

# ON THE EXISTENCE OF WEAK SOLUTIONS FOR THE INITIAL-BOUNDARY VALUE PROBLEM IN THE JEFFREYS MODEL OF MOTION OF A VISCOELASTIC MEDIUM

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This paper deals with the initial-boundary value problem for the system of motion equations of an incompressible viscoelastic medium with Jeffreys constitutive law in an arbitrary domain of two-dimensional or three-dimensional space. The existence of weak solutions of this problem is obtained.

## 1. Introduction

It is known [5, 7] that the motion of an incompressible medium with constant density  $\rho = \text{const}$  is determined by the system of differential equations in the form of Cauchy

$$\begin{aligned} \rho \left( \frac{\partial u}{\partial t} + \sum_{i=1}^n u_i \frac{\partial u}{\partial x_i} \right) + \text{grad } p = \text{Div } \sigma + \rho f, \quad (t, x) \in (0, T) \times \Omega, \\ \text{div } u = 0, \quad (t, x) \in (0, T) \times \Omega. \end{aligned} \quad (1.1)$$

Here  $u$  is the velocity vector,  $p$  is the pressure function,  $f$  is the body force, and  $\sigma$  is the deviator of the stress tensor (all of them depend on a point  $x$  of an arbitrary domain  $\Omega$  in the space  $\mathbb{R}^n$ ,  $n = 2, 3$ , and on a moment of time  $t$ ). The gradient  $\text{grad}$  and the divergence  $\text{div}$  are taken with respect to the variable  $x$ . The divergence  $\text{Div } \sigma$  of a tensor  $\sigma$  is the vector with the coordinates  $(\text{Div } \sigma)_j = \sum_{i=1}^n (\partial \sigma_{ij} / \partial x_i)$ .

Without loss of generality, we can consider the density  $\rho$  to be equal to 1.

The type of a medium is determined by the choice of the constitutive law between  $\sigma$  and the strain velocity tensor  $\mathcal{E}(u)$ ,  $\mathcal{E}(u) = (\mathcal{E}_{ij}(u))$ ,  $\mathcal{E}_{ij}(u) = (1/2)(\partial u_i / \partial x_j + \partial u_j / \partial x_i)$ . For instance, one class of mediums is connected with the Stokes conjecture that the deviator of the stress tensor in every point is completely determined by the strain velocity tensor in the same point at the same moment of time. It is the conception of linear- and nonlinear-viscous fluid [7].

However, this conception is not satisfactory for all incompressible media. In particular, it is not suitable for mediums "with memory": concrete, various polymers, the earth's crust, and so forth. To take into account the effects of memory one may introduce time

derivatives into the constitutive law. When this method was used the models of Maxwell, Jeffreys, Oldroyd, and a lot of other models [4, 8, 9] appeared.

In the present work we study the solvability in the weak sense of the initial-boundary value problem in the Jeffreys model [9] of motion of a viscoelastic medium in an arbitrary domain  $\Omega \subset \mathbb{R}^n$  ( $n = 2, 3$ ), which may also be unbounded. The corresponding constitutive law is

$$\sigma + \lambda_1 \frac{d}{dt} \sigma = 2\eta \left( \mathcal{E} + \lambda_2 \frac{d}{dt} \mathcal{E} \right). \tag{1.2}$$

Here  $\eta$  is the viscosity of the medium,  $\lambda_1$  is the relaxation time, and  $\lambda_2$  is the retardation time,  $0 < \lambda_2 < \lambda_1$ .

The main result of this paper is the existence theorem of weak solutions for the initial-boundary value problem for system (1.1)-(1.2) in an arbitrary domain  $\Omega \subset \mathbb{R}^n$  ( $n = 2, 3$ ).

We note that in a number of papers (see, e.g., [1, 2, 6]) the initial-boundary value problem was studied provided that the full derivative  $d/dt$  was replaced by the partial derivative  $\partial/\partial t$ , what essentially narrows the class of mediums satisfying the model [8]. In [11] the Jeffreys relation (1.2) was considered without such linearizations, but at the expression of the stress tensor through the strain velocity tensor the regularization of the velocity field with the help of averaging on the spatial variable was used. The feature of this work is that here there is no such regularization.

The plan of the paper is as follows. In Section 2, the statement of the problem, describing the motion of the viscoelastic fluid, is presented and the basic notations are introduced. Then we introduce the concept of a weak solution of this problem and formulate the main existence theorem. In Section 3, the existence of solutions of an auxiliary problem, depending on several parameters, is proved with the help of a priori estimates and the Leray-Schauder degree theory. In Section 4, the passage to the limit as one of these parameters tends to zero is carried out. With the help of the obtained result, in Section 5, we prove the existence of a weak solution of the initial-boundary value problem for the Jeffreys model and estimate this solution.

## 2. Statement of the problem and the main result

**2.1. Statement of the problem.** Let  $\Omega$  be an arbitrary domain in the space  $\mathbb{R}^n$ ,  $n = 2, 3$ , which may also be unbounded.

Consider the following initial-boundary value problem, which describes the motion of an incompressible viscoelastic medium, corresponding to the Jeffreys model:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^n u_i \frac{\partial u}{\partial x_i} + \text{grad } p = \text{Div } \sigma + f, \tag{2.1}$$

$$\sigma + \lambda_1 \left( \frac{\partial \sigma}{\partial t} + \sum_{i=1}^n u_i \frac{\partial \sigma}{\partial x_i} \right) = 2\eta \left( \mathcal{E} + \lambda_2 \left( \frac{\partial \mathcal{E}}{\partial t} + \sum_{i=1}^n u_i \frac{\partial \mathcal{E}}{\partial x_i} \right) \right), \tag{2.2}$$

$$\text{div } u = 0, \tag{2.3}$$

$$u|_{\partial\Omega} = 0, \tag{2.4}$$

$$u|_{t=0} = a, \quad \sigma|_{t=0} = \sigma_0. \tag{2.5}$$

**2.2. Basic notations.** We will use the following notations. Most of them are standard.

Denote by  $\mathbb{R}^{n \times n}$  the space of matrices of order  $n \times n$  with the following scalar product: for  $A = (A_{ij}), B = (B_{ij})$ ,

$$(A, B)_{\mathbb{R}^{n \times n}} = \sum_{i,j=1}^n A_{ij}B_{ij}, \tag{2.6}$$

and denote by  $\mathbb{R}_S^{n \times n}$  its subspace of symmetric matrices.

