GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO THE CAUCHY PROBLEM FOR A WAVE EQUATION WITH A WEAKLY NONLINEAR DISSIPATION

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We prove the global existence and study decay properties of the solutions to the wave equation with a weak nonlinear dissipative term by constructing a stable set in $H^1(\mathbb{R}^n)$.

1. Introduction

We consider the Cauchy problem for the nonlinear wave equation with a weak nonlinear dissipation and source terms of the type

$$\begin{gathered} u'' - \Delta_x u + \lambda^2(x)u + \sigma(t)g(u') = |u|^{p-1}u \quad \text{in } \mathbb{R}^n \times [0,+\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in } \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function and λ and σ are positive functions.

When we have a bounded domain instead of $\mathbb{R}^n$, and for the case $g(x) = \delta x$ $(\delta > 0)$ (without the term $\lambda^2(x)u$), Ikehata and Suzuki [8] investigated the dynamics, they have shown that for sufficiently small initial data (u_0,u_1), the trajectory $(u(t),u'(t))$ tends to $(0,0)$ in $H_0^1(\Omega) \times L^2(\Omega)$ as $t \to +\infty$. When $g(x) = \delta|x|^{m-1}x$ $(m \geq 1, \lambda \equiv 0, \sigma \equiv 1)$, Georgiev and Todorova [4] introduced a new method and determined suitable relations between m and p, for which there is global existence or alternatively finite-time blow up. Precisely they showed that the solutions continue to exist globally in time if $m \geq p$ and blow up in finite time if $m < p$ and the initial energy is sufficiently negative. This result was later generalized to an abstract setting by Levine and Serrin [12] and Levine et al. [11]. In these papers, the authors showed that no solution with negative initial energy can be extended on $[0,\infty[$, if the source term dominates over the damping term $(p > m)$. This generalization allowed them also to apply their result to quasilinear wave equations (see [1, 17]). Quite recently, Ikehata [7] proved that a global solution exists with no relation between p and m by the use of a stable set method due to Sattinger [18].

For the Cauchy problem (1.1) with $\lambda \equiv 1$ and $\sigma \equiv 1$, when $g(x) = \delta|x|^{m-1}x$ $(m \geq 1)$ Todorova [21] (see [16]) proved that the energy decay rate is $E(t) \leq (1+t)^{-(2-n(m-1))/(m-1)}$

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for $t \geq 0$. She used a general method introduced by Nakao [14] on condition that the data have compact support. Unfortunately, this method does not seem to be applicable in the case of more general functions λ and σ.

Our purpose in this paper is to give a global solvability in the class H^1 and energy decay estimates of the solutions to the Cauchy problem (1.1) for a weak linear perturbation and a weak nonlinear dissipation.

We use a new method recently introduced by Martinez [13] (see also [2]) to study the decay rate of solutions to the wave equation $u'' - \Delta_x u + g(u') = 0$ in $\Omega \times \mathbb{R}^+$, where Ω is a bounded domain of $\mathbb{R}^n$. This method is based on a new integral inequality that generalizes a result of Haraux [6]. So we proceed with the argument combining the method in [13] with the concept of modified stable set on $H^1(\mathbb{R}^n)$. Here the modified stable set is the extended $\mathbb{R}^n$ version of Sattinger's stable set.

2. Preliminaries and main results

$\lambda(x)$, $\sigma(t)$, and g satisfy the following hypotheses.

(i) $\lambda(x)$ is a locally bounded measurable function defined on $\mathbb{R}^n$ and satisfies

$$\lambda(x) \geq d(|x|), \tag{2.1}$$

where d is a decreasing function such that $\lim_{y\to\infty} d(y) = 0$.

(ii) $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function of class C^1 on $\mathbb{R}_+$.

Consider $g : \mathbb{R} \to \mathbb{R}$ a nondecreasing C^0 function and suppose that there exist $C_i > 0$, $i = 1,2,3,4$, such that

$$c_3'|v|^m \leq \left|g(v)\right| \leq c_4'|v|^{1/m} \quad \text{if } |v| \leq 1, \tag{2.2}$$

$$c_1|v| \leq \left|g(v)\right| \leq c_2|v|^r \quad \forall\, |v| \geq 1, \tag{2.3}$$

where $m \geq 1$ and $1 \leq r \leq (n+2)/(n-2)^+$.

