A CHARACTERIZATION OF THE GENERATORS OF ANALYTIC C_0-SEMIGROUPS IN THE CLASS OF SCALAR TYPE SPECTRAL OPERATORS

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To my beloved grandmothers, Polina Khokhmovich-Ryklina and Berta Krasnova-Ryklina

In the class of scalar type spectral operators in a complex Banach space, a characterization of the generators of analytic C_0-semigroups in terms of the analytic vectors of the operators is found.

1. Introduction

Let A be a linear operator in a Banach space X with norm $\|\cdot\|$,

$$C^\infty(A) \stackrel{\text{def}}{=} \bigcap_{n=0}^{\infty} D(A^n), \tag{1.1}$$

and $0 \leq \beta < \infty$.

The sets of vectors

$$\begin{aligned} \mathscr{E}^{\{\beta\}}(A) &\stackrel{\text{def}}{=} \{f \in C^\infty(A) \mid \exists \alpha > 0,\ \exists c > 0 : \|A^n f\| \leq c\alpha^n [n!]^\beta,\ n = 0,1,\dots\}, \\ \mathscr{E}^{(\beta)}(A) &\stackrel{\text{def}}{=} \{f \in C^\infty(A) \mid \forall \alpha > 0\ \exists c > 0 : \|A^n f\| \leq c\alpha^n [n!]^\beta,\ n = 0,1,\dots\} \end{aligned} \tag{1.2}$$

are called the βth-order *Gevrey classes* of the operator A of *Roumie's* and *Beurling's types*, respectively.

In particular, $\mathscr{E}^{\{1\}}(A)$ and $\mathscr{E}^{(1)}(A)$ are, correspondingly, the celebrated classes of *analytic* and *entire* vectors [6, 17].

Obviously,

$$\mathscr{E}^{(1)}(A) \subseteq \mathscr{E}^{\{1\}}(A). \tag{1.3}$$

In [7, 8] and later in [19, 20], it was established that, for a *selfadjoint nonpositive* operator A in a complex Hilbert space H,

$$\mathscr{E}^{(1)}(A) = \bigcup_{t>0} R(e^{tA}), \qquad \mathscr{E}^{\{1\}}(A) = \bigcap_{t>0} R(e^{tA}), \tag{1.4}$$

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where $R(\cdot)$ is the range of an operator, the exponentials understood in the sense of the *operational calculus* (o.c.) for normal operators

$$e^{tA} := \int_{\mathbb{C}} e^{t\lambda} dE_A(\lambda), \quad t > 0, \tag{1.5}$$

$E_A(\cdot)$ is the operator's *resolution of the identity* (see, e.g., [3, 18]).

In [9], it was proved that the second equality in (1.4) holds in a more general case, namely, when A generates an analytic C_0-semigroup $\{e^{tA} \mid t \geq 0\}$ in a complex Banach space X.

Later, in [12], it was demonstrated that, in the class of normal operators in a complex Hilbert space, each of the equalities (1.4) characterizes the generators of the analytic semigroups.

The purpose of the present paper is to stretch out the results of [12] to the case of *scalar type spectral operators* in a complex Banach space.

It is absolutely fair of the reader to anticipate that abandoning the comforts of a Hilbert space would inevitably require introducing new approaches and techniques.

2. Preliminaries

Henceforth, unless specified otherwise, A is a scalar type spectral operator in a complex Banach space X with norm $\|\cdot\|$ and $E_A(\cdot)$ is its *spectral measure* (s.m.) (the resolution of the identity), the operator's spectrum $\sigma(A)$ being the *support* for the latter [1, 4].

Note that, in a Hilbert space, the scalar type spectral operators are those similar to the *normal* ones [21].

