A MASSERA-TYPE CRITERION FOR ALMOST PERIODIC SOLUTIONS OF HIGHER-ORDER DELAY OR ADVANCE ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS

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Let u be a given bounded uniformly continuous mild solution of a higher-order abstract functional differential equation of delay or advance type. We give a so-called Massera-type criterion for the existence of a mild solution, which is a "spectral component" of u with spectrum similar to the one of the forcing term f. Various spectral criteria for the existence of almost periodic and quasiperiodic mild solutions are given.

1. Introduction

The asymptotic behavior of solutions of differential equations in Banach spaces has been extensively studied since the last two decades. One of the interesting topics in this study is to find conditions for the existence of almost periodic solutions. We refer the reader to [1, 2, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] for more information.

The equation which we consider in this paper is of the form

$$A_0 u(t) + \sum_{k=1}^{n} A_k u^{(k)}(t) + \sum_{l=0}^{m} \int_a^b dB_l(\eta) u^{(l)}(t+\eta) = f(t), \quad t \in \mathbb{R}, \qquad (1.1)$$

where A_0 is a closed linear operator acting on a Banach space $\mathbb{X}$; $A_k \in L(\mathbb{X})$, for all $k = 1,\dots,n$; $B_l \in \mathrm{BV}([a,b],L(\mathbb{X}))$, for all $l = 0,\dots,m$; f is an $\mathbb{X}$-valued almost periodic function on $\mathbb{R}$.

The problem with which we are concerned is to find (spectral) conditions for the existence of an almost periodic mild solution of (1.1) with the same structure of spectrum as the one of f. Such conditions are commonly referred to as Massera-type criteria for the existence of solutions with special structure of spectrum (see, e.g., [7, 9, 16]).

There have been several methods so far to deal with this problem for different kinds of equations. Our method of study is to use the one in [9] combined with the notion of almost periodic spectrum of bounded functions. As a result, we obtain a new Massera-type criterion for the existence of almost periodic mild solutions (see Corollary 3.7) which improves several known ones in [4, 7, 9].

2. Preliminaries

2.1. Notation. Throughout the paper, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{X}$ stand for the sets of real, complex numbers, and a complex Banach space, respectively; $L(\mathbb{X})$, $BC(\mathbb{R}, \mathbb{X})$, $BUC(\mathbb{R}, \mathbb{X})$, $AP(\mathbb{X})$ denote the spaces of linear bounded operators on $\mathbb{X}$, all $\mathbb{X}$-valued bounded continuous, bounded uniformly continuous, and almost periodic functions in Bohr's sense (see [6, page 4]) with sup-norm, respectively; $\mathcal{S}$ denotes the Schwartz space of all functions $\varphi \in C^\infty(\mathbb{R})$ such that each derivative of φ decays faster than any polynomial. By $BV([a,b], L(\mathbb{X}))$, we denote the space of all functions from $[a,b]$ into $L(\mathbb{X})$ which are of bounded variation. For a linear operator A, we denote by $D(A)$, $\sigma(A)$ the domain and the spectrum of A. Let $(S(t))_{t \in \mathbb{R}}$ be the translation group on $BUC(\mathbb{R}, \mathbb{X})$ given by

$$(S(t)u)(s) := u(t+s), \quad \forall s, t \in \mathbb{R}, \ \forall u \in BUC(\mathbb{R}, \mathbb{X}), \tag{2.1}$$

whose infinitesimal generator is $\mathcal{D} := d/dt$, defined on $D(\mathcal{D}) := BUC^1(\mathbb{R}, \mathbb{X})$ which consists of all functions $f \in BUC(\mathbb{R}, \mathbb{X})$ such that the derivative f' exists as an element of $BUC(\mathbb{R}, \mathbb{X})$. Similarly, we define the space $BUC^k(\mathbb{R}, \mathbb{X})$ for every natural number k.

2.2. Spectral theory of functions. In this paper, $sp(u)$ stands for the Beurling spectrum of a given bounded uniformly continuous function u, which is defined by

$$sp(u) := \{\xi \in \mathbb{R} : \forall \varepsilon > 0, \ \exists \varphi \in L^1(\mathbb{R}) : \operatorname{supp} \tilde{\varphi} \subset (\xi - \varepsilon, \xi + \varepsilon), \ \varphi * u \neq 0\}, \tag{2.2}$$

where

$$\tilde{\varphi}(s) := \int_{-\infty}^{\infty} e^{-ist} f(t)\,dt, \qquad \varphi * u(s) := \int_{-\infty}^{\infty} \varphi(s-t)u(t)\,dt. \tag{2.3}$$

In the case of $u \in BUC(\mathbb{R}, \mathbb{X})$, the Beurling spectrum coincides with the Carleman spectrum which consists of all $\xi \in \mathbb{R}$ such that the Carleman-Fourier transform of u, defined by

$$\hat{u}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t} u(t)\,dt & (\operatorname{Re}\lambda > 0), \\ -\int_0^\infty e^{\lambda t} u(-t)\,dt & (\operatorname{Re}\lambda < 0), \end{cases} \tag{2.4}$$

has no holomorphic extension to any neighborhood of $i\xi$ (see [11, Proposition 0.5, page 22]). In turn, the Beurling spectrum of u coincides with its Arveson spectrum, defined by (see [1, Section 2])

$$isp(u) = \sigma(\mathcal{D}_u), \tag{2.5}$$

where $\mathcal{D}_u$ is the infinitesimal generator of the restriction of the translation group $(S(t)|_{\mathcal{M}_u})_{t \in \mathbb{R}}$ to the closed subspace $\mathcal{M}_u := \overline{\operatorname{span}}\{S(\tau)u, \tau \in \mathbb{R}\}$.

Below we list some properties of the spectra of functions which we will need in the sequel.