We first state two well-known lemmas, and then we state and prove two other lemmas that will be needed later.

Lemma 2.1. *Let q be a number with $2 \leq q < +\infty$ $(n = 1,2)$ or $2 \leq q \leq 2n/(n-2)(n \geq 3)$. Then there is a constant $c_* = c(q)$ such that*

$$\|u\|_q \leq c_* \|u\|_{H^1(\mathbb{R}^n)} \quad \text{for } u \in H^1(\mathbb{R}^n). \tag{2.4}$$

Lemma 2.2 (Gagliardo-Nirenberg). *Let $1 \leq r < q \leq +\infty$ and $p \geq 2$. Then, the inequality*

$$\|u\|_p \leq C\left|\left|\nabla_x^m u\right|\right|_2^{\theta} \|u\|_r^{1-\theta} \quad \text{for } u \in \mathfrak{D}\left((-\Delta)^{m/2}\right)L^r \tag{2.5}$$

holds with some constant $C > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{n} + \frac{1}{r} - \frac{1}{2}\right)^{-1} \tag{2.6}$$

provided that $0 < \theta \leq 1$ (assuming that $0 < \theta < 1$ if $m - n/2$ is a nonnegative integer).

Lemma 2.3 [10]. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and assume that there are two constants $p \geq 1$ and $A > 0$ such that*

$$\int_S^{+\infty} E^{(p+1)/2}(t)\,dt \leq AE(S), \quad 0 \leq S < +\infty, \tag{2.7}$$

then

$$\begin{aligned} &E(t) \leq cE(0)(1+t)^{-2/(p-1)} \quad \forall t \geq 0, \text{ if } p > 1, \\ &E(t) \leq cE(0)e^{-\omega t} \quad \forall t \geq 0, \text{ if } p = 1, \end{aligned} \tag{2.8}$$

where c and ω are positive constants independent of the initial energy $E(0)$.

Lemma 2.4 [13]. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^2 function such that*

$$\phi(0) = 0, \qquad \phi(t) \longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty. \tag{2.9}$$

Assume that there exist $p \geq 1$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{(p+1)/2}(t)\phi'(t)\,dt \leq AE(S), \quad 0 \leq S < +\infty, \tag{2.10}$$

then

$$\begin{aligned} &E(t) \leq cE(0)\big(1+\phi(t)\big)^{-2/(p-1)} \quad \forall t \geq 0, \text{ if } p > 1, \\ &E(t) \leq cE(0)e^{-\omega\phi(t)} \quad \forall t \geq 0, \text{ if } p = 1, \end{aligned} \tag{2.11}$$

where c and ω are positive constants independent of the initial energy $E(0)$.

Proof of Lemma 2.4. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we remark that ϕ^{-1} has a sense by the hypotheses assumed on ϕ). f is nonincreasing, $f(0) = E(0)$, and if we set $x := \phi(t)$, we obtain

$$\begin{aligned} \int_{\phi(S)}^{\phi(T)} f(x)^{(p+1)/2}\,dx &= \int_{\phi(S)}^{\phi(T)} E(\phi^{-1}(x))^{(p+1)/2}\,dx = \int_S^T E(t)^{(p+1)/2}\phi'(t)\,dt \\ &\leq AE(S) = Af\big(\phi(S)\big), \quad 0 \leq S < T < +\infty. \end{aligned} \tag{2.12}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\int_s^{+\infty} f(x)^{(p+1)/2}\,dx \leq Af(s), \quad 0 \leq s < +\infty. \tag{2.13}$$