For such operators, there has been developed an o.c. for complex-valued Borel measurable functions on $\mathbb{C}$ [1, 4], $F(\cdot)$ being such a function, a new scalar type spectral operator,

$$F(A) = \int_{\mathbb{C}} F(\lambda) dE_A(\lambda), \tag{2.1}$$

is defined as follows:

$$\begin{gathered} F(A)f := \lim_{n\to\infty} F_n(A)f, \quad f \in D(F(A)), \\ D(F(A)) := \left\{ f \in X \mid \lim_{n\to\infty} F_n(A)f \text{ exists} \right\}, \end{gathered} \tag{2.2}$$

$D(\cdot)$ is the *domain* of an operator, where

$$F_n(\cdot) := F(\cdot)\chi_{\{\lambda\in\mathbb{C} \mid |F(\lambda)|\leq n\}}(\cdot), \quad n = 1, 2, \ldots, \tag{2.3}$$

$\chi_\alpha(\cdot)$ is the *characteristic function* of a set α, and

$$F_n(A) := \int_{\mathbb{C}} F_n(\lambda) dE_A(\lambda), \quad n = 1, 2, \ldots, \tag{2.4}$$

being the integrals of *bounded* Borel measurable functions on $\mathbb{C}$, are *bounded scalar type spectral operators* on X defined in the same manner as for normal operators (see, e.g., [3, 18]).

The properties of the s.m., $E_A(\cdot)$, and the o.c. underlying the entire subsequent argument are exhaustively delineated in [1, 4]. We just observe here that, due to its *strong countable additivity*, the s.m. $E_A(\cdot)$ is bounded, that is, there is an $M > 0$ such that, for any Borel set δ,

$$\|E_A(\delta)\| \le M, \tag{2.5}$$

see [2].

Observe that, in (2.5), the notation $\|\cdot\|$ was used to designate the norm in the space of bounded linear operators on X. We will adhere to this rather common economy of symbols in what follows, adopting the same notation for the norm in the dual space X^* as well.

With $F(\cdot)$ being an arbitrary complex-valued Borel measurable function on $\mathbb{C}$, for any $f \in D(F(A))$, $g^* \in X^*$ and arbitrary Borel sets $\delta \subseteq \sigma$, we have (see [2])

$$\begin{aligned}
&\int_\sigma |F(\lambda)|\,dv(f,g^*,\lambda)\\
&\quad\le 4\sup_{\delta\subseteq\sigma}\left|\int_\delta F(\lambda)\,d\langle E_A(\lambda)f,g^*\rangle\right|\\
&\quad= 4\sup_{\delta\subseteq\sigma}\left|\int_\sigma \chi_\delta(\lambda)F(\lambda)\,d\langle E_A(\lambda)f,g^*\rangle\right| \quad \text{(by the properties of the o.c.)}\\
&\quad= 4\sup_{\delta\subseteq\sigma}\left|\left\langle\int_\sigma \chi_\delta(\lambda)F(\lambda)\,dE_A(\lambda)f,g^*\right\rangle\right| \quad \text{(by the properties of the o.c.)}\\
&\quad= 4\sup_{\delta\subseteq\sigma}\left|\langle E_A(\delta)E_A(\sigma)F(A)f,g^*\rangle\right|\\
&\quad\le 4\sup_{\delta\subseteq\sigma}\|E_A(\delta)E_A(\sigma)F(A)f\|\,\|g^*\|\\
&\quad\le 4\sup_{\delta\subseteq\sigma}\|E_A(\delta)\|\,\|E_A(\sigma)F(A)f\|\,\|g^*\| \quad \text{(by (2.5))}\\
&\quad\le 4M\|E_A(\sigma)F(A)f\|\,\|g^*\|.
\end{aligned} \tag{2.6}$$

For the reader's convenience, we reformulate here Proposition 3.1 of [14], heavily relied upon in what follows, which allows to characterize the domains of the Borel measurable functions of a scalar type spectral operator in terms of positive measures (see [14] for a complete proof).