PROPOSITION 2.1. *Let* $u, u_n, v \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$ *such that* $\lim_{n\to\infty} \|u_n - u\| = 0$, $\psi \in \mathcal{S}$, *and* $\Lambda \subset \mathbb{R}$. *Then*

 (i) $\mathrm{sp}(u)$ *is closed;*
 (ii) $\mathrm{sp}(u + v) \subset \mathrm{sp}(u) \cup \mathrm{sp}(v)$;
 (iii) $\mathrm{sp}(\psi * u) \subset \mathrm{sp}(u) \cap \mathrm{supp}\,\tilde{\psi}$;
 (iv) $\mathrm{sp}(u - \psi * u) \subset \mathrm{sp}(u) \cap \mathrm{supp}(1 - \tilde{\psi})$;
 (v) *if* $\tilde{\psi} \equiv 1$ *on a neighbourhood of* $\mathrm{sp}(u)$ *then* $\psi * u = u$;
 (vi) *if* $\mathrm{sp}(u) \cap \mathrm{supp}\,\tilde{\psi} = \varnothing$, *then* $\psi * u = 0$;
 (vii) *if* $\mathrm{sp}(u_n) \subset \Lambda$, *for all n, then* $\mathrm{sp}(u) \subset \overline{\Lambda}$;
(viii) *if* $\mathrm{sp}(u)$ *is countable, and* $\mathbb{X}$ *does not contain any subspace which is isomorphic to the space of numerical null-sequences* c_0, *then u is almost periodic;*
 (ix) *if u is uniformly continuous and* $\mathrm{sp}(u)$ *is discrete, then u is almost periodic.*

Proof. We refer the reader to [11, Propositions 0.4, 0.6, Theorem 0.8, pages 20–25] and [6, Chapter 6] for the proofs of (i)–(viii) and [3] for (ix). □

Remark 2.2. The condition $\mathbb{X} \not\supset c_0$ in Proposition 2.1 (viii) can be replaced by one of the following conditions:

 (a) $u(\mathbb{R})$ is relatively weakly compact in $\mathbb{X}$;
 (b) u is totally ergodic, that is, the limit

$$M_\eta u = \lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} e^{i\eta s} S(s) u\, ds \qquad (2.6)$$

exists in $\mathrm{BUC}(\mathbb{R}, \mathbb{X})$, for all $\eta \in \mathbb{R}$.

For more information, we refer the reader to [6, page 92], [1, Section 3], and [15].

Now we recall that the almost periodic spectrum of a bounded uniformly continuous function u, (which will be called AP-*spectrum* for the sake of simplicity) is defined by

$$\mathrm{sp}_{\mathrm{AP}}(u) := \{\xi \in \mathbb{R} : \forall \varepsilon > 0, \ \exists \varphi \in L^1(R) : \ \mathrm{supp}\,\tilde{\varphi} \subset (\xi - \varepsilon, \xi + \varepsilon), \ \varphi * u \notin \mathrm{AP}(\mathbb{X})\}. \qquad (2.7)$$

To proceed, we recall some known facts in [1] about the notion of AP-spectrum. Let $Y := \mathrm{BUC}(\mathbb{R}, \mathbb{X})/\mathrm{AP}(\mathbb{X})$ and let $\pi : \mathrm{BUC}(\mathbb{R}, \mathbb{X}) \to Y$ be the quotient map. Since $\mathrm{AP}(\mathbb{X})$ is invariant with respect to the group $(S(t))_{t\in\mathbb{R}}$, $(S(t))_{t\in\mathbb{R}}$ induces a strongly continuous group on Y, denoted by $(\bar{S}(t))_{t\in\mathbb{R}}$, which is defined by the formula

$$\bar{S}(t)\pi(u) := \pi(S(t)u), \quad \forall t \in \mathbb{R}, \ \forall u \in \mathrm{BUC}(\mathbb{R}, \mathbb{X}). \qquad (2.8)$$

Let $\tilde{\mathscr{D}}$ denote the infinitesimal generator of the group $(\bar{S}(t))_{t\in\mathbb{R}}$ (see [2]).

Similarly to the Beurling spectrum of a function u, its AP-spectrum coincides with the Arveson spectrum of $\pi(u)$, that is,

$$\mathrm{isp}_{\mathrm{AP}}(u) = \sigma(\tilde{\mathscr{D}}_{\pi(u)}), \qquad (2.9)$$

where $\tilde{\mathcal{D}}_{\pi(u)}$ is the infinitesimal generator of the restriction of the translation group $(\bar{S}(t)|_{\mathcal{M}_{\pi(u)}})_{t\in\mathbb{R}}$ to the subspace $\mathcal{M}_{\pi(u)} := \overline{\mathrm{span}\{\bar{S}(\tau)\pi(u), \tau \in \mathbb{R}\}}$ (see [1] for details).

To verify when a point belongs to $\mathrm{sp}_{\mathrm{AP}}(u)$, we may use [1, Section 2, Lemma 2.2], which can be stated in our terminology as follows.

LEMMA 2.3. *Let $\pi(u) \in Y$ and $\eta \in \mathbb{R}$. Assume that $R(\lambda, \tilde{\mathcal{D}})\pi(u)(\mathrm{Re}\,\lambda > 0)$ has a holomorphic extension to a neighborhood of $i\eta$. Then $i\eta \notin \sigma(\tilde{\mathcal{D}}_{\pi(u)})$, and equivalently $\eta \notin \mathrm{sp}_{\mathrm{AP}}(u)$.*

We list below some main properties of AP-spectra of functions.

PROPOSITION 2.4. *Let $u, v, u_n \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$ such that $\lim_{n\to\infty} \|u_n - u\| = 0$, $\psi \in \mathcal{S}$, and Λ is a subset of $\mathbb{R}$. Then*

(i) $\mathrm{sp}_{\mathrm{AP}}(u+v) \subset \mathrm{sp}_{\mathrm{AP}}(u) \cup \mathrm{sp}_{\mathrm{AP}}(v)$;
(ii) $\mathrm{sp}_{\mathrm{AP}}(\psi * u) \subset \mathrm{sp}_{\mathrm{AP}}(u) \cap \mathrm{supp}\,\tilde{\psi}$;
(iii) $\mathrm{sp}_{\mathrm{AP}}(u - \psi * u) \subset \mathrm{sp}_{\mathrm{AP}}(u) \cap \mathrm{supp}(1 - \tilde{\psi})$;
(iv) *if* $\mathrm{sp}_{\mathrm{AP}}(u_n) \subset \Lambda$, *for all n, then* $\mathrm{sp}_{\mathrm{AP}}(u) \subset \overline{\Lambda}$;
(v) *if* $\mathrm{sp}_{\mathrm{AP}}(u)$ *is countable, and $\mathbb{X}$ does not contain c_0, then u is almost periodic.*

Proof. (i)-(ii) Using two basic properties of almost periodic functions: $u + v \in \mathrm{AP}(\mathbb{X})$ and $\psi * u \in \mathrm{AP}(\mathbb{X})$ whenever $u, v \in \mathrm{AP}(\mathbb{X})$, $\psi \in \mathcal{S}$, from the definition of AP-spectrum, we can verify that (i) holds and

$$\mathrm{sp}_{\mathrm{AP}}(\psi * u) \subset \mathrm{sp}_{\mathrm{AP}}(u). \tag{2.10}$$

It remains to prove that $\mathrm{sp}_{\mathrm{AP}}(\psi * u) \subset \mathrm{supp}\,\tilde{\psi}$. In fact, by Proposition 2.1(iii), we have

$$\mathrm{sp}_{\mathrm{AP}}(\psi * u) \subset \mathrm{sp}(\psi * u) \subset \mathrm{supp}\,\tilde{\psi}, \tag{2.11}$$

thus proving (ii).