Thanks to Lemma 2.3, we deduce the desired results. □

Before stating the global existence theorem and decay property of problem (1.1), we will introduce the notion of the modified stable set. Let

$$\begin{aligned} K(u) &= ||\nabla_x u||_2^2 + \|u\|_2^2 - \|u\|_{p+1}^{p+1} \quad \text{if } \lambda \equiv 1, \\ I(u) &= ||\nabla_x u||_2^2 - \|u\|_{p+1}^{p+1} \quad \text{if } \lambda \not\equiv \text{const}, \end{aligned} \tag{2.14}$$

for $u \in H^1(\mathbb{R}^n)$. Then we define the modified stable set $\widetilde{\mathcal{W}}^*$ and $\widetilde{\mathcal{W}}^{**}$ by

$$\begin{aligned} \widetilde{\mathcal{W}}^* &\equiv \{u \in H^1(\mathbb{R}^n) \backslash K(u) > 0\} \cup \{0\} \quad \text{if } \lambda \equiv 1, \\ \widetilde{\mathcal{W}}^{**} &\equiv \{u \in H^1(\mathbb{R}^n) \backslash I(u) > 0\} \cup \{0\} \quad \text{if } \lambda \not\equiv \text{const}. \end{aligned} \tag{2.15}$$

Next, let $J(u)$ and $E(t)$ be the potential and energy associated with problem (1.1), respectively:

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla_x u\|_2^2 + \frac{1}{2}\|\lambda(x)u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \quad \text{for } u \in H^1(\mathbb{R}^n), \\ E(t) &= \frac{1}{2}\|u'\|_2^2 + J(u). \end{aligned} \tag{2.16}$$

We get the local existence solution.

Theorem 2.5. *Let $1 < p \le (n+2)/(n-2)$ ($1 < p < \infty$ if $n = 1,2$) and assume that $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and u_0 belong to the modified stable set $\widetilde{\mathcal{W}}^*$. Then there exists $T > 0$ such that the Cauchy problem (1.1) has a unique solution $u(t)$ on $\mathbb{R}^n \times [0,T)$ in the class*

$$u(t,x) \in C([0,T);H^1(\mathbb{R}^n)) \cap C^1([0,T);L^2(\mathbb{R}^n)), \tag{2.17}$$

satisfying

$$u(t) \in \widetilde{\mathcal{W}}^*, \tag{2.18}$$

and this solution can be continued in time as long as $u(t) \in \widetilde{\mathcal{W}}^*$.

When $\lambda \not\equiv$ const, we use the following theorem of local existence in the space $H^2 \times H^1$, and the decay property of the energy $E(t)$ is necessarily required for the local solution to remain in $\widetilde{\mathcal{W}}^{**}$ as $t \to \infty$; this fact of course guarantees the global existence in $H^2 \times H^1$ and by approximation, we obtain global existence in $H^1 \times L^1$.

Theorem 2.6 [15]. *Let $(u_0, u_1) \in H^2 \times H^1$. Suppose that*

$$1 \le p \le \frac{n}{n-4} \quad (1 \le \infty \text{ if } N \le 4). \tag{2.19}$$

Then under the hypotheses (2.1), (2.2), and (2.3), problem (1.1) admits a unique local solution $u(t)$ on some interval $[0,T[$, $T \equiv T(u_0,u_1) > 0$, in the class $W^{2,\infty}([0,T[;L^2) \cap W^{1,\infty}([0,T[;H^1) \cap L^\infty([0,T[;H^2)$, satisfying the finite propagation property.

Proof of Theorem 2.5 (see [15, 18]*).* Since the argument is standard, we only sketch the main idea of the proof. Let $(u_0, u_1) \in H^1 \times L^2$ and $u_0 \in \widetilde{\mathcal{W}}^*$. Then we have a unique local solution $u(t)$ for some $T > 0$. Indeed, taking suitable approximate functions f_j such that (see [20])

$$f_j(u) = f(u) \quad \text{if } |u| \le j, \qquad |f_j(u)| \le |f(u)|, \qquad |f_j(u)| \le c_j|u|, \tag{2.20}$$