Proposition 2.1 [14]. *Let A be a scalar type spectral operator in a complex Banach space X and let $F(\cdot)$ be a complex-valued Borel measurable function on $\mathbb{C}$. Then, $f \in D(F(A))$ if and only if the following hold:*

(i) *for any $g^* \in X^*$,*

$$\int_{\mathbb{C}} |F(\lambda)|\,dv(f,g^*,\lambda) < \infty, \tag{2.7}$$

(ii)

$$\sup_{\{g^*\in X^* \mid \|g^*\|=1\}} \int_{\{\lambda\in\mathbb{C} \mid |F(\lambda)|>n\}} |F(\lambda)|\, dv(f,g^*,\lambda) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

As was shown in [13], a scalar type spectral operator A in a complex Banach space X generates an analytic C_0-semigroup, if and only if, for some real ω and $0 < \theta \le \pi/2$,

$$\sigma(A) \subseteq \left\{\lambda \in \mathbb{C} \mid |\arg(\lambda-\omega)| \ge \frac{\pi}{2}+\theta\right\}, \tag{2.9}$$

where $\arg\cdot$ is the *principal value* of the argument from the interval $(-\pi,\pi]$ (see [15] for generalizations), in which case the semigroup consists of the exponentials

$$e^{tA} = \int_{\mathbb{C}} e^{t\lambda}\, dE_A(\lambda), \quad t \ge 0. \tag{2.10}$$

It is also to be noted that, according to [16], for a scalar type spectral operator A in a complex Banach space X,

$$\mathscr{E}^{\{1\}}(A) \supseteq \bigcup_{t>0} D(e^{t|A|}), \qquad \mathscr{E}^{(1)}(A) \supseteq \bigcap_{t>0} D(e^{t|A|}), \tag{2.11}$$

the inclusions turning into equalities provided the space X is *reflexive*.

3. The principal statement

Theorem 3.1. *Let A be a scalar type spectral operator in a complex Banach space X. Then, each of equalities (1.4), the operator exponentials e^{tA}, $t > 0$, defined in the sense of the o.c. for scalar type spectral operators, is necessary and sufficient for A to be the generator of an analytic C_0-semigroup.*

Proof

Necessity. We consider the general of A being a generator of an analytic C_0-semigroup $\{e^{tA} \mid t \ge 0\}$ in a complex Banach space X, without the assumption of A being a scalar type spectral operator.

First, note that the inclusions

$$\mathscr{E}^{\{1\}}(A) \supseteq \bigcup_{t>0} R(e^{tA}), \qquad \mathscr{E}^{(1)}(A) \supseteq \bigcap_{t>0} R(e^{tA}), \tag{3.1}$$

immediately follow from the estimate

$$||A^n e^{tA}|| \le e^{\omega t}\frac{M^n}{t^n}n!, \quad n = 1,2,\ldots, \ t > 0 \tag{3.2}$$

with some positive ω and M, known for analytic C_0-semigroups (see, e.g., [11]).

We show now that the inverse inclusions hold even in a more general case, when A generates a C_0-semigroup $\{e^{tA} \mid t \ge 0\}$ not necessarily analytic.

Let f be an *analytic* (*entire*) vector of the operator A, then, for some (any) $\delta > 0$, the power series

$$\sum_{n=0}^{\infty} \frac{(-A)^n f}{n!}\lambda^n \tag{3.3}$$

converges whenever $|\lambda| < \delta$.

Formally designating the series by $e^{\lambda(-A)}f$ and differentiating it termwise, with the closedness of A in view, we obtain

$$e^{\lambda(-A)}f \in D(A), \quad \frac{d}{d\lambda}e^{\lambda(-A)}f = -Ae^{\lambda(-A)}f, \quad |\lambda| < \delta. \tag{3.4}$$

Considering that for any $g \in D(A)$,

$$\frac{d}{dt}e^{tA}g = Ae^{tA}g = e^{tA}Ag, \quad t \ge 0, \tag{3.5}$$

(see [5, 10]), we have, for all $0 \le t < \delta$,

$$\begin{aligned} \frac{d}{dt}e^{tA}e^{t(-A)}f &= \frac{d}{ds}e^{As}e^{t(-A)}f|_{s=t} + e^{At}\frac{d}{dt}e^{t(-A)}f \\ &= Ae^{tA}e^{t(-A)}f + e^{At}(-Ae^{t(-A)}f) \\ &= Ae^{At}e^{-At}f - Ae^{At}e^{-At}f = 0. \end{aligned} \tag{3.6}$$

This implies that, for all $0 \le t < \delta$,

$$e^{tA}e^{t(-A)}f = e^{As}e^{s(-A)}f|_{s=0} = f. \tag{3.7}$$

Therefore,

$$\mathscr{E}^{\{1\}}(A) \subseteq \bigcup_{t>0} R(e^{At}) \left(\mathscr{E}^{(1)}(A) \subseteq \bigcap_{t>0} R(e^{At})\right). \tag{3.8}$$

Sufficiency. We prove this part by *contrapositive*.