(iii) By Proposition 2.1(iv), Proposition 2.4(i), and (2.10), we have

$$\mathrm{sp}_{\mathrm{AP}}(u - \psi * u) \subset \mathrm{sp}_{\mathrm{AP}}(u) \cup \mathrm{sp}_{\mathrm{AP}}(\psi * u) \subset \mathrm{sp}_{\mathrm{AP}}(u),$$
$$\mathrm{sp}_{\mathrm{AP}}(u - \psi * u) \subset \mathrm{sp}(u - \psi * u) \subset \mathrm{supp}(1 - \tilde{\psi}). \tag{2.12}$$

Hence, $\mathrm{sp}_{\mathrm{AP}}(u - \psi * u) \subset \mathrm{sp}_{\mathrm{AP}}(u) \cap \mathrm{supp}(1 - \tilde{\psi})$.

(iv) First, we will prove that $\mathrm{sp}_{\mathrm{AP}}(f) \subset \overline{\Lambda}$ if and only if $\varphi * f \in \mathrm{AP}(\mathbb{X})$ whenever $\varphi \in L^1(\mathbb{R})$ and $\mathrm{supp}\,\tilde{\varphi} \cap \overline{\Lambda} = \varnothing$. In fact, in what follows, we prove the "if" and "only if" parts.

"*If*" part. Suppose $\xi \notin \overline{\Lambda}$. Then, we show that $\xi \notin \mathrm{sp}_{\mathrm{AP}}(f)$. Since $\xi \notin \overline{\Lambda}$, there exists $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \cap \overline{\Lambda} = \varnothing$. Therefore, by the assumption that for all $\varphi \in L^1(\mathbb{R})$ such that $\mathrm{supp}\,\tilde{\varphi} \subset (\xi - \varepsilon, \xi + \varepsilon)$, we have $\varphi * f \notin \mathrm{AP}(\mathbb{X})$, we get $\xi \notin \mathrm{sp}_{\mathrm{AP}}(f)$. This proves the "if" part.

"*Only if*" part. Suppose that $\mathrm{sp}_{\mathrm{AP}}(f) \subset \overline{\Lambda}$. We now show that for all $\varphi \in L^1(\mathbb{R}) : \mathrm{supp}\,\tilde{\varphi} \subset (\xi - \varepsilon, \xi + \varepsilon)$, $\varphi * f \in \mathrm{AP}(\mathbb{X})$. Indeed, we have

$$\mathrm{sp}_{\mathrm{AP}}(\varphi * f) \subset \mathrm{sp}_{\mathrm{AP}}(f) \cap \mathrm{supp}\,\tilde{\varphi} \subset \overline{\Lambda} \cap \mathrm{supp}\,\tilde{\varphi} = \varnothing. \tag{2.13}$$

Hence, $\varphi * f \in \mathrm{AP}(\mathbb{X})$.

Now we prove (iv). From the above claim and by assumption, we have $\varphi * u_n \in \mathrm{AP}(\mathbb{X})$ whenever $\varphi \in L^1(\mathbb{R})$ and $\operatorname{supp} \tilde{\varphi} \cap \bar{\Lambda} = \varnothing$. On the other hand, since $\lim_{n \to \infty} \|u_n - u\| = 0$, it follows that

$$\mathrm{AP}(\mathbb{X}) \ni \varphi * u_n \longrightarrow \varphi * u, \tag{2.14}$$

as $n \to \infty$. Therefore, as $\mathrm{AP}(\mathbb{X})$ is closed, $\varphi * u \in \mathrm{AP}(\mathbb{X})$, that is, $\operatorname{sp}_{\mathrm{AP}}(u) \subset \bar{\Lambda}$.

(v) For the proof, the reader is referred to [6, page 92]. $\qquad \square$

Remark 2.5. In analogy to Remark 2.2, we can replace the condition $\mathbb{X} \not\supset c_0$ in Proposition 2.4(v) by one of conditions (a) or (b) in Remark 2.2. We refer the reader to [6, page 92], [1, Section 3], and [15] for more information.

Definition 2.6. A function $u \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$ is called a *strong solution* of (1.1) if $u(t) \in D(A_0)$, for all $t \in \mathbb{R}$ and $u \in \mathrm{BUC}^k(\mathbb{R}, \mathbb{X})$, with $k = \max(m, n)$ and (1.1) holds for all $t \in \mathbb{R}$.

A function $u \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$ is a *mild solution* on $\mathbb{R}$ of (1.1), if, for each $\psi \in L^1(\mathbb{R})$ such that its Fourier transform $\tilde{\psi}$ has compact support, the function $\psi * u$ is a strong solution of (1.1) with f replaced by $\psi * f$.

Remark 2.7. (i) By the superposition principle, if u is a mild solution of (1.1) and $\psi \in L^1(\mathbb{R})$ has $\operatorname{supp} \tilde{\psi}$ compact, then $u - \psi * u$ is also a mild solution of (1.1) with f replaced by $f - \psi * f$.

(ii) In the case where A_0 generates a strongly continuous semigroup of linear operators and $A_1 = -I$, $n = 1$, $m = 0$, the above-defined notion of mild solutions coincides with the usual one of mild solutions to inhomogeneous linear evolution equations, that is, bounded continuous solutions to the integral equation

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\xi)f(\xi)d\xi, \quad \forall t \geq s; \ t, s \in \mathbb{R}, \tag{2.15}$$

where $(T(t))_{t \geq 0}$ is the semigroup generated by A_0. In the light of this, the above-defined notion of mild solutions can be regarded as a generalization of the usual one. There are several generalizations which can be found in [8, 10, 11].

3. Main results

We begin this section with the notion of spectrum of (1.1), which is described as follows.

Definition 3.1. For every $\lambda \in \mathbb{C}$, define an operator

$$R(\lambda) := A_0 + \sum_{k=1}^n \lambda^k A_k + \sum_{l=0}^m \lambda^l \int_a^b e^{\lambda \eta} dB_l(\eta) \tag{3.1}$$

on $\mathbb{X}$ with $D(R(\lambda)) = D(A_0)$, which is called *a spectral operator* of (1.1). The set

$$\Sigma := \{\lambda \in \mathbb{C} : R(\lambda) \text{ is not invertible}\} \tag{3.2}$$

is called *the spectrum* of (1.1).