Denote by  $\mathbb{R}^{n \times n \times n}$  the space of ordered collections of  $n$  matrices of order  $n \times n$  with the following scalar product: for  $A = (A_1, \dots, A_n), B = (B_1, \dots, B_n)$ ,

$$(A, B)_{\mathbb{R}^{n \times n \times n}} = \sum_{i=1}^n (A_i, B_i)_{\mathbb{R}^{n \times n}}. \tag{2.7}$$

The symbol  $\nabla u$  will stand for the Jacobi matrix of a vector function  $u : \Omega \rightarrow \mathbb{R}^n$ . The symbol  $\nabla \tau$  will denote the ordered collection of the Jacobi matrices of the columns of a matrix function  $\tau : \Omega \rightarrow \mathbb{R}^{n \times n}$ .

Below,  $E$  stands for one of the spaces  $\mathbb{R}^n, \mathbb{R}^{n \times n}, \mathbb{R}_S^{n \times n}, \mathbb{R}^{n \times n \times n}$ .

We will use the standard notations  $L_p(\Omega, E), W_p^m(\Omega, E), H^m(\Omega, E) = W_2^m(\Omega, E), H_0^m(\Omega, E) = \overset{\circ}{W}_2^m(\Omega, E)$  for Lebesgue and Sobolev spaces of functions with values in  $E$ . Sometimes we will write simply  $L_p$  instead of  $L_p(\Omega, E)$ , and so forth, if it is clear from the context which space  $E$  is used.

The scalar products in  $L_2(\Omega, E)$  and  $H^1(\Omega, E)$  may be given by the following bilinear forms:

$$(u, v) = \int_{\Omega} (u(x), v(x))_E dx, \tag{2.8}$$

$$(u, v)_1 = (u, v) + \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right).$$

The norms in these spaces will be denoted by  $\| \cdot \|$  and  $\| \cdot \|_1$ , respectively. The Euclid norm in  $E$  will be denoted by  $| \cdot |$ . By the symbol  $C_0^\infty(\Omega, E)$  we will denote the space of smooth functions with compact support in  $\Omega$  and with values in  $E$ .

For brevity, we will denote by  $C_0^\infty$  the space  $C_0^\infty(\Omega, \mathbb{R}_S^{n \times n})$ . Denote by  $\mathcal{V}$  the set  $\{u \in C_0^\infty(\Omega, \mathbb{R}^n), \operatorname{div} u = 0\}$ . Let the symbols  $H$  and  $V$  denote the closures of  $\mathcal{V}$  in  $L_2(\Omega, \mathbb{R}^n)$  and  $W_2^1(\Omega, \mathbb{R}^n)$ , respectively. Following [10], we will identify the space  $H$  and its conjugate space  $H^*$ . Therefore we have the embedding

$$V \subset H \equiv H^* \subset V^*. \tag{2.9}$$

The value of a functional  $\nu$  from  $V^*$  or  $H^{-m}$  ( $m = 1, 2$ ) on an element  $\varphi$  from  $V$  or  $H_0^m$ , respectively, will be denoted as  $\langle \nu, \varphi \rangle$ .

The symbols  $C([0, T]; X), C_w([0, T]; X), L_2(0, T; X), \dots$ , will denote the Banach spaces of continuous, weakly continuous, quadratically integrable, and so forth functions on the segment  $[0, T]$  with values in some Banach space  $X$ .

The symbol  $K_i(\cdot, \dots, \cdot)$ ,  $i = 0, 1, 2, \dots$  will stand for positive constants, depending continuously on arguments, which will be enumerated. Other constants will be denoted by the symbol  $C$ .

**2.3. Weak formulation of the problem.** Let  $f$  belong to the space  $L_2(0, T; V^*)$ .

*Definition 2.1.* A pair of functions  $(u, \sigma)$ ,

$$\begin{aligned} u &\in L_2(0, T; V) \cap C_w([0, T]; H), \quad \frac{du}{dt} \in L_1(0, T; V^*), \\ \sigma &\in L_2(0, T; L_2(\Omega, \mathbb{R}_S^{N \times N})) \cap C_w([0, T]; H^{-1}(\Omega, \mathbb{R}_S^{N \times N})), \end{aligned} \tag{2.10}$$

is a weak solution of problem (2.1)–(2.5) if it satisfies condition (2.5) and the equalities

$$\begin{aligned} \frac{d}{dt}(u, \varphi) + (\sigma, \nabla \varphi) - \sum_{i=1}^n \left( u_i u, \frac{\partial \varphi}{\partial x_i} \right) &= \langle f, \varphi \rangle, \\ (\sigma, \Phi) + \lambda_1 \frac{d}{dt}(\sigma, \Phi) - \lambda_1 \sum_{i=1}^n \left( u_i \sigma, \frac{\partial \Phi}{\partial x_i} \right) & \\ = -2\eta(u, \text{Div } \Phi) - 2\eta\lambda_2 \left( \frac{d}{dt}(u, \text{Div } \Phi) + \sum_{i=1}^n \left( u_i \mathcal{E}(u), \frac{\partial \Phi}{\partial x_i} \right) \right) & \end{aligned} \tag{2.11}$$

are true for all  $\varphi \in \mathcal{V}$  and  $\Phi \in C_0^\infty$  in the sense of distributions on  $(0, T)$ .

*Remark 2.2.* Equalities (2.11) appear from the following reasoning. Let  $(u, \sigma, p)$  be a classical solution of problem (2.1)–(2.5). Taking the  $L_2$ -scalar product of equalities (2.1) and (2.2) with  $\varphi \in \mathcal{V}$  and  $\Phi \in C_0^\infty$ , respectively, and integrating the obtained equalities by parts, we obtain identities (2.11).

**2.4. The main result.** The main result of this paper is the following.

**THEOREM 2.3.** *Given  $f \in L_2(0, T; V^*)$ ,  $a \in H$ ,  $\sigma_0 \in W_2^{-1}(\Omega, \mathbb{R}_S^{n \times n})$ ,  $\sigma_0 - 2\eta(\lambda_2/\lambda_1)\mathcal{E}(a) \in L_2(\Omega, \mathbb{R}_S^{n \times n})$ , there exists a weak solution of problem (2.1)–(2.5) in class (2.10).*

**3. Auxiliary problem**

Before proving [Theorem 2.3](#), we study an auxiliary problem. We begin with an equivalent transformation of system (2.11). Denote  $\mu_1 = \eta(\lambda_2/\lambda_1)$ ,  $\mu_2 = (\eta - \mu_1)/\lambda_1$ , and  $\tau = \sigma - 2\mu_1 \mathcal{E}(u)$ . Then we can rewrite (2.11) as follows:

$$\frac{d}{dt}(\tau, \Phi) + \frac{1}{\lambda_1}(\tau, \Phi) - \sum_{i=1}^n \left( u_i \tau, \frac{\partial \Phi}{\partial x_i} \right) + 2\mu_2(u, \text{Div } \Phi) = 0, \tag{3.1}$$

$$\frac{d}{dt}(u, \varphi) - \sum_{i=1}^n \left( u_i u, \frac{\partial \varphi}{\partial x_i} \right) + \mu_1(\nabla u, \nabla \varphi) + (\tau, \nabla \varphi) = \langle f, \varphi \rangle \tag{3.2}$$

for all  $\varphi \in \mathcal{V}$  and  $\Phi \in C_0^\infty$ .