As was noted in Section 2, for a scalar type spectral operator A, its being the generator of an analytic C_0-semigroup is equivalent to inclusion (2.9) with some real ω and $0 < \theta \le \pi/2$.

Hence, as is easily seen, the negation of the fact that A generates an analytic C_0-semigroup implies that for any $b > 0$, the set

$$\sigma(A) \setminus \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \le -b|\operatorname{Im}\lambda|\} \tag{3.9}$$

is *unbounded*.

In particular, for any natural n, the set

$$\sigma(A) \setminus \left\{\lambda \in C \mid \operatorname{Re}\lambda \le -\frac{1}{n^2}|\operatorname{Im}\lambda|\right\} \tag{3.10}$$

is unbounded.

Hence, we can choose a sequence of points of the complex plane $\{\lambda_n\}_{n=1}^{\infty}$ in the following way:

$$\begin{gathered} \lambda_n \in \sigma(A), \quad n = 1,2,\ldots; \\ \operatorname{Re}\lambda_n > -\frac{1}{n^2}|\operatorname{Im}\lambda|, \quad n = 1,2,\ldots; \\ \lambda_0 := 0, \quad |\lambda_n| > \max[n, |\lambda_{n-1}|], \quad n = 1,2,\ldots. \end{gathered} \tag{3.11}$$

The latter, in particular, implies that the points λ_n are *distinct*:

$$\lambda_i \ne \lambda_j, \quad i \ne j. \tag{3.12}$$

Since the set

$$\left\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > -\frac{1}{n^2}|\operatorname{Im}\lambda|\right\} \tag{3.13}$$

is *open* in $\mathbb{C}$ for any $n = 1,2,\ldots$, there exists such an $\varepsilon_n > 0$ that this set contains together with the point λ_n the *open disk* centered at λ_n:

$$\Delta_n = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_n| < \varepsilon_n\}, \tag{3.14}$$

that is, for any $\lambda \in \Delta_n$,

$$\begin{gathered} \operatorname{Re}\lambda > -\frac{1}{n^2}|\operatorname{Im}\lambda|, \\ |\lambda| > \max[n, |\lambda_{n-1}|]. \end{gathered} \tag{3.15}$$

Moreover, since the points λ_n are distinct, we can regard that the radii of the disks, ε_n, are chosen to be small enough so that

$$\begin{gathered} 0 < \varepsilon_n < \frac{1}{n}, \quad n = 1,2,\ldots; \\ \Delta_i \cap \Delta_j = \varnothing, \quad i \ne j \quad (\text{the disks are } \textit{pairwise disjoint}). \end{gathered} \tag{3.16}$$

Note that, by the properties of the *s.m.*, the latter implies that the subspaces $E_A(\Delta_n)X$, $n = 1,2,\dots$, are *nontrivial*, since $\Delta_n \cap \sigma(A) \neq \emptyset$ and Δ_n is open and

$$E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j. \tag{3.17}$$

Thus, choosing a unit vector e_n in each subspace $E_A(\Delta_n)X$, we obtain a vector sequence such that

$$E_A(\Delta_i)e_j = \delta_{ij}e_i \tag{3.18}$$

(δ_{ij} is the *Kronecker delta symbol*).

The latter, in particular, implies that the vectors $\{e_1,e_2,\dots\}$ are linearly independent and that

$$d_n := \operatorname{dist}(e_n, \operatorname{span}(\{e_k \mid k \in \mathbb{N},\ k \neq n\})) > 0, \quad n = 1,2,\dots. \tag{3.19}$$

Furthermore,

$$d_n \not\longrightarrow 0 \quad n \longrightarrow \infty. \tag{3.20}$$

Indeed, assuming the opposite, $d_n \to 0$ as $n \to \infty$, would imply that, for any $n = 1,2,\dots$, there is an $f_n \in \operatorname{span}(\{e_k \mid k \in \mathbb{N},\ k \neq n\})$ such that $\|e_n - f_n\| < d_n + 1/n$, whence $e_n = E_A(\Delta_n)(e_n - f_n) \to 0$, which is a contradiction.