In the case where (1.1) is of the form $\dot{x} = A_0 x - f$, here A_0 is a linear closed operator, the spectral operator and the spectrum of (1.1), respectively, are

$$R(\lambda) = \lambda - A_0, \qquad \Sigma = \sigma(A_0). \tag{3.3}$$

As is well known, Σ is a closed subset of $\mathbb{C}$ and $R^{-1}(\lambda)$ is a holomorphic function from $\mathbb{C} \backslash \sigma(A_0)$ to $L(\mathbb{X})$. It is natural to ask if these properties hold true in the general case. The following lemma gives an answer to this question.

LEMMA 3.2. *Let A_0 be a linear closed operator on $\mathbb{X}$, $A_k \in L(\mathbb{X})$, for all $k = 1,\dots,n$, and $B_l \in \mathrm{BV}([a,b],L(\mathbb{X}))$, for all $l = 0,\dots,m$. Then*

  (i) *Σ is a closed set;*
  (ii) *$R^{-1} : \mathbb{C} \backslash \Sigma \to L(\mathbb{X})$ is a holomorphic function.*

Proof. (i) To prove the closedness of Σ, we will show that $\mathbb{C} \backslash \Sigma$ is open. For every $\lambda_0 \in \mathbb{C} \backslash \Sigma$, the operator $R^{-1}(\lambda_0)$ exists as an element of $L(\mathbb{X})$. Since A_0 is closed, $R^{-1}(\lambda_0)$ is a bounded operator by the closed graph theorem.

Setting

$$G(\lambda) := \sum_{k=1}^{n} \lambda^k A_k + \sum_{l=0}^{m} \lambda^l \int_a^b e^{\lambda \eta} dB_l(\eta), \tag{3.4}$$

we have

$$R(\lambda)R^{-1}(\lambda_0) = I + [R(\lambda) - R(\lambda_0)]R^{-1}(\lambda_0) = I + [G(\lambda) - G(\lambda_0)]R^{-1}(\lambda_0). \tag{3.5}$$

It is easily seen that $[G(\lambda) - G(\lambda_0)]R^{-1}(\lambda_0) \in L(\mathbb{X})$, for all $\lambda \in \mathbb{C}$. Moreover, we have the estimate

$$\left\| [G(\lambda) - G(\lambda_0)]R^{-1}(\lambda_0) \right\| \le M \|G(\lambda) - G(\lambda_0)\|$$

$$= M \left\| \sum_{k=1}^{n} (\lambda^k - \lambda_0^k) A_k + \sum_{l=0}^{m} \int_a^b (\lambda^l e^{\lambda \eta} - \lambda_0^l e^{\lambda_0 \eta}) dB_l(\eta) \right\|$$

$$\le M \sum_{k=1}^{n} |\lambda^k - \lambda_0^k| \|A_k\| + M \sum_{l=0}^{m} \int_a^b |\lambda^l e^{\lambda \eta} - \lambda_0^l e^{\lambda_0 \eta}| dV_l(\eta), \tag{3.6}$$

where $M := \|R^{-1}(\lambda_0)\|$; $V_l(\eta) := \mathrm{Var}_a^\eta[B_l]$ is the variation of B_l on the interval $[a, \eta]$. It follows that there exists a neighborhood U_{λ_0} of λ_0 such that $\|[G(\lambda) - G(\lambda_0)]R^{-1}(\lambda_0)\| < 1$, for all $\lambda \in U_{\lambda_0}$. This shows that there exists the bounded inverse operator

$$[I + [G(\lambda) - G(\lambda_0)]R^{-1}(\lambda_0)]^{-1}, \quad \forall \lambda \in U_{\lambda_0}; \tag{3.7}$$

and hence, $R^{-1}(\lambda)$ exists in U_{λ_0} and is given by the following formula:

$$R^{-1}(\lambda) = R^{-1}(\lambda_0)[I + [G(\lambda) - G(\lambda_0)]R^{-1}(\lambda_0)]^{-1}. \tag{3.8}$$

By the definition of Σ, we have $U_{\lambda_0} \subset \mathbb{C} \backslash \Sigma$, so $\mathbb{C} \backslash \Sigma$ is an open subset of $\mathbb{C}$. This completes the proof of (i).

(ii) Assume that $\lambda_0 \in \mathbb{C} \backslash \Sigma$. By (i), we have proved

$$R^{-1}(\lambda) = R^{-1}(\lambda_0)[I + [G(\lambda) - G(\lambda_0)]R^{-1}(\lambda_0)]^{-1}, \quad \forall \lambda \in U_{\lambda_0}. \tag{3.9}$$

It may be noted that $G(\lambda)$ is holomorphic on the whole complex plane. Therefore, $R^{-1} : \mathbb{C} \backslash \Sigma \to L(\mathbb{X})$ is a holomorphic function. $\qquad \square$

Set

$$\Lambda := \{\xi \in \mathbb{R} : R(i\xi) \text{ is not invertible}\}. \tag{3.10}$$

We are going to estimate the spectra of a bounded mild solution of (1.1) which will play an important role in proving the main results of the paper.

LEMMA 3.3. *Assume that A_0 is a linear closed operator on $\mathbb{X}$, $A_k \in L(\mathbb{X})$, for all $k = 1, \dots, n$, and $B_l \in \mathrm{BV}([a, b], L(\mathbb{X}))$, for all $l = 0, \dots, m$. Let $u \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$ be a mild solution of (1.1). Then*

(i) $\mathrm{sp}(u) \subset \Lambda \cup \mathrm{sp}(f)$;
(ii) $\mathrm{sp}_{\mathrm{AP}}(u) \subset \Lambda$ *if $f \in \mathrm{AP}(\mathbb{X})$.*

Proof. We first prove that *if (i) and (ii) hold for every strong solution, then so do they for any mild solution.* In fact, let $u \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$ be a mild solution of (1.1). Let $K_n \in L^1(\mathbb{R})$ be a sequence of functions such that

(i) $\mathrm{supp}\, \tilde{K}_n = [-n, n]$;
(ii) for every $g \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$, if we set $g_n = g * K_n$, then $\lim_{n \to \infty} g_n = g$.

We refer the reader to [6, pages 87-88] for the construction of such a sequence of functions K_n. Since $u \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$ is a mild solution of (1.1), it follows that $K_n * u$ is a strong solution of (1.1) with f replaced by $K_n * f$. By the assumption that (i) holds for strong solutions, we have

$$\begin{aligned}
\mathrm{sp}(K_n * u) &\subset \Lambda \cup \mathrm{sp}(K_n * f) \\
&\subset \Lambda \cup (\mathrm{sp}(f) \cap \mathrm{supp}\, \tilde{K}_n) \\
&\subset \Lambda \cup \mathrm{sp}(f), \quad \forall n = 1, 2, \dots.
\end{aligned} \tag{3.11}$$

This shows that (i) holds for any mild solution.