Consider the following auxiliary problem:

$$\frac{d}{dt}(\tau, \Phi) + \frac{1}{\lambda_1}(\tau, \Phi) - \xi \sum_{i=1}^n \left( \frac{u_i \tau}{1 + \delta(|\tau|^2/2\mu_2 + |u|^2)}, \frac{\partial \Phi}{\partial x_i} \right) + 2\mu_2 \xi(u, \text{Div } \Phi) + \frac{\varepsilon}{\lambda_1}(\nabla \tau, \nabla \Phi) = 0, \tag{3.3}$$

$$\frac{d}{dt}(u, \varphi) - \xi \sum_{i=1}^n \left( \frac{u_i u}{1 + \delta(|\tau|^2/2\mu_2 + |u|^2)}, \frac{\partial \varphi}{\partial x_i} \right) + \mu_1(\nabla u, \nabla \varphi) + \xi(\tau, \nabla \varphi) = \langle f, \varphi \rangle \tag{3.4}$$

for all  $\varphi \in V, \Phi \in H_0^1$  a.e. in  $(0, T)$ ;

$$u|_{t=0} = a, \quad \tau|_{t=0} = \tau_0. \tag{3.5}$$

The numbers  $\delta > 0, 0 \leq \xi \leq 1, 0 < \varepsilon \leq 1$  are parameters.

We introduce the following spaces:

$$W = \left\{ u \in L_2(0, T; V), \frac{du}{dt} \in L_2(0, T; V^*) \right\}, \tag{3.6}$$

$$W_M = \left\{ \tau \in L_2(0, T; H_0^1(\Omega, \mathbb{R}_S^{n \times n})), \frac{d\tau}{dt} \in L_2(0, T; H^{-1}(\Omega, \mathbb{R}_S^{n \times n})) \right\}$$

with natural intersection norms.

It follows from [10, Chapter III, Theorem 1.2] that  $W$  and  $W_M$  are embedded into  $C([0, T]; H)$  and  $C([0, T]; L_2)$ , respectively. If  $\Omega$  is bounded, (see [10, Chapter III, Theorem 2.1] and [10, Chapter II, Theorem 1.1]), the embeddings  $W$  into  $L_2(0, T; H)$  and  $W_M$  into  $L_2(0, T; L_2)$  are compact.

LEMMA 3.1. *Let  $a \in H, \tau_0 \in L_2, f \in L_2(0, T; V^*)$ , and let a pair  $(u \in W, \tau \in W_M)$  be a solution of problem (3.3)–(3.5). Then the following estimate holds:*

$$\begin{aligned} & \max_{t \in [0, T]} (\|u\|(t) + \|\tau\|(t)) + \int_0^T \|u\|_1^2(t) dt + \varepsilon \int_0^T \|\tau\|_1^2(t) dt \\ & \leq K_0(\|a\|, \|\tau_0\|, \|f\|_{L_2(0, T; V^*)}), \end{aligned} \tag{3.7}$$

where  $K_0$  does not depend on  $\Omega, \varepsilon, \delta, \xi$ .

*Proof.* First we show that the following identities hold a.e. in  $(0, T)$ :

$$(\tau, \nabla u) + (u, \text{Div } \tau) = 0, \tag{3.8}$$

$$\sum_{i=1}^n \left( \frac{u_i u}{1 + \delta(|\tau|^2/2\mu_2 + |u|^2)}, \frac{\partial u}{\partial x_i} \right) + \frac{1}{2\mu_2} \left( \frac{u_i \tau}{1 + \delta(|\tau|^2/2\mu_2 + |u|^2)}, \frac{\partial \tau}{\partial x_i} \right) = 0. \tag{3.9}$$

In fact, from Green's formula we have

$$\sum_{i,j=1}^n \left[ \left( \tau_{ij}, \frac{\partial u_i}{\partial x_j} \right) + \left( u_i, \frac{\partial \tau_{ij}}{\partial x_j} \right) \right] = 0 \tag{3.10}$$

and it is the same as (3.8). We transform the left part of (3.9) as follows:

$$\begin{aligned} & \frac{1}{2} \left( \sum_{i=1}^n \frac{u_i}{1 + \delta(|\tau|^2/2\mu_2 + |u|^2)}, \frac{\partial |u|^2}{\partial x_i} + \frac{1}{2\mu_2} \frac{\partial |\tau|^2}{\partial x_i} \right) \\ &= \frac{1}{2\delta} \sum_{i=1}^n \left( u_i, \frac{\partial}{\partial x_i} \ln \left( 1 + \delta \left( \frac{|\tau|^2}{2\mu_2} + |u|^2 \right) \right) \right). \end{aligned} \tag{3.11}$$

Since  $u(t) \in V$  for a.a.  $t$ , from the formula of integration by parts it follows that the last expression is equal to zero.

It appears from [10, Chapter III, Lemma 1.1] that

$$\left\langle \frac{du}{dt}, \varphi \right\rangle = \frac{d}{dt}(u, \varphi), \quad \left\langle \frac{d\tau}{dt}, \Phi \right\rangle = \frac{d}{dt}(\tau, \Phi). \tag{3.12}$$

Therefore

$$\left. \frac{d}{dt}(u, \varphi) \right|_{\varphi=u(t)} = \left\langle \frac{du(t)}{dt}, u(t) \right\rangle = \frac{1}{2} \frac{d}{dt}(u(t), u(t)). \tag{3.13}$$

Analogously

$$\left. \frac{d}{dt}(\tau, \Phi) \right|_{\Phi=\tau(t)} = \frac{1}{2} \frac{d}{dt}(\tau, \tau). \tag{3.14}$$

Putting  $\Phi = \tau(t)/2\mu_2$  in (3.3) and  $\varphi = u(t)$  in (3.4) for a.a.  $t \in [0, T]$ , adding the results, and taking into account (3.8) and (3.9), we obtain

$$\frac{1}{2} \frac{d}{dt}(u, u) + \frac{1}{4\mu_2} \frac{d}{dt}(\tau, \tau) + \mu_1(\nabla u, \nabla u) + \frac{1}{2\lambda_1\mu_2}(\tau, \tau) + \frac{\varepsilon}{2\lambda_1\mu_2}(\nabla \tau, \nabla \tau) = \langle f, u \rangle. \tag{3.15}$$