problem (1.1) with $f(u) \equiv |u|^{p-1}u$ replaced by $f_j(u)$ admits a unique solution $u_j(t) \in C([0,T);H^1(\mathbb{R}^n)) \cap C^1([0,T);L^2(\mathbb{R}^n))$. Further, we can prove that $u_j(t) \in \widetilde{\mathcal{W}}^*$, $0 < t < T$, for sufficiently large j, and there exists a subsequence of $\{u_j(t)\}$ which converges to a function $\tilde{u}(t)$ in certain senses. $\tilde{u}(t)$ is, in fact, a weak solution in $C([0,T);H^1(\mathbb{R}^n)) \cap C^1([0,T);L^2(\mathbb{R}^n))$ (see [19, 20]) and such a solution is unique by Ginibre and Velo [5] and Brenner [3]. We can also construct such a solution which meets moreover the finite propagation property, if we assume that the initial data $u_0(x)$ and $u_1(x)$ are of compact support:

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \{x \in \mathbb{R}^n,\ |x| < L\}, \quad \text{for some } L > 0. \tag{2.21}$$

Applying [9, Appendix 1] of John, then the solution is also of compact support: supp $u_j(t) \subset \{x \in \mathbb{R}^n, |x| < L + t\}$. So, we have $\operatorname{supp} \tilde{u}(t) \subset \{x \in \mathbb{R}^n, |x| < L + t\}$. □

We denote the life span of the solution $u(t,x)$ of the Cauchy problem (1.1) by $T_{\max}$. First we consider the case $\lambda(x) \equiv \text{const}$ ($\lambda(x) \equiv 1$ without loss of generality). And construct a stable set in $H^1(\mathbb{R}^n)$.

Setting

$$C_0 \equiv K\left\{\frac{2(p+1)}{(p-1)}\right\}^{(p-1)/2}, \tag{2.22}$$

$$\int_0^\infty \sigma(\tau)\,d\tau = +\infty \quad \text{if } m = 1, \tag{2.23}$$

$$\int_0^\infty (1+\tau)^{-n(m-1)/2}\sigma(\tau)\,d\tau = +\infty \quad \text{if } m > 1. \tag{2.24}$$

Theorem 2.7. *Let $u(t,x)$ be a local solution of problem (1.1) on $[0,T_{\max})$ with initial data $u_0 \in \widetilde{\mathcal{W}}^*$, $u_1 \in L^2(\mathbb{R}^n)$ with sufficiently small initial energy E(0) so that*

$$C_0 E(0)^{(p-1)/2} < 1. \tag{2.25}$$

Then $T_{\max} = \infty$. Furthermore, the global solution of the Cauchy problem (1.1) has the following energy decay property. Under (2.22), (2.3), and (2.23),

$$E(t) \le E(0)\exp\left(1 - \omega\int_0^t \sigma(\tau)\,d\tau\right) \quad \forall t > 0. \tag{2.26}$$

Under (2.2), (2.3), and (2.24),

$$E(t) \le \left(\frac{C(E(0))}{\int_0^t (1+\tau)^{-n(m-1)/2}\sigma(\tau)\,d\tau}\right)^{2/(m-1)} \quad \forall t > 0. \tag{2.27}$$

Secondly, we consider the case $\lambda(x) \not\equiv \text{const}$ and we assume that

$$\frac{n+4}{n} \le p \le \frac{n}{n-2}. \tag{2.28}$$

(1) If $\sigma(t) = \mathbb{O}(\tilde{d}(t))$, where $\tilde{d}(t) = d(L+t)$.

If $m = 1$, we suppose that

$$\int_0^\infty \sigma(\tau)\,d\tau = +\infty \tag{2.29}$$

with

$$\begin{gathered}
(\tilde{d}(t))^{-(4-(n-2)(p-1))/2} \exp\left(1 - \omega \int_0^t \sigma(\tau)\,d\tau\right)^{(p-1)/2} < \infty, \\
(\tilde{d}(t))^{-1} \exp\left(\frac{1}{2} - \frac{\omega}{2} \int_0^t \sigma(\tau)\,d\tau\right) < \infty.
\end{gathered} \tag{2.30}$$