Now, assuming that (ii) holds for strong solutions, we show that so does (ii) for mild solutions. By definition, it suffices to show that $K_n * f \in AP(\mathbb{X})$. But this is obvious because $K_n \in L^1(\mathbb{R})$ and f is an almost periodic function. Thus,

$$\text{sp}_{\text{AP}}\left(K_n * u\right) \subset \Lambda, \quad \forall n = 1, 2, \ldots. \tag{3.12}$$

Since $K_n * u$ converges uniformly to u and by Propositions 2.1(iv) and 2.4(i), we have

$$\text{sp}(u) \subset \Lambda \cup \text{sp}(f), \qquad \text{sp}_{\text{AP}}(u) \subset \Lambda, \tag{3.13}$$

which proves our claim.

Now, *it remains to prove (i) and (ii) for strong solutions.*

(i) Suppose that u is a strong solution of (1.1). For a fixed $\eta \in \mathbb{R}$, let $v(t) := u(t + \eta)$, $b_l(t) := \int_a^b dB_l(\eta)u^{(l)}(t + \eta)$, for all $l = 0, \ldots, m$, $t \in \mathbb{R}$. Since u is a strong solution, it follows that $u^{(k)}$, $k = 0, \ldots, n$, and $v^{(l)}, b_l, l = 0, \ldots, m$, are bounded functions. Therefore, their Fourier-Carleman transforms exist and can be easily computed. In fact, by induction, we have

$$\widehat{u^{(k)}}(\lambda) = \lambda^k \hat{u}(\lambda) - \sum_{j=0}^{k-1} \lambda^j u^{(k-j-1)}(0), \quad \forall k = 1, \ldots, n. \tag{3.14}$$

Similarly,

$$\begin{aligned}
\widehat{v^{(l)}}(\lambda) &= \lambda^l \hat{v}(\lambda) - \sum_{j=0}^{l-1} \lambda^j v^{(l-j-1)}(0) \\
&= \lambda^l e^{\lambda\eta} \left(\hat{u}(\lambda) - \int_0^\eta e^{-\lambda t} u(t) dt \right) - \sum_{j=0}^{l-1} \lambda^j u^{(l-j-1)}(\eta), \quad \forall l = 1, \ldots, m.
\end{aligned} \tag{3.15}$$

Now, we compute the Fourier-Carleman transforms of b_l, for all $l = 0, \ldots, m$. For $\text{Re}\,\lambda > 0$,

$$\begin{aligned}
\hat{b}_l(\lambda) &= \int_0^\infty e^{-\lambda t} \left(\int_a^b dB_l(\eta) u^{(l)}(t + \eta) \right) dt \\
&= \int_a^b dB_l(\eta) \left(\int_0^\infty e^{-\lambda t} u^{(l)}(t + \eta) dt \right) \\
&= \int_a^b dB_l(\eta) \widehat{v^{(l)}}(\lambda), \quad \forall l = 1, \ldots, m,
\end{aligned} \tag{3.16}$$

and for $\text{Re}\,\lambda < 0$,

$$\begin{aligned}
\hat{b}_l(\lambda) &= -\int_0^\infty e^{\lambda t} \left(\int_a^b dB_l(\eta) u^{(l)}(-t + \eta) \right) dt \\
&= -\int_a^b dB_l(\eta) \left(\int_0^\infty e^{\lambda t} u^{(l)}(-t + \eta) dt \right) \\
&= \int_a^b dB_l(\eta) \widehat{v^{(l)}}(\lambda), \quad \forall l = 1, \ldots, m.
\end{aligned} \tag{3.17}$$

Finally, for $\operatorname{Re}\lambda \neq 0$,

$$
\begin{aligned}
\widehat{b_l}(\lambda) &= \int_a^b dB_l(\eta)\widehat{v^{(l)}}(\lambda) \\
&= \lambda^l \int_a^b e^{\lambda\eta} dB_l(\eta)\hat{u}(\lambda) - \lambda^l \int_a^b dB_l(\eta)\left(e^{\lambda\eta}\int_0^\eta e^{-\lambda t}u(t)dt\right) \\
&\quad - \sum_{j=0}^{l-1}\lambda^j\int_a^b e^{\lambda\eta}dB_l(\eta)u^{(l-j-1)}(\eta), \quad \forall l = 1,\ldots,m.
\end{aligned}
\tag{3.18}
$$

In the case of $l = 0$,

$$
\widehat{b_0}(\lambda) = \int_a^b e^{\lambda\eta}dB_0(\eta)\hat{u}(\lambda) - \int_a^b dB_0(\eta)\left(e^{\lambda\eta}\int_0^\eta e^{-\lambda t}u(t)dt\right).
\tag{3.19}
$$

Taking Fourier-Carleman transform in (1.1), by (3.14), (3.15), (3.18), (3.19), we see that $\hat{u}(\lambda) \in D(A_0)$, for all $\lambda \in \mathbb{C}$, and

$$
\begin{aligned}
A_0\hat{u}(\lambda) &+ \sum_{k=1}^n A_k\left[\lambda^k\hat{u}(\lambda) - \sum_{j=0}^{k-1}\lambda^j u^{(k-j-1)}(0)\right] \\
&+ \left[\int_a^b e^{\lambda\eta}dB_0(\eta)\hat{u}(\lambda) - \int_a^b dB_0(\eta)\left(e^{\lambda\eta}\int_0^\eta e^{-\lambda t}u(t)dt\right)\right] \\
&+ \sum_{l=1}^m\left[\lambda^l\int_a^b e^{\lambda\eta}dB_l(\eta)\hat{u}(\lambda) - \lambda^l\int_a^b dB_l(\eta)\left(e^{\lambda\eta}\int_0^\eta e^{-\lambda t}u(t)dt\right)\right. \\
&\quad \left. - \sum_{j=0}^{l-1}\lambda^j\int_a^b e^{\lambda\eta}dB_l(\eta)u^{(l-j-1)}(\eta)\right] = \hat{f}(\lambda).
\end{aligned}
\tag{3.20}
$$