Integrate this equality from 0 to  $t$ :

$$\begin{aligned} & \frac{1}{2} \|u\|^2(t) + \frac{1}{4\mu_2} \|\tau\|^2(t) + \int_0^t \frac{1-\varepsilon}{2\lambda_1\mu_2} \|\tau\|^2 ds + \int_0^t \frac{\varepsilon}{2\lambda_1\mu_2} \|\tau\|_1^2 ds + \int_0^t \mu_1 (\|u\|_1^2 - \|u\|^2) ds \\ & \leq \frac{1}{2} \|a\|^2 + \frac{1}{4\mu_2} \|\tau_0\|^2 + \int_0^t \|f\|_{V^*} \|u\|_1 ds. \end{aligned} \tag{3.16}$$

Take the maxima on the segment  $[0, T]$  of the left and the right parts:

$$\begin{aligned} \max_{t \in [0, T]} \left( \frac{1}{2} \|u\|^2(t) + \frac{1}{4\mu_2} \|\tau\|^2(t) \right) + \int_0^T \frac{1-\varepsilon}{2\lambda_1\mu_2} \|\tau\|^2 dt + \int_0^T \frac{\varepsilon}{2\lambda_1\mu_2} \|\tau\|_1^2 dt \\ + \int_0^T \mu_1 (\|u\|_1^2 - \|u\|^2) dt \leq \frac{1}{2} \|a\|^2 + \frac{1}{4\mu_2} \|\tau_0\|^2 + \int_0^T \|f\|_{V^*} \|u\|_1 dt. \end{aligned} \tag{3.17}$$

Note that the following inequality is valid:

$$\max_{t \in [0, T]} \frac{1}{4} \|u\|^2(t) + \int_0^T \mu_1 (\|u\|_1^2 dt - \|u\|^2) \geq \gamma \int_0^T \|u\|_1^2 dt, \tag{3.18}$$

where  $\gamma = \min(1/4T, \mu_1)$ .

For its proof it is enough to add the inequalities

$$\begin{aligned} \max_{t \in [0, T]} \frac{1}{4} \|u\|^2(t) \geq \frac{1}{4T} \int_0^T \|u\|^2 dt \geq \gamma \int_0^T \|u\|^2 dt, \\ \int_0^T \mu_1 (\|u\|_1^2 - \|u\|^2) dt \geq \gamma \int_0^T (\|u\|_1^2 - \|u\|^2) dt. \end{aligned} \tag{3.19}$$

Now, from (3.17) and (3.18) we have

$$\gamma \int_0^T \|u\|_1^2 dt \leq \frac{1}{2} \|a\|^2 + \frac{1}{4\mu_2} \|\tau_0\|^2 + \|f\|_{L_2(0, T; V^*)} \left( \int_0^T \|u\|_1^2 dt \right)^{1/2}. \tag{3.20}$$

This yields  $(\int_0^T \|u\|_1^2 dt)^{1/2} \leq y_2$ , where  $y_2$  is the greater root of the quadratic equation

$$\gamma y^2 = \frac{1}{2} \|a\|^2 + \frac{1}{4\mu_2} \|\tau_0\|^2 + \|f\|_{L_2(0, T; V^*)} y. \tag{3.21}$$

Then, from (3.17) and (3.18), it follows that

$$\begin{aligned} \max_{t \in [0, T]} \left( \frac{1}{4} \|u\|^2(t) + \frac{1}{4\mu_2} \|\tau\|^2(t) \right) + \gamma \int_0^T \|u\|_1^2 dt + \frac{\varepsilon}{2\lambda_1\mu_2} \int_0^T \|\tau\|_1^2 dt \\ \leq \frac{1}{2} \|a\|^2 + \frac{1}{4\mu_2} \|\tau_0\|^2 + \|f\|_{L_2(0, T; V^*)} y_2, \end{aligned} \tag{3.22}$$

which yields the statement of the lemma. □

**THEOREM 3.2.** *Let  $\Omega$  be bounded and let  $a, \tau_0, f$  satisfy the conditions of Lemma 3.1. Then problem (3.3)–(3.5) possesses a solution  $u \in W, \tau \in W_M$ .*

*Proof.* Introduce auxiliary operators by the following formulas (in these formulas  $\varphi$  and  $\Phi$  are arbitrary elements of  $V$  and  $H_0^1(\Omega, \mathbb{R}_S^{n \times n})$ , resp.):

$$\begin{aligned}
 N_1 : W_M &\longrightarrow L_2(0, T; V^*), & \langle N_1(\tau), \varphi \rangle &= (\tau, \nabla \varphi), \\
 N_2 : W &\longrightarrow L_2(0, T; H^{-1}), & \langle N_2(u), \Phi \rangle &= 2\mu_2(u, \text{Div } \Phi), \\
 K_\delta : W \times W_M &\longrightarrow L_2(0, T; V^*), \\
 \langle K_\delta(u, \tau), \varphi \rangle &= - \sum_{i=1}^n \left( \frac{u_i u}{1 + \delta(|u|^2 + |\tau|^2/2\mu_2)}, \frac{\partial \varphi}{\partial x_i} \right), \\
 \tilde{K}_\delta : W \times W_M &\longrightarrow L_2(0, T; H^{-1}), \\
 \langle \tilde{K}_\delta(u, \tau), \Phi \rangle &= - \sum_{i=1}^n \left( \frac{u_i \tau}{1 + \delta(|u|^2 + |\tau|^2/2\mu_2)}, \frac{\partial \Phi}{\partial x_i} \right), \\
 A : W &\longrightarrow L_2(0, T; V^*), & (3.23) \\
 \langle A(u), \varphi \rangle &= \mu_1(\nabla u, \nabla \varphi), \\
 A_\varepsilon : W_M &\longrightarrow L_2(0, T; H^{-1}), \\
 \langle A_\varepsilon(\tau), \Phi \rangle &= \varepsilon(\nabla \tau, \nabla \Phi) + \frac{1}{\lambda_1}(\tau, \Phi), \\
 \tilde{A} : W \times W_M &\longrightarrow L_2(0, T; V^*) \times L_2(0, T; H^{-1}) \times H \times L_2, \\
 \tilde{A}(u, \tau) &= \left( \frac{du}{dt} + A(u), \frac{d\tau}{dt} + A_\varepsilon(\tau), u|_{t=0}, \tau|_{t=0} \right), \\
 Q : W \times W_M &\longrightarrow L_2(0, T; V^*) \times L_2(0, T; H^{-1}) \times H \times L_2, \\
 Q(u, \tau) &= (K_\delta(u, \tau) + N_1(\tau), \tilde{K}_\delta(u, \tau) + N_2(u), 0, 0).
 \end{aligned}$$

Then problem (3.3)–(3.5) is equivalent to the operator equation

$$\tilde{A}(u, \tau) + \xi Q(u, \tau) = (f, 0, a, \tau_0). \tag{3.24}$$

The operator  $\tilde{A}$  is invertible by [3, Chapter VI, Theorem 1.1]. Moreover, since the embeddings  $W$  into  $L_2(0, T; H)$  and  $W_M$  into  $L_2(0, T; L_2)$  are compact, the operators  $N_1, N_2$  are compact. The operators  $K_\delta$  and  $\tilde{K}_\delta$  are compact, which may be shown as in [2, Theorem 2.2]. Hence, the operator  $Q$  is also compact.