If $m > 1$, we suppose that

$$\int_0^\infty (1+\tau)^{-n(m-1)/2}\sigma(\tau)\,d\tau = \infty \tag{2.31}$$

with

$$\begin{gathered}
\frac{(\tilde{d}(t))^{-(4-(n-2)(p-1))/2}}{\left(\int_0^t (1+\tau)^{-n(m-1)/2}\sigma(\tau)\,d\tau\right)^{(p-1)/(m-1)}} < \infty, \\
\frac{(\tilde{d}(t))^{-1}}{\left(\int_0^t (1+\tau)^{-n(m-1)/2}\sigma(\tau)\,d\tau\right)^{1/(m-1)}} < \infty.
\end{gathered} \tag{2.32}$$

(2) If $\tilde{d}(t) = \mathbb{O}(\sigma(t))$.
If $m = 1$, we suppose that for some $0 \le \alpha < 1$,

$$\int_0^\infty \frac{\tilde{d}^2(\tau)}{\sigma^\alpha(\tau)}\,d\tau = +\infty \tag{2.33}$$

with

$$\begin{gathered}
(\tilde{d}(t))^{-(4-(n-2)(p-1))/2} \exp\left(1 - \omega \int_0^t \frac{\tilde{d}^2(\tau)}{\sigma^\alpha(\tau)}\,d\tau\right)^{(p-1)/2} < \infty, \\
(\tilde{d}(t))^{-1} \exp\left(\frac{1}{2} - \frac{\omega}{2} \int_0^t \frac{\tilde{d}^2(\tau)}{\sigma^\alpha(\tau)}\,d\tau\right) < \infty.
\end{gathered} \tag{2.34}$$

If $m > 1$, we suppose that for some $0 \le \alpha < 1$,

$$\int_0^\infty (1+\tau)^{-n(m-1)/2}\sigma^{-((1+\alpha)(1+m)-2)/2}(\tau)\tilde{d}^{m+1}(\tau)\,d\tau = \infty \tag{2.35}$$

with

$$\frac{\left(\tilde{d}(t)\right)^{-(4-(n-2)(p-1))/2}}{\left(\int_0^t (1+\tau)^{-n(m-1)/2}\sigma^{-((1+\alpha)(1+m)-2)/2}(\tau)\tilde{d}^{m+1}(\tau)\,d\tau\right)^{(p-1)/(m-1)}} < \infty,$$
$$\frac{\left(\tilde{d}(t)\right)^{-1}}{\left(\int_0^t (1+\tau)^{-n(m-1)/2}\sigma^{-((1+\alpha)(1+m)-2)/2}(\tau)\tilde{d}^{m+1}(\tau)\,d\tau\right)^{1/(m-1)}} < \infty. \tag{2.36}$$

We have the following theorem.

Theorem 2.8. *Let $(u_0,u_1) \in H^1 \times L^2$, $u_0 \in \widetilde{\mathcal{W}}^{**}$, and let the initial energy $E(0)$ be sufficiently small. The following cases are considered.*

(i) $\sigma(t) = \mathbb{O}(\tilde{d}(t))$.

Suppose (2.2), (2.3), (2.29), and (2.30) or (2.2), (2.3), (2.31), and (2.32). Then problem (1.1) admits a unique solution $u(t) \in C([0,\infty);H^1) \cap C^1([0,\infty);L^2)$ and has the same decay property as Theorem 2.7.

(ii) $\tilde{d}(t) = \mathbb{O}(\sigma(t))$.

Suppose (2.2), (2.3), (2.33), and (2.34) or (2.2), (2.3), (2.35), and (2.36). Then problem (1.1) admits a unique solution $u(t) \in C([0,\infty);H^1) \cap C^1([0,\infty);L^2)$. Furthermore, the global solution of the Cauchy problem (1.1) has the following energy decay property:

$$E(t) \le E(0)\exp\left(1-\omega\int_0^t \frac{\tilde{d}^2(\tau)}{\sigma^\alpha(\tau)}\,d\tau\right) \quad \forall t>0 \text{ if } m=1, \tag{2.37}$$

$$E(t) \le \left(\frac{C(E(0))}{\int_0^t (1+\tau)^{-n(m-1)/2}\sigma^{-((1+\alpha)(1+m)-2)/2}(\tau)\tilde{d}^{m+1}(\tau)\,d\tau}\right)^{2/(m-1)} \quad \forall t>0 \text{ if } m>1. \tag{2.38}$$

Remark 2.9. In Theorem 2.7, the global existence and energy decay are independent, but in Theorem 2.8, we need the estimation of the energy decay for a local solution to prove global existence.