Therefore, there is a positive ε such that

$$d_n \geq \varepsilon, \quad n = 1,2,\dots. \tag{3.21}$$

As follows from the *Hahn-Banach theorem*, for each $n = 1,2,\dots$, there is an $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1, \qquad \langle e_i, e_j^* \rangle = \delta_{ij}d_i. \tag{3.22}$$

Let

$$g^* := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n^*. \tag{3.23}$$

On one hand, for any $n = 1,2,\dots$,

$$\begin{aligned} v(e_n,g^*,\Delta_n) &\geq |\langle E_A(\Delta_n)e_n,g^*\rangle| \quad (\text{by } (3.18)) \\ &= |\langle e_n,g^*\rangle| = \frac{d_n}{n^2} \quad (\text{by } (3.21)) \\ &\geq \frac{\varepsilon}{n^2}. \end{aligned} \tag{3.24}$$

On the other hand, for any $n = 1, 2, \ldots$,

$$\begin{aligned} &v(e_n, g^*, \Delta_n) \quad (\delta \text{ being an arbitrary Borel subset of } \Delta_n, [2]) \\ &\quad \le 4 \sup_{\delta} |\langle E_A(\delta)e_n, g^*\rangle| \le 4 \sup_{\delta} \|E_A(\delta)\| \, \|e_n\| \, \|g^*\| \quad (\text{by } (2.5)) \\ &\quad \le 4M\|g^*\|. \end{aligned} \tag{3.25}$$

Concerning the sequence of the real parts, $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$, there are two possibilities: it is either *bounded below*, or not. We consider each of them separately.

First, assume that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is bounded below, that is, there is such an $\omega > 0$ that

$$\operatorname{Re}\lambda_n \ge \omega, \quad n = 1, 2, \ldots. \tag{3.26}$$

Observe that this fact immediately implies that the operators e^{-tA}, $t > 0$, are bounded and, thus, defined on the entire X [1, 4].

Therefore, $R(e^{tA}) = D(e^{-tA}) = X$, $t > 0$.

Let

$$f := \sum_{n=1}^{\infty} \frac{1}{n^2} e_n. \tag{3.27}$$

As can be easily deduced from (3.17),

$$\begin{gathered} E_A(\Delta_n) f = \frac{1}{n^2} e_n, \quad n = 1, 2, \ldots, \\ E_A\left(\bigcup_{n=1}^{\infty} \Delta_n\right) f = f. \end{gathered} \tag{3.28}$$

For an arbitrary $t > 0$, we have

$$\begin{aligned} &\int_{\mathbb{C}} e^{t|\lambda|} dv(f, g^*, \lambda) \quad \text{by } (3.28); \\ &\quad = \int_{\mathbb{C}} e^{t|\lambda|} dv\left(E_A\left(\bigcup_{n=1}^{\infty} \Delta_n\right) f, g^*, \lambda\right) \quad (\text{by the properties of the o.c.}) \\ &\quad = \int_{\bigcup_{n=1}^{\infty} \Delta_n} e^{t|\lambda|} dv(E_A(\Delta_n) f, g^*, \lambda) \\ &\quad = \sum_{n=1}^{\infty} \int_{\Delta_n} e^{t|\lambda|} dv(E_A(\Delta_n) f, g^*, \lambda) \quad (\text{by } (3.28)) \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\Delta_n} e^{t|\lambda|} dv(e_n, g^*, \lambda) \quad \text{for } \lambda \in \Delta_n, \ (\text{by } (3.15), |\lambda| \ge n) \\ &\quad \ge \sum_{n=1}^{\infty} \frac{1}{n^2} e^{tn} v(f, g^*, \Delta_n) \quad (\text{by } (3.24)) \\ &\quad \ge \sum_{n=1}^{\infty} \frac{\varepsilon e^{tn}}{n^4} = \infty. \end{aligned} \tag{3.29}$$

This, by [14, Proposition 3.1], implies that

$$f \notin \bigcup_{t>0} D(e^{t|A|}). \tag{3.30}$$

Then, by (2.11), moreover,

$$f \notin \mathscr{E}^{\{1\}}(A). \tag{3.31}$$

Therefore, equalities (1.4) do not hold.

