrow \infty$ by [3, Lemma 2.3], while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(\Omega))^N$ by (3.17).

Since $\nabla T_k(v_j)\chi_j^s \rightarrow \nabla T_k(u)\chi^s$ strongly in $(E_M(\Omega))^N$ as $j \rightarrow \infty$, where χ^s denotes the characteristic function of $\Omega_s = \{x \in \Omega : |\nabla T_k(u)| \leq s\}$, it is easy to see that

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j)\chi_j^s)[\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s]\varphi'(\theta^j)dx \rightarrow 0 \text{ as } j \rightarrow \infty, \tag{3.34}$$

and thus

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\varphi'(\theta_n^j)dx = \varepsilon_2(n, j). \tag{3.35}$$

Concerning the third term on the right-hand side of (3.24), we have

$$-\int_{\Omega \setminus \Omega_s^j} a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)\varphi'(\theta_n^j)dx \rightarrow -\int_{\Omega \setminus \Omega_s^j} h_k\nabla T_k(v_j)\varphi'(\theta^j)dx \tag{3.36}$$

as $n \rightarrow \infty$ by using the fact that $\nabla T_k(v_j)$ belongs to $(E_M(\Omega))^N$.

In view of the modular convergence of (∇v_j) in $(L_M(\Omega))^N$, we have

$$-\int_{\Omega \setminus \Omega_s^j} h_k\nabla T_k(v_j)\varphi'(\theta^j)dx \rightarrow -\int_{\Omega \setminus \Omega_s} h_k\nabla T_k(u)dx \text{ as } j \rightarrow \infty \tag{3.37}$$

and thus

$$-\int_{\Omega \setminus \Omega_s^j} a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)\varphi'(\theta_n^j)dx = \varepsilon_3(n, j) - \int_{\Omega \setminus \Omega_s} h_k\nabla T_k(u)dx. \tag{3.38}$$

Combining now (3.32), (3.35), and (3.38), we obtain

$$\begin{aligned} \langle A(u_n), z_n^j \rangle &= \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)] \\ &\quad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] \varphi'(\theta_n^j) dx - \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx + \varepsilon_4(n, j). \end{aligned} \tag{3.39}$$

We now turn to the second term on the left-hand side of (3.22). We have

$$\begin{aligned} &\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) z_n^j dx \right| \\ &= \left| \int_{\{|u_n| \leq k\}} g_n(x, T_k(u_n), \nabla T_k(u_n)) z_n^j dx \right| \\ &\leq \int_{\Omega} b(k) c'(x) |\varphi(\theta_n^j)| dx + b(k) \int_{\Omega} M(|\nabla T_k(u_n)|) |\varphi(\theta_n^j)| dx \\ &\leq \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\varphi(\theta_n^j)| dx + \varepsilon_5(n, j). \end{aligned} \tag{3.40}$$

The first term of the right-hand side of this inequality reads as

$$\begin{aligned} &\frac{b(k)}{\alpha} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)] \\ &\quad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] |\varphi(\theta_n^j)| dx \\ &\quad + \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s) [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] |\varphi(\theta_n^j)| dx \\ &\quad - \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j)\chi_j^s |\varphi(\theta_n^j)| dx \end{aligned} \tag{3.41}$$

and, as above, it is easy to see that

$$\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s) [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] |\varphi(\theta_n^j)| dx = \varepsilon_6(n, j) \tag{3.42}$$

and that

$$-\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j)\chi_j^s |\varphi(\theta_n^j)| dx = \varepsilon_7(n, j) \tag{3.43}$$

so that

$$\begin{aligned} &\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) z_n^j dx \right| \\ &\leq \frac{b(k)}{\alpha} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)] \\ &\quad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] |\varphi(\theta_n^j)| dx + \varepsilon_8(n, j). \end{aligned} \tag{3.44}$$

Combining this inequality with (3.22) and (3.39), we obtain

$$\int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)] [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] \times \left[\varphi'(\theta_n^j) - \frac{b(k)}{\alpha} |\varphi(\theta_n^j)| \right] dx \leq \varepsilon_9(n, j) + \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx. \tag{3.45}$$

Consequently,

$$\int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)] [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] dx \leq 2\varepsilon_9(n, j) + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx. \tag{3.46}$$