Rewrite (3.24) as

$$(u, \tau) - \xi \tilde{A}^{-1} Q(u, \tau) = \tilde{A}^{-1}(f, 0, a, \tau_0). \tag{3.25}$$

By Lemma 3.1 equation (3.25) has no solutions on the boundary of a sufficiently large ball  $B$  in  $W \times W_M$ , independent on  $\xi, \delta, \varepsilon$ . Without loss of generality,  $a_0 = \tilde{A}^{-1}(f, 0, a, \tau_0)$  belongs to this ball. Then  $\text{deg}_{\text{LS}}(I - \xi \tilde{A}^{-1} Q, B, a_0)$ , the Leray-Schauder degree of the map  $I - \xi \tilde{A}^{-1} Q$  on the ball  $B$  with respect to the point  $a_0$ , is defined, where  $I$  is the identity



operator. By the homotopic invariance property of the degree we have

$$\text{deg}_{\text{LS}}(I - \xi \tilde{A}^{-1}Q, B, a_0) = \text{deg}_{\text{LS}}(I, B, a_0) = 1. \tag{3.26}$$

Hence, (3.25) (and therefore, problem (3.3)–(3.5)) has a solution in  $B$  for every  $\xi$ .  $\square$

We need the following estimates on the time derivatives of the solutions of problem (3.3)–(3.5).

LEMMA 3.3. *In the conditions of the previous theorem the following estimates of the solutions are valid:*

$$\left\| \frac{du}{dt} \right\|_{L_1(0,T;V^*)} \leq K_1(\|a\|, \|\tau_0\|, \|f\|_{L_2(0,T;V^*)}), \tag{3.27}$$

$$\left\| \frac{d\tau}{dt} \right\|_{L_1(0,T;H^{-1})} \leq K_2(\|a\|, \|\tau_0\|, \|f\|_{L_2(0,T;V^*)}, \varepsilon), \tag{3.28}$$

where  $K_1, K_2$  do not depend on  $\Omega, \delta, \xi$ , and  $K_1$  also does not depend on  $\varepsilon$ .

*Proof.* From (3.4) we have

$$\int_0^T \left| \frac{d}{dt}(u, \varphi) \right| dt \leq \int_0^T \left| \xi \sum_{i=1}^n \left( \frac{u_i u}{1 + \delta(|\tau|^2/2\mu_2 + |u|^2)}, \frac{\partial \varphi}{\partial x_i} \right) \right| + |\mu_1(\nabla u, \nabla \varphi)| + |\xi(\tau, \nabla \varphi)| + |\langle f, \varphi \rangle| dt. \tag{3.29}$$

Applying Hölder’s inequality and estimate (3.7), and taking into account the embedding  $V \subset L_4$  (provided  $N = 2, 3$ ) and the inequality  $\xi/(1 + \delta(|\tau|^2/2\mu_2 + |u|^2)) \leq 1$ , we see that the right part does not exceed

$$\begin{aligned} & \|\varphi\|_V \int_0^T (\|u(t)\|_{L_4}^2 + \mu_1\|u(t)\|_1 + \|\tau(t)\| + \|f(t)\|_{V^*}) dt \\ & \leq C\|\varphi\|_V (1 + \|u\|_{L_2(0,T;V)}^2 + \|\tau\|_{L_\infty(0,T;L_2)} + \|f\|_{L_2(0,T;V^*)}) \\ & \leq K_1(\|a\|, \|\tau_0\|, \|f\|_{L_2(0,T;V^*)}) \|\varphi\|_V. \end{aligned} \tag{3.30}$$

These estimates and (3.12) yield (3.27).

Analogously, from (3.3), one obtains (3.28).  $\square$

#### 4. Passage to the limit

Consider one more auxiliary system

$$\begin{aligned} \frac{d}{dt}(\tau, \Phi) + \frac{1}{\lambda_1}(\tau, \Phi) - \sum_{i=1}^n \left( u_i \tau, \frac{\partial \Phi}{\partial x_i} \right) + 2\mu_2(u, \text{Div } \Phi) + \frac{\varepsilon}{\lambda_1}(\nabla \tau, \nabla \Phi) &= 0, \\ \frac{d}{dt}(u, \varphi) - \sum_{i=1}^n \left( u_i u, \frac{\partial \varphi}{\partial x_i} \right) + \mu_1(\nabla u, \nabla \varphi) + (\tau, \nabla \varphi) &= \langle f, \varphi \rangle, \end{aligned} \tag{4.1}$$

for all  $\varphi \in V, \Phi \in H_0^1$  a.e. in  $(0, T), 0 < \varepsilon \leq 1$ .

THEOREM 4.1. Let  $\Omega, a, \tau_0, f$  satisfy the conditions of [Theorem 3.2](#). Then problem (4.1), (3.5) possesses a solution in the class

$$\begin{aligned} u &\in L_2(0, T; V), & \tau &\in L_2(0, T; H_0^1), \\ \frac{du}{dt} &\in L_1(0, T; V^*), & \frac{d\tau}{dt} &\in L_1(0, T; H^{-1}), \end{aligned} \tag{4.2}$$

which satisfies estimates (3.27), (3.28), and

$$\begin{aligned} \text{vrai max}_{t \in [0, T]} (\|u\|(t) + \|\tau\|(t)) + \int_0^T \|u\|_1^2(t) dt + \varepsilon \int_0^T \|\tau\|_1^2(t) dt \\ \leq K_0 (\|a\|, \|\tau_0\|, \|f\|_{L_2(0, T; V^*)}). \end{aligned} \tag{4.3}$$

*Proof.* Consider problems (3.3)–(3.5) with  $\xi = 1$  and  $\delta = 1/m, m = 1, 2, \dots$ . By [Theorem 3.2](#) there exist solutions  $(u_m, \tau_m)$  of these problems. Taking into account estimate (3.7), without loss of generality, we may assume that

$$\begin{aligned} u_m &\rightharpoonup u_* \quad \text{weakly in } L_2(0, T; V), \\ u_m &\rightharpoonup u_* \quad \text{* -weakly in } L_\infty(0, T; H), \\ \tau_m &\rightharpoonup \tau_* \quad \text{weakly in } L_2(0, T; H_0^1), \\ \tau_m &\rightharpoonup \tau_* \quad \text{* -weakly in } L_\infty(0, T; L_2), \end{aligned} \tag{4.4}$$

as  $m \rightarrow \infty$ .