Examples 2.10. (1) If $\sigma(t) = 1/t^\theta$, by applying Theorem 2.7 we obtain

$$\begin{aligned} &E(t) \le E(0)e^{1-\omega t^{1-\theta}} \quad \text{if } m=1,\\ &E(t) \le C(E(0))(1+t)^{-(2-n(m-1)-2\theta)/(m-1)} \quad \text{if } 1<m<1+\frac{2-2\theta}{n},\ 0<\theta<1,\\ &E(t) \le C(E(0))(\ln t)^{-2/(m-1)} \quad \text{if } m=1+\frac{2-2\theta}{n},\ 0<\theta<1. \end{aligned} \tag{2.39}$$

(2) If $\sigma(t) = 1/t^\theta \ln t \ln_2 t \cdots \ln_p t$, by applying Theorem 2.7, we obtain

$$E(t) \le E(0)\left(\ln_p t\right)^{-\omega} \quad \text{if } m=1,\ \theta=1. \tag{2.40}$$

For example, if $n(m-1)/2+\theta = 1$, that is, $1<m<1+2/n$,

$$E(t) \le C(E(0))\left(\ln_p t\right)^{-2/(m-1)}. \tag{2.41}$$

(3) If $\sigma(t) = 1/t^{\theta}$ and $d(r) = 1/r^{\gamma}$ with $\theta \geq \gamma$ by applying Theorem 2.8, we obtain

$$
\begin{aligned}
E(t) &\leq C(E(0))(1+t)^{-(2-n(m-1)-2\theta)/(m-1)} \quad \text{if } 1 < m < 1 + \frac{2-2\theta}{2\gamma+n},\ 0<\theta<1, \\
E(t) &\leq C(E(0))(\ln t)^{-2/(m-1)} \quad \text{if } m = 1 + \frac{2-2\theta}{2\gamma+n},\ 0<\theta<1.
\end{aligned}
\tag{2.42}
$$

In order to show the global existence, it suffices to obtain the a priori estimates for $E(t)$ and $\|u(t)\|_2$ in the interval of existence.

To prove Theorem 2.7 we first have the following energy identity to problem (1.1).

Lemma 2.11 (energy identity). *Let $u(t,x)$ be a local solution to problem (1.1) on $[0, T_{\max})$ as in Theorem 2.5. Then*

$$E(t) + \int_{\mathbb{R}^n}\int_0^t \sigma(s)u'(s)g(u'(s))\,ds\,dx = E(0) \tag{2.43}$$

for all $t \in [0, T_{\max})$.

Next we state several facts about the modified stable set $\widetilde{\mathcal{W}}^*$.

Lemma 2.12. *Suppose that*

$$1 < p \leq \frac{n+2}{n-2}. \tag{2.44}$$

Then

(i) *$\widetilde{\mathcal{W}}^*$ is a neighborhood of 0 in $H^1(\mathbb{R}^n)$,*

(ii) *for $u \in \widetilde{\mathcal{W}}^*$,*

$$J(u) \geq \frac{p-1}{2(p+1)}\left(\|\nabla_x u\|_2^2 + \|u\|_2^2\right). \tag{2.45}$$

Proof of Lemma 2.12. (i) From Lemma 2.1 we have

$$\|u\|_{p+1}^{p+1} \leq K\|u\|_{H^1}^{p+1} \leq K\|u\|_{H^1}^{p-1}\left(\|u\|_2^2 + \|\nabla_x u\|_2^2\right). \tag{2.46}$$

Let

$$U(0) \equiv \left\{u \in H^1(\mathbb{R}^N) \setminus \|u\|_{H^1}^{p-1} < \frac{1}{K}\right\}. \tag{2.47}$$

Then, for any $u \in U(0)\setminus\{0\}$, we deduce from (2.46) that

$$\|u\|_{p+1}^{p+1} < \|u\|_2^2 + \|\nabla_x u\|_2^2, \tag{2.48}$$

that is, $K(u) > 0$. This implies $U(0) \subset \widetilde{\mathcal{W}}^*$.