On the other hand,

$$\begin{aligned} & \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)] [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] dx \\ &= \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)] [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] dx \\ & \quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s] dx \\ & \quad - \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] dx \\ & \quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s) [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] dx. \end{aligned} \tag{3.47}$$

We will pass to the limit in n and in j in the last three terms on the right-hand side of the above equality. Similar tools as in (3.24) and (3.41) give

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s] dx = \varepsilon_{10}(n, j), \tag{3.48}$$

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] dx = \varepsilon_{11}(n, j), \tag{3.49}$$

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s) [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] dx = \varepsilon_{12}(n, j) \tag{3.50}$$

which imply that

$$\begin{aligned} & \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)] [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] dx \\ &= \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)] [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] dx \\ & \quad + \varepsilon_{13}(n, j). \end{aligned} \tag{3.51}$$

For $r \leq s$, one has

$$\begin{aligned}
 0 &\leq \int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] dx \\
 &\leq \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] dx \\
 &= \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)] [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] dx \\
 &\leq \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)] [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] dx \\
 &= \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)] [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s] dx \\
 &\quad + \varepsilon_{13}(n, j) \\
 &\leq \varepsilon_{14}(n, j) + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx.
 \end{aligned} \tag{3.52}$$

This implies that, by passing at first to the limit sup over n and next over j ,

$$\begin{aligned}
 0 &\leq \limsup_{n \rightarrow \infty} \int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] dx \\
 &\leq 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx.
 \end{aligned} \tag{3.53}$$

Using the fact that $h_k \nabla T_k(u) \in L^1(\Omega)$ and letting $s \rightarrow \infty$, we get

$$\int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] dx \rightarrow 0 \tag{3.54}$$

as $n \rightarrow \infty$.

As in [3], we deduce that there exists a subsequence still denoted by u_n such that

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e. in } \Omega, \tag{3.55}$$

which implies that

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \quad \text{for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{3.56}$$

Step 3 (modular convergence of the truncations). Going back to (3.46), we can write

$$\begin{aligned}
 \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx &\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx \\
 &\quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \\
 &\quad \times [\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s] dx \\
 &\quad + 2\varepsilon_9(n, j) + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx,
 \end{aligned} \tag{3.57}$$

which implies, by using (3.50),

$$\begin{aligned} & \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ & \leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx + \varepsilon_{15}(n, j) + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx. \end{aligned} \tag{3.58}$$

Passing to the limit sup over n in both sides of this inequality yields

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ & \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(v_j) \chi_j^s dx + \lim_{n \rightarrow \infty} \varepsilon_{15}(n, j) + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx, \end{aligned} \tag{3.59}$$

in which we can pass to the limit in j to obtain

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ & \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \chi^s dx + 2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u) dx \end{aligned} \tag{3.60}$$

which gives, by letting $s \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx. \tag{3.61}$$

On the other hand, we have, by using Fatou's lemma,

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx, \tag{3.62}$$

which implies that

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \longrightarrow \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \quad \text{as } n \longrightarrow \infty, \tag{3.63}$$

and by using [4, Lemma 2.4], we conclude that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \longrightarrow a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \quad \text{in } L^1(\Omega). \tag{3.64}$$

This implies, by using (3.4), that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in } W_0^1 L_M(\Omega) \tag{3.65}$$

for the modular convergence.

Step 4 (equi-integrability of the nonlinearities and passage to the limit). We will prove that $g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u)$ strongly in $L^1(\Omega)$ by using Vitali’s theorem.

Since $g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u)$ a.e. in Ω , thanks to (3.55), it suffices to prove that $g_n(x, u_n, \nabla u_n)$ are uniformly equi-integrable in Ω . Let $E \subset \Omega$ be a measurable subset of Ω . We have, for any $m > 0$,

$$\begin{aligned} \int_E |g_n(x, u_n, \nabla u_n)| dx &= \int_{E \cap \{|u_n| \leq m\}} |g_n(x, u_n, \nabla u_n)| dx + \int_{E \cap \{|u_n| > m\}} |g_n(x, u_n, \nabla u_n)| dx \\ &\leq b(m) \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) dx \\ &\quad + b(m) \int_E c'(x) dx + \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx. \end{aligned} \tag{3.66}$$

Standard arguments allow to deduce, using the strong convergence (3.64), that there exists $\mu > 0$ such that

$$|E| < \mu \implies \int_E |g_n(x, u_n, \nabla u_n)| dx \leq \varepsilon, \quad \forall n, \tag{3.67}$$

which shows that $g_n(x, u_n, \nabla u_n)$ are uniformly equi-integrable in Ω as required.