Now, suppose that the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ is *unbounded below*, that is, there is a subsequence $\{\operatorname{Re}\lambda_{n(k)}\}_{k=1}^{\infty}$ $(k \le n(k))$ such that

$$\operatorname{Re}\lambda_{n(k)} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty. \tag{3.32}$$

Without the loss of generality, we can regard that

$$\operatorname{Re}\lambda_{n(k)} \le -k, \quad k = 1,2,\ldots. \tag{3.33}$$

Let

$$f := \sum_{k=1}^{\infty} e^{k\operatorname{Re}\lambda_{n(k)}} e_{n(k)}. \tag{3.34}$$

Similarly to (3.17), we have

$$\begin{gathered} E_A(\Delta_{n(k)}) f = e^{k\operatorname{Re}\lambda_{n(k)}} e_{n(k)}, \quad n = 1,2,\ldots, \\ E_A\left(\bigcup_{n=1}^{\infty} \Delta_{n(k)}\right) f = f. \end{gathered} \tag{3.35}$$

For any $t > 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned} &\int_{\mathbb{C}} e^{-t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \\ &\quad = \int_{\bigcup_{k=1}^{\infty}\Delta_{n(k)}} e^{-t\operatorname{Re}\lambda} dv(f,g^*,\lambda) \quad \text{(by the properties of the o.c.)} \\ &\quad = \sum_{k=1}^{\infty} \int_{\Delta_{n(k)}} e^{t|\lambda|} dv(E_A(\Delta_{n(k)}) f,g^*,\lambda) \quad \text{(by (3.35))} \\ &\quad = \sum_{k=1}^{\infty} e^{k\operatorname{Re}\lambda_{n(k)}} \int_{\Delta_{n(k)}} e^{-t\operatorname{Re}\lambda} dv(e_{n(k)},g^*,\lambda) \quad \text{(by (3.16))} \\ &\quad \le \sum_{k=1}^{\infty} e^{k\operatorname{Re}\lambda_{n(k)}} e^{t(-\operatorname{Re}\lambda_{n(k)}+1)} v(e_{n(k)},g^*,\Delta_{n(k)}) \quad \text{(by (3.25))} \\ &\quad \le 4M\|g^*\| e^t \sum_{k=1}^{\infty} e^{(k-t)\operatorname{Re}\lambda_{n(k)}} < \infty. \end{aligned} \tag{3.36}$$

Indeed, for $\lambda \in \Delta_{n(k)}$, by (3.16), $-\operatorname{Re}\lambda = -\operatorname{Re}\lambda_{n(k)} + (\operatorname{Re}\lambda_{n(k)} - \operatorname{Re}\lambda) \leq -\operatorname{Re}\lambda_{n(k)} + |\lambda_{n(k)} - \lambda| \leq -\operatorname{Re}\lambda_{n(k)} + \varepsilon_{n(k)} \leq -\operatorname{Re}\lambda_{n(k)} + 1$ and for all natural k's large enough so that $k - t \geq 1$, due to (3.33),