Therefore,

$$
\begin{aligned}
&\left[A_0 + \sum_{k=1}^n\lambda^k A_k + \sum_{l=0}^m\lambda^l\int_a^b e^{\lambda\eta}dB_l(\eta)\right]\hat{u}(\lambda) - \sum_{l=0}^m\lambda^l\int_a^b dB_l(\eta)\left(e^{\lambda\eta}\int_0^\eta e^{-\lambda t}u(t)dt\right) \\
&- \sum_{k=1}^n\sum_{j=0}^{k-1}\lambda^j u^{(k-j-1)}(0) - \sum_{l=1}^m\sum_{j=0}^{l-1}\lambda^j\int_a^b e^{\lambda\eta}dB_l(\eta)u^{(l-j-1)}(\eta) = \hat{f}(\lambda).
\end{aligned}
\tag{3.21}
$$

Finally, we arrive at

$$
R(\lambda)\hat{u}(\lambda) - p(\lambda) = \hat{f}(\lambda),
\tag{3.22}
$$

where

$$
\begin{aligned}
p(\lambda) &:= \sum_{l=0}^m\lambda^l\int_a^b dB_l(\eta)\left(e^{\lambda\eta}\int_0^\eta e^{-\lambda t}u(t)dt\right) \\
&\quad + \sum_{k=1}^n\sum_{j=0}^{k-1}\lambda^j A_k u^{(k-j-1)}(0) + \sum_{l=1}^m\sum_{j=0}^{l-1}\lambda^j\int_a^b e^{\lambda\eta}dB_l(\eta)u^{(l-j-1)}(\eta).
\end{aligned}
\tag{3.23}
$$

Take $\xi_0 \in \mathbb{R}$ such that $\xi_0 \notin (\Lambda \cup \mathrm{sp}(f))$, that is, $\xi_0 \notin \Lambda$ and $\xi_0 \notin \mathrm{sp}(f)$. Since $\xi_0 \notin \Lambda$, by Lemma 3.2, it follows that $R^{-1}(\lambda)$ exists and is a holomorpic function in a neighborhood of $i\xi_0$. Thus,

$$\hat{u}(\lambda) = R^{-1}(\lambda)\big[\hat{f}(\lambda) + p(\lambda)\big]. \tag{3.24}$$

On the other hand, since $\xi_0 \notin \mathrm{sp}(f)$, $\hat{f}(\lambda)$ has a holomorphic extension to a neighborhood of $i\xi_0$ and $p(\lambda)$ is a holomorphic function. Hence, from (3.24), we see that $\hat{u}(\lambda)$ also has a holomorphic extension to a neighboorhood of $i\xi_0$, that is, $\xi_0 \notin \mathrm{sp}(u)$. This proves (i).

(ii) Suppose that $\xi_0 \in \mathbb{R}$ such that $\xi_0 \notin \Lambda$. We have to prove that $\xi_0 \notin \mathrm{sp}_{\mathrm{AP}}(u)$. To this end, we apply $S(t)$ to both sides of (1.1). Since the derivation operator commutes with shift operators, we obtain

$$\mathcal{A}_0 S(t)u + \sum_{k=1}^{n} \mathcal{A}_k \big(S(t)u\big)^{(k)} + \sum_{l=0}^{m} \int_a^b d\mathcal{B}_l(\eta)\big(S(t+\eta)u\big)^{(l)} = S(t)f, \quad t \in \mathbb{R}, \tag{3.25}$$

where $\mathcal{A}_k,\ k = 0,\dots,n$, and $\mathcal{B}_l(\eta), \eta \in [a,b],\ l = 0,\dots,m$, are multiplication operators defined by the formulas

$$\begin{aligned}
(\mathcal{A}_k u)(t) &= A_k u(t), \quad k = 0,\dots,n,\\
(\mathcal{B}_l(\eta)u)(t) &= B_l(\eta)u(t), \quad \eta \in [a,b],\ l = 0,\dots,m.
\end{aligned} \tag{3.26}$$

It is easily seen that for the chosen $i\xi_0 := \lambda_0$, the operator of multiplication by $R(\lambda)$,

$$\mathcal{R}(\lambda_0) := \mathcal{A}_0 + \sum_{k=1}^{n} \lambda_0^k \mathcal{A}_k + \sum_{l=0}^{m} \lambda_0^l \int_a^b e^{\lambda_0 \eta} d\mathcal{B}_l(\eta), \tag{3.27}$$

is invertible as well. Let $U(t) := S(t)u$, $F(t) := S(t)f$, for all $t \in \mathbb{R}$. We can rewrite (3.25) as follows:

$$\mathcal{A}_0 U(t) + \sum_{k=1}^{n} \mathcal{A}_k U^{(k)}(t) + \sum_{l=0}^{m} \int_a^b d\mathcal{B}_l(\eta) U^{(l)}(t+\eta) = F(t), \quad t \in \mathbb{R}. \tag{3.28}$$

As in the proof of (i), by taking Fourier-Carleman transforms in (3.28), we get

$$\mathcal{R}(\lambda)\hat{U}(\lambda) - q(\lambda) = \hat{F}(\lambda), \tag{3.29}$$

where

$$\begin{aligned}
q(\lambda) := &\sum_{l=0}^{m} \lambda^l \int_a^b d\mathcal{B}_l(\eta)\left(e^{\lambda\eta}\int_0^\eta e^{-\lambda t} U(t)dt\right)\\
&+ \sum_{k=1}^{n}\sum_{j=0}^{k-1} \lambda^j \mathcal{A}_k U^{(k-j-1)}(0) + \sum_{l=1}^{m}\sum_{j=0}^{l-1} \lambda^j \int_a^b e^{\lambda\eta} d\mathcal{B}_l(\eta) U^{(l-j-1)}(\eta).
\end{aligned} \tag{3.30}$$

By Lemma 2.3, it suffices to show that $R(\lambda, \tilde{\mathcal{D}})\pi(u)(\operatorname{Re}\lambda > 0)$ has a holomorphic extension to a neighborhood of $i\xi_0$. In fact, by Lemma 3.2, $R^{-1}(\lambda)$ exists as a holomorpic function in the neighborhood of $i\xi_0 := \lambda_0$. Thus, around λ_0, from (3.29),

$$\hat{U}(\lambda) = \mathscr{R}^{-1}(\lambda)\big[\hat{F}(\lambda) + q(\lambda)\big]. \tag{3.31}$$

For $\operatorname{Re}\lambda > 0$, by integral representation of the resolvent formula, we get

$$\hat{U}(\lambda) = \widehat{(S(\cdot)u)}(\lambda) = \int_0^\infty e^{-\lambda t}S(t)u\,dt = R(\lambda, \mathcal{D})u, \tag{3.32}$$

and similarly, $\hat{F}(\lambda) = \widehat{(S(\cdot)f)}(\lambda) = R(\lambda, \mathcal{D})f$. Hence, around λ_0,

$$R(\lambda, \mathcal{D})u = \mathscr{R}^{-1}(\lambda)R(\lambda, \mathcal{D})f + \mathscr{R}^{-1}(\lambda)q(\lambda), \quad \operatorname{Re}\lambda > 0. \tag{3.33}$$