By [Lemma 3.3](#) the sequence  $\{du_m/dt\}$  is bounded in  $L_1(0, T; V^*)$ , and the sequence  $\{d\tau_m/dt\}$  is bounded in  $L_1(0, T; H^{-1})$ . Then, by the compactness theorem (see [10, Chapter III, Theorem 2.3]),

$$\begin{aligned} u_m &\longrightarrow u_* \quad \text{strongly in } L_2(0, T; H), \\ \tau_m &\longrightarrow \tau_* \quad \text{strongly in } L_2(0, T; L_2), \end{aligned} \tag{4.5}$$

and we may assume that

$$\begin{aligned} u_m(t)(x) &\longrightarrow u_*(t)(x) \quad \text{a.e. in } (0, T) \times \Omega, \\ \tau_m(t)(x) &\longrightarrow \tau_*(t)(x) \quad \text{a.e. in } (0, T) \times \Omega. \end{aligned} \tag{4.6}$$

It is obvious that estimate (4.3) is valid for  $(u_*, \tau_*)$ .

Substitute  $(u_m, \tau_m)$  in equalities (3.3) and (3.4) with  $\delta = 1/m, \xi = 1$ . Taking the scalar product of these equalities in  $L_2(0, T)$  with a smooth scalar function  $\psi(t), \psi(T) = 0$ , and

integrating by parts the first terms, we obtain

$$\begin{aligned}
 & - \int_0^T (\tau_m, \Phi \psi'(t)) dt + \int_0^T \left( \frac{1}{\lambda_1} (\tau_m, \psi \Phi) - \sum_{i=1}^n \left( \frac{(u_m)_i \tau_m}{1 + (1/m)(|\tau_m|^2/2\mu_2 + |u_m|^2)}, \psi \frac{\partial \Phi}{\partial x_i} \right) \right. \\
 & \qquad \qquad \qquad \left. + 2\mu_2 (u_m, \psi \operatorname{Div} \Phi) + \frac{\varepsilon}{\lambda_1} (\nabla \tau_m, \psi \nabla \Phi) \right) dt = (\tau_0, \Phi) \psi(0), \\
 & - \int_0^T (u_m, \varphi \psi'(t)) dt \\
 & + \int_0^T \left( \mu_1 (\nabla u_m, \psi \nabla \varphi) - \sum_{i=1}^n \left( \frac{(u_m)_i u_m}{1 + (1/m)(|\tau_m|^2/2\mu_2 + |u_m|^2)}, \psi \frac{\partial \Phi}{\partial x_i} \right) + (\tau_m, \psi \nabla \varphi) \right) dt \\
 & = \int_0^T \langle f, \varphi \psi \rangle dt + (a, \varphi) \psi(0).
 \end{aligned} \tag{4.7}$$

As in [2, page 42], we may see that

$$\begin{aligned}
 & \int_0^T \sum_{i=1}^n \left( \frac{(u_m)_i \tau_m}{1 + (1/m)(|\tau_m|^2/2\mu_2 + |u_m|^2)}, \psi \frac{\partial \Phi}{\partial x_i} \right) dt \longrightarrow \int_0^T \sum_{i=1}^n \left( (u_*)_i \tau_*, \psi \frac{\partial \Phi}{\partial x_i} \right) dt, \\
 & \int_0^T \sum_{i=1}^n \left( \frac{(u_m)_i u_m}{1 + (1/m)(|\tau_m|^2/2\mu_2 + |u_m|^2)}, \psi \frac{\partial \varphi}{\partial x_i} \right) dt \longrightarrow \int_0^T \sum_{i=1}^n \left( (u_*)_i u_*, \psi \frac{\partial \varphi}{\partial x_i} \right) dt.
 \end{aligned} \tag{4.8}$$

Passing to the limit in (4.7) as  $m \rightarrow \infty$ , we have

$$\begin{aligned}
 & - \int_0^T (\tau_*, \Phi \psi'(t)) dt + \int_0^T \left( \frac{1}{\lambda_1} (\tau_*, \psi \Phi) - \sum_{i=1}^n \left( (u_*)_i \tau_*, \psi \frac{\partial \Phi}{\partial x_i} \right) \right. \\
 & \qquad \qquad \qquad \left. + 2\mu_2 (u_*, \psi \operatorname{Div} \Phi) + \frac{\varepsilon}{\lambda_1} (\nabla \tau_*, \psi \nabla \Phi) \right) dt = (\tau_0, \Phi) \psi(0), \\
 & - \int_0^T (u_*, \Phi \psi'(t)) dt + \int_0^T \left( \mu_1 (\nabla u_*, \psi \nabla \varphi) - \sum_{i=1}^n \left( (u_*)_i u_*, \psi \frac{\partial \Phi}{\partial x_i} \right) \right. \\
 & \qquad \qquad \qquad \left. + (\tau_*, \psi \nabla \varphi) \right) dt = \int_0^T \langle f, \varphi \psi \rangle dt + (a, \varphi) \psi(0).
 \end{aligned} \tag{4.9}$$

Since it has been carried out, in particular, for every  $\psi \in C_0^\infty(0, T)$ , the function  $(u_*, \tau_*)$  satisfies (4.1) in the sense of distributions on  $(0, T)$ .

Substitute  $(u_*, \tau_*)$  into equalities (4.1). Since all terms in the obtained equalities are integrable on  $(0, T)$ , these equalities are valid a.e. on  $(0, T)$ . Taking the scalar product of these equalities in  $L_2(0, T)$  with a smooth scalar function  $\psi(t)$ ,  $\psi(T) = 0$ ,  $\psi(0) \neq 0$ , and

comparing the result with (4.9), we see that

$$\begin{aligned} (u_*|_{t=0}, \varphi)\psi(0) &= (a, \varphi)\psi(0), \\ (\tau_*|_{t=0}, \Phi)\psi(0) &= (\tau_0, \Phi)\psi(0). \end{aligned} \tag{4.10}$$

Since  $\Phi$  and  $\varphi$  are arbitrary,  $u_*$  and  $\tau_*$  satisfy (3.5). Repeating the proof of Lemma 3.3 with  $\delta = 0, \xi = 1$ , we see that the solutions of problem (4.1), (3.5) satisfy estimates (3.27), (3.28). Thus,  $(u_*, \tau_*)$  is the desirable solution.  $\square$

### 5. Existence of a weak solution for the Jeffreys model and its estimation

First we will prove a statement from which Theorem 2.3 immediately follows.