$$e^{(k-t)\operatorname{Re}\lambda_{n(k)}} \leq e^{-k}. \tag{3.37}$$

Similarly, for any $t > 0$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \mathbb{C} \mid e^{-t\operatorname{Re}\lambda} > n\}} e^{-t\operatorname{Re}\lambda} dv(f, g^*, \lambda) \\
&\quad = \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^t \sum_{k=1}^{\infty} e^{k\operatorname{Re}\lambda_{n(k)}} \int_{\{\lambda \in \Delta_{n(k)} \mid e^{-t\operatorname{Re}\lambda} > n\}} e^{-t\operatorname{Re}\lambda} dv(e_{n(k)}, g^*, \lambda) \\
&\quad \leq e^t \sum_{k=1}^{\infty} e^{(k-t)\operatorname{Re}\lambda_{n(k)}} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} v(f, g^*, \{\lambda \in \Delta_{n(k)} \mid e^{-t\operatorname{Re}\lambda} > n\}) \quad (\text{by } (2.6)) \\
&\quad \leq e^t \sum_{k=1}^{\infty} e^{(k-t)\operatorname{Re}\lambda_{n(k)}} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \| E_A(\{\lambda \in \Delta_{n(k)} \mid e^{t\operatorname{Re}\lambda} > n\}) f \| \, \|g^*\| \\
&\quad \leq 4Me^t \sum_{k=1}^{\infty} e^{(k-t)\operatorname{Re}\lambda_{n(k)}} \| E_A(\{\lambda \in \mathbb{C} \mid e^{-t\operatorname{Re}\lambda} > n\}) f \| \\
&\qquad (\text{by the strong continuity of the s.m.} \longrightarrow 0 \text{ as } n \longrightarrow \infty).
\end{aligned} \tag{3.38}$$

According to [14, Proposition 3.1], (3.36) and (3.38) imply that

$$f \in \bigcap_{t>0} D(e^{-tA}) = \bigcap_{t>0} R(e^{tA}). \tag{3.39}$$

However, for an arbitrary $t > 0$, we have

$$\begin{aligned}
&\int_{\mathbb{C}} e^{t|\lambda|} dv(f, g^*, \lambda) \\
&\quad = \sum_{k=1}^{\infty} e^{k\operatorname{Re}\lambda_{n(k)}} \int_{\Delta_{n(k)}} e^{t|\lambda|} dv(e_{n(k)}, g^*, \lambda) \quad (\text{by the properties of the o.c. and } (3.35)) \\
&\quad \geq \sum_{k=1}^{\infty} e^{k\operatorname{Re}\lambda_{n(k)}} e^{-tn(k)^2(\operatorname{Re}\lambda_{n(k)}+1)} dv(e_{n(k)}, g^*, \Delta_{n(k)}) \quad (\text{by } (3.15) \text{ and } (3.16)) \\
&\quad = \sum_{k=1}^{\infty} e^{-tn(k)^2} e^{(tn(k)^2-k)(-\operatorname{Re}\lambda_{n(k)})} dv(e_{n(k)}, g^*, \Delta_{n(k)}) \quad (\text{by } (3.24)) \\
&\quad \geq \sum_{k=1}^{\infty} e^{-tn(k)^2} e^{(tn(k)^2-k)(-\operatorname{Re}\lambda_{n(k)})} \frac{\varepsilon}{n(k)^2} = \infty.
\end{aligned} \tag{3.40}$$

Indeed, for $\lambda \in \Delta_{n(k)}$, by (3.15) and (3.16), $|\lambda| \geq |\operatorname{Im}\lambda| \geq -n(k)^2 \operatorname{Re}\lambda \geq -n(k)^2(\operatorname{Re}\lambda_{n(k)} + |\operatorname{Re}\lambda - \operatorname{Re}\lambda_{n(k)}|) \geq -n(k)^2(\operatorname{Re}\lambda_{n(k)} + 1)$, and for all natural k's large enough so that $tn(k)^2 - k > 0$, due to (3.33), we have

$$e^{-tn(k)^2} e^{(tn(k)^2-k)(-\operatorname{Re}\lambda_{n(k)})} \frac{\varepsilon}{n(k)^2} \geq \varepsilon \frac{e^{tn(k)^3-tn(k)^2-kn(k)}}{n(k)^2} \longrightarrow \infty, \quad \text{as } k \longrightarrow \infty. \tag{3.41}$$

Whence, by [14, Proposition 3.1], we infer that $f \notin \bigcup_{t>0} D(e^{t|A|})$. Then, by (2.11), moreover $f \notin \mathscr{E}^{\{1\}}(A)$. Therefore, equalities (1.4) do not hold in this case either.

With all the possibilities concerning $\{\operatorname{Re}\lambda_n\}_{n=1}^{\infty}$ having been analyzed, we conclude that the sufficiency part has been proved by *contrapositive*. □

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