In order to pass to the limit, we have, by going back to approximate equations (3.13),

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla w dx + \int_\Omega g_n(x, u_n, \nabla u_n) w dx = \langle f, w \rangle \tag{3.68}$$

for all $w \in \mathcal{D}(\Omega)$, in which, we can easily pass to the limit as $n \rightarrow \infty$ to get

$$\int_\Omega a(x, u, \nabla u) \nabla w dx + \int_\Omega g(x, u, \nabla u) w dx = \langle f, w \rangle. \tag{3.69}$$

Let now $v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$. There exists $(w_j) \subset \mathcal{D}(\Omega)$ such that $\|w_j\|_{\infty, \Omega} \leq (N + 1)\|v\|_{\infty, \Omega}$ for all $j \in \mathbb{N}$ and

$$w_j \longrightarrow v \tag{3.70}$$

for the modular convergence in $W_0^1 L_M(\Omega)$. Taking $w = w_j$ in (3.69) and letting $j \rightarrow \infty$ yields

$$\int_\Omega a(x, u, \nabla u) \nabla v dx + \int_\Omega g(x, u, \nabla u) v dx = \langle f, v \rangle. \tag{3.71}$$

By choosing $v = T_k(u)$ in the last equality, we get

$$\int_\Omega a(x, u, \nabla u) \nabla T_k(u) dx + \int_\Omega g(x, u, \nabla u) T_k(u) dx = \langle f, T_k(u) \rangle. \tag{3.72}$$

From (3.16), we deduce by Fatou’s lemma that $g(x, u, \nabla u)u \in L^1(\Omega)$ and since $|g(x, u, \nabla u)T_k(u)| \leq g(x, u, \nabla u)u$ and $T_k(u) \rightarrow u$ in $W_0^1 L_M(\Omega)$ for the modular convergence and

a.e. in Ω as $k \rightarrow \infty$, it is easy to pass to the limit in both sides of (3.72) (by using Lebesgue theorem) to obtain

$$\int_{\Omega} a(x, u, \nabla u) \nabla u \, dx + \int_{\Omega} g(x, u, \nabla u) u \, dx = \langle f, u \rangle. \quad (3.73)$$

This completes the proof of [Theorem 3.1](#). \square

Remark 3.4. If we replace, as in [5], (3.2) by the general growth condition

$$|a(x, s, \xi)| \leq \bar{b}(|s|) (c(x) + \bar{M}^{-1} M(\gamma |\xi|)), \quad (3.74)$$

where $\gamma > 0$, $c \in E_{\bar{M}}(\Omega)$, and $\bar{b}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous nondecreasing function, we prove the existence of solutions for the following problem:

$$\begin{aligned} u \in W_0^1 L_M(\Omega), \quad g(x, u, \nabla u) \in L^1(\Omega), \quad g(x, u, \nabla u) u \in L^1(\Omega), \\ \langle A(u), T_k(u - v) \rangle + \int_{\Omega} g(x, u, \nabla u) T_k(u - v) \, dx \leq \langle f, T_k(u - v) \rangle \\ \forall v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega). \end{aligned} \quad (3.75)$$

Indeed, we consider the following approximate problems:

$$\begin{aligned} u_n \in W_0^1 L_M(\Omega), \\ -\operatorname{div} a(x, T_n(u_n), \nabla u_n) + g_n(x, u_n, \nabla u_n) = f \quad \text{in } \Omega, \end{aligned} \quad (3.76)$$

and we conclude by adapting the same steps.

As an application of this result, we can treat the following model equations:

$$-\operatorname{div} \left((1 + |u|)^m \frac{\exp(|\nabla u|) - 1}{|\nabla u|^2} \nabla u \right) + u \cos^2 u \exp(|\nabla u|) = f, \quad m \geq 0. \quad (3.77)$$

Remark that the solutions of (3.77) belong to $L^\infty(\Omega)$ so that (3.77) holds in the distributional sense.

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