THEOREM 5.1. *Given  $f \in L_2(0, T; V^*)$ ,  $a \in H$ ,  $\tau_0 \in L_2$ , there exists a pair of functions  $(u, \tau)$ ,*

$$\begin{aligned} u \in L_2(0, T; V) \cap L_\infty(0, T; H) \cap C_w([0, T], H), \quad \frac{du}{dt} \in L_1(0, T; V^*), \\ \tau \in L_\infty(0, T; L_2) \cap C_w([0, T], L_2), \quad \frac{d\tau}{dt} \in L_2(0, T; H^{-2}), \end{aligned} \tag{5.1}$$

satisfying (3.1), (3.2) a.e. in  $(0, T)$ , the initial condition (3.5), and the estimate

$$\begin{aligned} \|u\|_{L_2(0, T; V)} + \|u\|_{L_\infty(0, T; H)} + \left\| \frac{du}{dt} \right\|_{L_1(0, T; V^*)} + \|\tau\|_{L_\infty(0, T; L_2)} + \left\| \frac{d\tau}{dt} \right\|_{L_2(0, T; H^{-2})} \\ \leq K_3(\|a\|, \|\tau_0\|, \|f\|_{L_2(0, T; V^*)}), \end{aligned} \tag{5.2}$$

where  $K_3$  does not depend on  $\Omega$ .

*Proof.* Denote by  $\Omega_m$  the intersection of  $\Omega$  with the ball  $B_m$  of radius  $m$  with the center in the origin in the space  $\mathbb{R}^n$ ,  $m = 1, 2, \dots$ . From the definition of the space  $H$  it follows that there exists a sequence  $a_m \in C_0^\infty(\Omega)$ ,  $\operatorname{div} a_m = 0$ ,  $\operatorname{supp} a_m \subset \Omega_m$ , which converges to  $a$  in  $L_2(\Omega)$ . Without loss of generality,  $\|a_m\| \leq \|a\|$ . Consider, for every  $m$  on  $\Omega_m$ , problem (4.1) with  $\varepsilon = 1/m$  and the following initial condition:

$$u|_{t=0} = a_m, \quad \tau|_{t=0} = \tau_0|_{\Omega_m}. \tag{5.3}$$

By Theorem 4.1 there exists a solution  $(u_m, \tau_m)$  of this problem. Denote by  $\tilde{u}_m$  and  $\tilde{\tau}_m$  the functions which coincide with  $u_m$  and  $\tau_m$ , respectively, in  $\Omega_m$  and are identically zero in  $\Omega \setminus \Omega_m$ .

As in the proof of Theorem 4.1, without loss of generality, we may assume that

$$\begin{aligned} \tilde{u}_m &\rightharpoonup u_* \quad \text{weakly in } L_2(0, T; V), \\ \tilde{u}_m &\rightharpoonup u_* \quad \text{* -weakly in } L_\infty(0, T; H), \\ \tilde{\tau}_m &\rightharpoonup \tau_* \quad \text{* -weakly in } L_\infty(0, T; L_2), \\ \tilde{u}_m|_{\Omega_k} &\rightharpoonup u_*|_{\Omega_k} \quad \text{strongly in } L_2(0, T; L_2(\Omega_k)), \end{aligned} \tag{5.4}$$

for every  $k$  (there is no strong convergence of  $\tilde{\tau}_m$  now, because estimate (3.28) depends on  $\varepsilon$ ). Obviously,  $(u_*, \tau_*)$  satisfies estimate (4.3).

Take arbitrary  $\varphi \in \mathcal{V}$ ,  $\Phi \in C_0^\infty$ . Fix  $k$  large enough such that the supports of  $\varphi$  and  $\Phi$  are contained in  $\Omega_k$ .

Substitute  $(u_m, \tau_m)$  in equalities (4.1) with  $m \geq k$ ,  $\varepsilon = 1/m$ . Take the scalar product of these equalities in  $L_2(0, T)$  with a smooth scalar function  $\psi(t)$ ,  $\psi(T) = 0$ , and integrate by parts the first terms. Because of the choice of  $k$ , we can replace  $u_m$  and  $\tau_m$  in these equalities by  $\tilde{u}_m$  and  $\tilde{\tau}_m$ . We have

$$\begin{aligned}
 & - \int_0^T (\tilde{\tau}_m, \Phi \varphi'(t)) dt + \int_0^T \left( \frac{1}{\lambda_1} (\tilde{\tau}_m, \psi \Phi) - \sum_{i=1}^n \left( (\tilde{u}_m)_i \tilde{\tau}_m, \psi \frac{\partial \Phi}{\partial x_i} \right) \right. \\
 & \qquad \qquad \qquad \left. + 2\mu_2 (\tilde{u}_m, \psi \operatorname{Div} \Phi) + \frac{1}{\lambda_1 n} (\nabla \tilde{\tau}_m, \psi \nabla \Phi) \right) dt = (\tau_0, \Phi) \psi(0), \\
 & - \int_0^T (\tilde{u}_m, \varphi \psi'(t)) dt + \int_0^T \left( \mu_1 (\nabla \tilde{u}_m, \psi \nabla \varphi) - \sum_{i=1}^n \left( (\tilde{u}_m)_i \tilde{u}_m, \psi \frac{\partial \varphi}{\partial x_i} \right) + (\tilde{\tau}_m, \psi \nabla \varphi) \right) dt \\
 & \qquad = \int_0^T \langle f, \varphi \psi \rangle dt + (a_m, \varphi) \psi(0).
 \end{aligned} \tag{5.5}$$

Observe that

$$\begin{aligned}
 \left| \frac{1}{m} \int_0^T (\nabla \tilde{\tau}_m, \psi \nabla \Phi) \right| &= \left| \frac{1}{m} \int_0^T (\tilde{\tau}_m, \psi \Delta \Phi) \right| \\
 &\leq \frac{1}{m} \|\tau_m\|_{L_\infty(0, T; L_2)} \int_0^T \|\psi \Delta \Phi\| \xrightarrow{m \rightarrow \infty} 0,
 \end{aligned} \tag{5.6}$$

where  $\Delta$  is the Laplacian.