We have $\pi(\mathscr{R}^{-1}(\lambda)R(\lambda, \mathcal{D})f) = 0$, because $f \in \mathrm{AP}(\mathbb{X})$ and $\mathscr{R}^{-1}(\lambda), R(\lambda, \mathcal{D})$ are bounded operators for $\operatorname{Re}\lambda > 0$ and around λ_0. Thus, by applying π to the above formula, we obtain

$$\pi(R(\lambda, \mathcal{D})u) = \pi(\mathscr{R}^{-1}(\lambda)q(\lambda)). \tag{3.34}$$

Hence,

$$R(\lambda, \tilde{\mathcal{D}})\pi(u) = \pi(\mathscr{R}^{-1}(\lambda)q(\lambda)), \quad \operatorname{Re}\lambda > 0. \tag{3.35}$$

Since $\mathscr{R}^{-1}(\lambda)$ and $q(\lambda)$ are holomorpic in the neighborhood U_{ξ_0} of $i\xi_0$, it follows that $R(\lambda, \tilde{\mathcal{D}})\pi(u)$ $(\lambda \in U_{\xi_0}, \operatorname{Re}\lambda > 0)$ has a holomorphic extension to U_{ξ_0}. This proves the lemma. $\square$

We are in a position to state one of the main results of this paper.

Theorem 3.4. *Let A_0 be a linear closed operator on $\mathbb{X}$, $A_k \in L(\mathbb{X})$, for all $k = 1, \dots, n$, and let $B_l \in \mathrm{BV}([a,b], L(\mathbb{X}))$, for all $l = 0, \dots, m$. Assume further that Λ is bounded and $\mathrm{dist}(\Lambda \backslash \mathrm{sp}(f), \mathrm{sp}(f)) > 0$. Then, whenever equation (1.1) has a mild solution $u \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$, it has a mild solution $w \in \mathrm{BUC}(\mathbb{R}, \mathbb{X})$ such that*

$$\mathrm{sp}(w) \subset \mathrm{sp}(f). \tag{3.36}$$

Proof. Set $d = \mathrm{dist}(\Lambda \backslash \mathrm{sp}(f), \mathrm{sp}(f))$. Since Λ is compact, we can choose a continuous function $\varphi \in L^1(\mathbb{R})$ such that

$$\tilde{\varphi} = 1 \quad \text{on } \Lambda \backslash \mathrm{sp}(f),$$
$$\mathrm{supp}\,\tilde{\varphi} \subset \left\{\lambda \in \mathbb{R} : \mathrm{dist}\,(\lambda, \Lambda \backslash \mathrm{sp}(f)) < \frac{d}{2}\right\}. \tag{3.37}$$

Set $w = u - \varphi * u$. According to Remark 2.7(i), the function w is a mild solution of (1.1) with forcing term $f - \varphi * f$. Since $\mathrm{supp}\,\tilde{\varphi} \cap \mathrm{sp}(f) = \varnothing$, by Proposition 2.1(v), we get $f - \varphi * f = f$. Hence, w is a mild solution of (1.1).

Next, we will show that $\mathrm{sp}(w) \subset \mathrm{sp}(f)$. Indeed, by Proposition 2.1(iv) and Lemma 3.3(i), we have

$$
\begin{aligned}
\mathrm{sp}(w) &= \mathrm{sp}(u - \varphi * u) \\
&\subset \mathrm{sp}(u) \cap \mathrm{supp}(1 - \tilde{\varphi}) \\
&\subset (\Lambda \cup \mathrm{sp}(f)) \cap \mathrm{supp}(1 - \tilde{\varphi}) \\
&\subset \mathrm{sp}(f). \qquad \square
\end{aligned}
\tag{3.38}
$$

Remark 3.5. The above result is an extension of the classical one due to Massera (see [7, 9] for more information in this direction).

Recall that f is quasiperiodic if it is of the form $f(t) = F(t, t, \ldots, t), t \in \mathbb{R}$, where $F(t_1, t_2, \ldots, t_n)$ is $\mathbb{X}$-valued continuous function of n variables which is periodic on each variable. By [6, page 48], if f is quasiperiodic and $\mathrm{sp}(f)$ is discrete, then $\mathrm{sp}(f)$ has an integer and finite basis, that is, there exists a finite subset $T \subset \mathrm{sp}(f)$ such that any element $\lambda \in \mathrm{sp}(f)$ can be represented in the form $\lambda = n_1 b_1 + \cdots + n_m b_m$, where $n_j \in \mathbb{Z}$, $b_j \in T$, $j = 1, \ldots, m$. Conversely, if f is almost periodic and $\mathrm{sp}(f)$ has an integer and finite basis, f is quasiperiodic. Finally, from Proposition 2.1(viii), we get immediately the following corollary.

COROLLARY 3.6. *Let all assumptions of Theorem 3.4 be satisfied. Moreover, assume that $\mathbb{X}$ does not contain c_0. Then*

  (i) *if $\mathrm{sp}(f)$ is countable, (1.1) has an almost periodic mild solution w such that $\mathrm{sp}(w) \subset \mathrm{sp}(f)$;*
  (ii) *if $\mathrm{sp}(f)$ has an integer and finite basis, (1.1) has a quasiperiodic mild solution w such that $\mathrm{sp}(w) \subset \mathrm{sp}(f)$.*

Proof. (i) The theorem follows immediately from Theorem 3.4 and Proposition 2.1(viii).

(ii) By (i), we have that w is almost periodic and $\mathrm{sp}(w) \subset \mathrm{sp}(f)$. Therefore, $\mathrm{sp}(w)$ has an integer and finite basis. Hence w is quasiperiodic. $\qquad \square$

Using the estimate of the AP-spectrum of bounded mild solutions, we can improve the above corollary as follows.