(ii) By the definition of $K(u)$ and $J(u)$, we have the identity

$$(p+1)J(u) = K(u) + \frac{(p-1)}{2}\left(\|\nabla_x u\|_2^2 + \|u\|_2^2\right). \tag{2.49}$$

Since $u \in \widetilde{\mathcal{W}}^*$, we have $K(u) \geq 0$. Therefore from (2.44) we get the desired in-equality (2.45). □

Lemma 2.13. *Let $u(t)$ be a solution to problem (1.1) on $[0, T_{\max})$. Suppose (2.44) holds. If $u_0 \in \widetilde{\mathcal{W}}^*$ and $u_1 \in L^2(\mathbb{R}^n)$ satisfy*

$$C_0 E(0)^{(p-1)/2} < 1, \tag{2.50}$$

then

(i) $u(t) \in \widetilde{\mathcal{W}}^*$ *on* $[0, T_{\max})$,

(ii) $\|u(t)\|_2 \leq I_0$ *on* $[0, T_{\max})$.

Proof of Lemma 2.13. Suppose that there exists a number $t^* \in [0, T_{\max}[$ such that $u(t) \in \widetilde{\mathcal{W}}^*$ on $[0, t^*[$ and $u(t^*) \notin \widetilde{\mathcal{W}}^*$. Then we have

$$K(u(t^*)) = 0, \quad u(t^*) \neq 0. \tag{2.51}$$

Since $u(t) \in \widetilde{\mathcal{W}}^*$ on $[0, t^*[$, it holds that

$$\frac{p-1}{2(p+1)} \left(\|\nabla_x u\|_2^2 + \|u\|_2^2 \right) \leq J(u) \leq E(t); \tag{2.52}$$

it follows from the nonincreasing of the energy that

$$\|\nabla_x u\|_2^2 + \|u\|_2^2 \leq \frac{2(p+1)}{p-1} E(0) \equiv I_0^2. \tag{2.53}$$

Hence, we obtain

$$\|u\|_2^2 \leq \frac{2(p+1)}{p-1} E(0) \equiv I_0^2 \quad \text{on } [0, t^*]. \tag{2.54}$$

Next, from Lemma 2.1 and (2.54) we have

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\leq K \|u(t)\|_{H^1(\mathbb{R}^n)}^{p+1} \\ &\leq K \|u(t)\|_{H^1(\mathbb{R}^n)}^{p-1} \left(\|\nabla_x u\|_2^2 + \|u\|_2^2 \right) \\ &\leq K I_0^{p-1} \left(\|\nabla_x u\|_2^2 + \|u\|_2^2 \right) \\ &\leq C_0 E(0)^{(p-1)/2} \left(\|u(t)\|_2^2 + \|\nabla_x u(t)\|_2^2 \right) \end{aligned} \tag{2.55}$$

for all $t \in [0, t^*]$, where C_0 is the constant defined by (2.22). Note that from (2.55) and our hypothesis

$$\eta_0 \equiv C_0 E(0)^{(p-1)/2} < 1, \tag{2.56}$$

it follows that

$$\|u(t)\|_{p+1}^{p+1} \leq (1 - \eta_0) \left(\|u(t)\|_2^2 + \|\nabla_x u(t)\|_2^2 \right). \tag{2.57}$$

Therefore, we obtain

$$K(u(t^*)) \geq \eta_0(||u(t^*)||_2^2 + ||\nabla_x u(t^*)||_2^2) \tag{2.58}$$

which contradicts (2.51). Thus, we conclude that $u(t) \in \widetilde{\mathcal{W}}^*$ on $[0, T_{\max}[$. The assertion (ii) can be obtained by the same argument as for (2.54). This completes the proof of Lemma 2.13. □

Lemma 2.14. *Under the same assumptions as in Lemma 2.13, there exists a constant M_2 depending on $\|u_0\|_{H^1}$ and $\|u_1\|_2$ such that*

$$||u(t)||_{H^1}^2 + ||u'(t)||_2^2 \leq M_2^2 \tag{2.59}$$

for all $t \in [0, T_{\max}[$.