Now, we show that

$$\int_0^T \sum_{i=1}^n \left( (\tilde{u}_m)_i \tilde{\tau}_m, \psi \frac{\partial \Phi}{\partial x_i} \right) dt \longrightarrow \int_0^T \sum_{i=1}^n \left( (\tilde{u}_*)_i \tilde{\tau}_*, \psi \frac{\partial \Phi}{\partial x_i} \right) dt, \tag{5.7}$$

$$\int_0^T \sum_{i=1}^n \left( (\tilde{u}_m)_i \tilde{u}_m, \psi \frac{\partial \varphi}{\partial x_i} \right) dt \longrightarrow \int_0^T \sum_{i=1}^n \left( (\tilde{u}_*)_i \tilde{u}_*, \psi \frac{\partial \varphi}{\partial x_i} \right) dt. \tag{5.8}$$

Really, using Hölder’s inequality, we see that

$$\begin{aligned}
 & \left| \int_0^T \sum_{i=1}^n \left( (\tilde{u}_m)_i \tilde{\tau}_m - (\tilde{u}_*)_i \tilde{\tau}_*, \psi \frac{\partial \Phi}{\partial x_i} \right) dt \right| \\
 & \leq \left| \int_0^T \sum_{i=1}^n \left( ((\tilde{u}_m)_i - (\tilde{u}_*)_i) \tilde{\tau}_m, \psi \frac{\partial \Phi}{\partial x_i} \right) dt \right| + \left| \int_0^T \sum_{i=1}^n \left( (\tilde{u}_*)_i (\tilde{\tau}_m - \tilde{\tau}_*), \psi \frac{\partial \Phi}{\partial x_i} \right) dt \right| \\
 & \leq \| (u_m - u_*) |_{\Omega_k} \|_{L_2(0, T; L_2)} \|\tau_m\|_{L_2(0, T; L_2)} \left\| \psi \frac{\partial \Phi}{\partial x_1} \right\|_{L_\infty(0, T; L_\infty)} + \left\langle \tilde{\tau}_m - \tilde{\tau}_*, \sum_{i=1}^n (\tilde{u}_*)_i \psi \frac{\partial \Phi}{\partial x_i} \right\rangle.
 \end{aligned} \tag{5.9}$$

The last brackets denote the action of the functional  $\tilde{\tau}_m - \tilde{\tau}_*$  from  $L_1(0, T; L_2)^* = L_\infty(0, T; L_2)$  on the element  $\sum_{i=0}^n (\tilde{u}_*)_i \psi(\partial\Phi/\partial x_i)$ . Both terms tend to zero, and (5.7) is proved. Similarly one shows (5.8).

Now, let  $m$  tend to infinity in (5.5). We obtain

$$\begin{aligned}
 & - \int_0^T (\tau_*, \Phi \psi'(t)) dt + \int_0^T \left( \frac{1}{\lambda_1} (\tau_*, \psi \Phi) - \sum_{i=1}^n \left( (u_*)_i \tau_*, \psi \frac{\partial \Phi}{\partial x_i} \right) + 2\mu_2 (u_*, \psi \operatorname{Div} \Phi) \right) dt \\
 & = (\tau_0, \Phi) \psi(0), \\
 & - \int_0^T (u_*, \varphi \psi'(t)) dt + \int_0^T \left( \mu_1 (\nabla u_*, \psi \nabla \varphi) - \sum_{i=1}^n \left( (u_*)_i u_*, \psi \frac{\partial \varphi}{\partial x_i} \right) + (\tau_*, \psi \nabla \varphi) \right) dt \\
 & = \int_0^T \langle f, \varphi \psi \rangle dt + (a, \varphi) \psi(0).
 \end{aligned} \tag{5.10}$$

As in the proof of Theorem 4.1, it follows from here that  $(u_*, \tau_*)$  is a solution of (3.1), (3.2), (3.5) and  $u_*$  satisfies estimate (3.27). To prove (5.2), it remains to estimate the fifth term in its left part.

Substitute  $(u_*, \tau_*)$  in (3.1). Using Hölder's inequality, the embedding  $V \subset L_4$ , and (4.3), we have

$$\begin{aligned}
 \left\| \frac{d}{dt} (\tau_*, \Phi) \right\|_{L_2(0, T)} & \leq \left\| \frac{1}{\lambda_1} (\tau_*, \Phi) \right\|_{L_2(0, T)} + \left\| \sum_{i=1}^n \left( (u_*)_i \tau_*, \frac{\partial \Phi}{\partial x_i} \right) \right\|_{L_2(0, T)} \\
 & \quad + \|2\mu_2 (u_*, \operatorname{Div} \Phi)\|_{L_2(0, T)} \\
 & \leq \frac{1}{\lambda_1} \|\tau_*\|_{L_2(0, T; L_2)} \|\Phi\|_{L_2} + \|u_*\|_{L_2(0, T; L_4)} \|\tau_*\|_{L_\infty(0, T; L_2)} \left\| \frac{\partial \Phi}{\partial x_i} \right\|_{L_4} \\
 & \quad + 2\mu_2 \|u_*\|_{L_2(0, T; L_2)} \|\operatorname{Div} \Phi\|_{L_2} \\
 & \leq C \|\Phi\|_{H_0^2} \left( \|\tau_*\|_{L_\infty(0, T; L_2)} + \|u_*\|_{L_2(0, T; V)} \|\tau_*\|_{L_\infty(0, T; L_2)} + \|u_*\|_{L_2(0, T; V)} \right) \\
 & \leq K_4 (\|a\|, \|\tau_0\|, \|f\|_{L_2(0, T; V^*)}) \|\Phi\|_{H_0^2}.
 \end{aligned} \tag{5.11}$$

Taking into account (3.12), we obtain the desirable estimate.

By [10, Chapter III, Lemma 1.1] a.e. in  $(0, T)$   $u_*$  is equal to a continuous function on  $[0, T]$  with values in  $V^*$ . Without loss of generality we may assume that  $u_*$  itself is continuous on  $[0, T]$  with values in this space. And since it belongs to  $L_\infty(0, T; H)$ , it is weakly continuous on  $[0, T]$  with values in  $H$  (see [10, Chapter III, Lemma 1.4]). Similarly  $\tau \in C_w([0, T; L_2])$ . The proof of the theorem is complete.  $\square$

*Remark 5.2.* By the same scheme it is easy to show that in Theorems 3.2 and 4.1 the condition of boundedness of  $\Omega$  is not necessary.

*Proof of Theorem 2.3.* Take  $\tau_0 = \sigma_0 - 2\mu_1 \mathcal{E}(a)$ . By the condition of the theorem  $\tau_0 \in L_2$ . By Theorem 5.1 there exists a solution  $(u, \tau)$  of problem (3.1), (3.2), (3.5) in class (5.1).

It is easy to see that  $\mathcal{E}(u) \in L_2(0, T; L_2) \cap C_w([0, T]; H^{-1})$ .

Take  $\sigma = \tau + 2\mu_1 \mathcal{E}(u)$ . Then the pair  $(u, \sigma)$  belongs to class (2.10) and satisfies (2.5), (2.11), that is, it is a weak solution of problem (2.1)–(2.5).  $\square$

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