COROLLARY 3.7. *Let all assumptions of Theorem 3.4 be satisfied. Moreover, assume that $\mathbb{X}$ does not contain c_0, $\Lambda \cap \mathrm{sp}(f)$ is countable, and $f \in \mathrm{AP}(\mathbb{X})$. Then, (1.1) has an almost periodic mild solution w such that $\mathrm{sp}(w) \subset \mathrm{sp}(f)$.*

Proof. Choose $\varphi \in L^1(\mathbb{R})$ as in the proof of Theorem 3.4 and set $w = u - \varphi * u$. In Theorem 3.4, we have proved that w is a mild solution of (1.1) such that (3.36) holds. We now show that $w \in \mathrm{AP}(\mathbb{X})$. In fact, by Proposition 2.4(iii) and Lemma 3.3(ii), we get

$$
\begin{aligned}
\mathrm{sp}_{\mathrm{AP}}(w) &= \mathrm{sp}_{\mathrm{AP}}(u - \varphi * u) \\
&\subset \mathrm{sp}_{\mathrm{AP}}(u) \cap \mathrm{supp}(1 - \tilde{\varphi}) \\
&\subset \Lambda \cap \mathrm{supp}(1 - \tilde{\varphi}) \\
&\subset \Lambda \cap \mathrm{sp}(f).
\end{aligned}
\tag{3.39}
$$

Since $\Lambda \cap \mathrm{sp}(f)$ is countable and $\mathbb{X}$ does not contain c_0, by Propositon 2.4(v), we have that w is almost periodic. $\qquad\square$

To illustrate the usefulness of the above criterion for the almost periodicity of mild solutions of higher-order delay or advance abstract functional differential equations, we will consider the following examples.

Example 3.8. We consider a simple ordinary differential equation of the form

$$\dot{x}(t) = f(t), \quad x(t) \in \mathbb{C}^n, \tag{3.40}$$

where f is a $\mathbb{C}^n$-valued continuous 2π-periodic function. Obviously, the existence of a 2π-periodic solution to (3.40) is equivalent to the 2π-periodicity of the primitive

$$F(t) = \int_0^t f(\xi)d\xi, \quad t \in \mathbb{R}. \tag{3.41}$$

By an elementary argument, we can show that $F(t)$ can be represented in the form

$$F(t) = mt + G(t), \tag{3.42}$$

where G is a continuous 2π-periodic function and $m = (1/2\pi)\int_0^{2\pi} f(\xi)d\xi$. Hence, for F to be 2π-periodic, it is necessary and sufficient that F be bounded. It is easy to see that under this condition, there is exactly one solution x of (3.40) such that $\mathrm{sp}(x) \subset \mathrm{sp}(f)$.

This justifies the essential of our condition on the existence of a bounded mild solution in Theorem 3.4.

Example 3.9. In this example, we consider the oscillation equation

$$\ddot{x}(t) + x(t) = \sin(1-\epsilon)t, \quad x(t) \in \mathbb{R}, \tag{3.43}$$

where $0 \le \epsilon < 1$ is a parameter. If $0 < \epsilon < 1$, the equation has the general solution that is of the form

$$x(t) = C_1 \cos t + C_2 \sin t + \frac{1}{1-(1-\epsilon)^2}\sin(1-\epsilon)t, \tag{3.44}$$

where C_1, C_2 are two constants. It is clear that every solution is almost periodic. In this case, since there is no resonance, there exists only one almost periodic solution, that is,

$$x_1(t) = \frac{1}{1-(1-\epsilon)^2}\sin(1-\epsilon)t \tag{3.45}$$

with the same frequencies as the forcing term $f(t) = \sin(1-\epsilon)t$.

Obviously, if $\epsilon = 0$, the general solution of this equation is of the form

$$x(t) = C_1 \cos t + C_2 \sin t - \frac{t}{2} \cos t, \qquad (3.46)$$

where C_1, C_2 are constants. Therefore, since every solution of (3.43) in this case is unbounded, it has no 2π-periodic solutions.

Example 3.10. Consider the equation

$$u_{jt}(t,x) = d_j \Delta u_j(t,x) + \sum_{k=1}^{n} \int_0^\infty db_{jk}(\tau) u_k(t-\tau,x) + f_j(x,t), \quad t \in \mathbb{R},\ x \in \Omega,$$
$$(3.47)$$
$$d_j \frac{\partial u_j}{\partial \nu}(t,x) = 0, \quad x \in \partial\Omega,\ j = 1,\dots,n,$$

where $u(x,t)$, $f(x,t)$ are scalar-valued functions, $d_j \geq 0$, $b_{jk} \in \mathrm{BV}(\mathbb{R}_+)$, for all $j,k = 1,\dots,n$, $\Omega \subset \mathbb{R}^n$ denotes a bounded domain with smooth boundary, and $\nu(x)$ the outer normal at $x \in \partial\Omega$.

This equation without the forcing term $f_j(t,x)$ has been investigated in [10, Section 6]. We define the space $\mathbb{X} := [L^2(\Omega)]^n$ and $A : \mathbb{X} \to \mathbb{X}$ by the formula

$$A = (\mathrm{diag}\, d_j)\Delta,$$
$$D(A) = \left\{ u \in \mathbb{X} : d_j u_j \in W^{2,2}(\Omega),\ d_j \frac{\partial u_j}{\partial \nu}(t,x) = 0 \text{ on } \partial\Omega \right\}. \qquad (3.48)$$

We define $B(t) : \mathbb{X} \to \mathbb{X}$ by the formula

$$(B(t)u)_j(x) = \sum_{k=1}^{n} b_{jk}(t) u_k(x), \quad u \in \mathbb{X}. \qquad (3.49)$$

Then, (3.47) can be rewritten in the form

$$\frac{du(t)}{dt} = Au(t) + \int_0^\infty B(\tau)u(t-\tau) + f(t), \quad u(t), f(t) \in \mathbb{X}. \qquad (3.50)$$

Let u be a bounded uniformly continuous solution of (3.50) and let f be an almost periodic function. Let $\mu_0 = 0 > \mu_1 \geq \mu_2 \geq \cdots$ denote the eigenvalues of the Laplacian with Neumann boundary condition and e_j the corresponding eigenvectors. By [10, Section 6], the imaginary spectrum of (3.50) is yielded as follows:

$$\Lambda = \{\lambda \in \mathbb{R} : \chi_m(i\lambda) = 0 \text{ for some } m \in \mathbb{N}_0\}, \qquad (3.51)$$

where

$$\chi_m(i\lambda) := \det\left(\lambda - \mu_m D - \widehat{dB}(\lambda)\right), \quad m \in \mathbb{N}_0. \qquad (3.52)$$

Moreover, one can prove that Λ has Lebesgue measure zero. However, from this and the spectral inclusion in Lemma 3.3(ii)

$$\mathrm{sp}_{\mathrm{AP}}(u) \subset \Lambda, \tag{3.53}$$

we cannot conclude that u is almost periodic. But if we set up more conditions on f such that $\Lambda \cap \mathrm{sp}(f)$ is countable, then according to Corollary 3.7, there exists an almost periodic solution w of equation (3.50) such that (3.36) holds.

Acknowledgments

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