Proof of Lemma 2.14. It follows from Lemma 2.13 that $u(t) \in \widetilde{\mathcal{W}}^*$ on $[0, T_{\max}[$. So Lemma 2.12(ii) implies that

$$J(u) \geq \frac{p-1}{2(p+1)}(||u(t)||_2^2 + ||\nabla_x u(t)||_2^2) \quad \text{on } [0, T_{\max}[. \tag{2.60}$$

Hence, from Lemma 2.11 and (2.60) we get

$$\frac{1}{2}||u'(t)||_2^2 + \frac{p-1}{2(p+1)}(\|u\|_2^2 + ||\nabla_x u(t)||_2^2) \leq E(t) \leq E(0). \tag{2.61}$$

So we get

$$||u(t)||_{H^1}^2 + ||u'(t)||_2^2 \leq M_2^2, \tag{2.62}$$

for some $M_2 > 0$.

The above inequality and the continuation principle lead to the global existence of the solution, that is, $T_{\max} = \infty$. □

Proof of the energy decay. From now on, we denote by c various positive constants which may be different at different occurrences. We multiply the first equation of (1.1) by $E^q\phi' u$, where ϕ is a function satisfying all the hypotheses of Lemma 2.4. We obtain

$$\begin{aligned}
0 &= \int_S^T E^q\phi' \int_{\mathbb{R}^n} u(u'' - \Delta u + u + \sigma(t)g(u') - |u|^{p-1}u)\,dx\,dt \\
&= \left[E^q\phi' \int_{\mathbb{R}^n} uu'\,dx\right]_S^T - \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_{\mathbb{R}^n} uu'\,dx\,dt - 2\int_S^T E^q\phi' \int_{\mathbb{R}^n} u'^2\,dx\,dt \\
&\quad + \int_S^T E^q\phi' \int_{\mathbb{R}^n}\left(u'^2 + |u|^2 + |\nabla u|^2 - \frac{2}{p+1}|u|^{p+1}\right)dx\,dt + \int_S^T E^q\phi' \int_{\mathbb{R}^n} \sigma(t)ug(u')\,dx\,dt \\
&\qquad \int_S^T E^q\phi' \int_{\mathbb{R}^n}\left(\frac{2}{p+1} - 1\right)|u|^{p+1}\,dx\,dt.
\end{aligned} \tag{2.63}$$

Since

$$\begin{aligned}\left(1-\frac{2}{p+1}\right)\int_{\mathbb{R}^n}|u|^{p+1}\,dx &\le (1-\eta_0)\frac{p-1}{p+1}\|u(t)\|^2_{H^1(\mathbb{R}^n)}\,dx\\ &\le (1-\eta_0)\frac{p-1}{p+1}\frac{2(p+1)}{p-1}E(t)\\ &= 2(1-\eta_0)E(t),\end{aligned}\tag{2.64}$$

we deduce that

$$\begin{aligned}2\eta_0\int_S^T E^{q+1}\phi'\,dt \le{}& -\left[E^q\phi'\int_{\mathbb{R}^n}uu'\,dx\right]_S^T+\int_S^T\left(qE'E^{q-1}\phi'+E^q\phi''\right)\int_{\mathbb{R}^n}uu'\,dx\,dt\\ &+2\int_S^T E^q\phi'\int_{\mathbb{R}^n}u'^2\,dx\,dt-\int_S^T E^q\phi'\int_{\mathbb{R}^n}\sigma(t)ug(u')\,dx\,dt\\ \le{}& -\left[E^q\phi'\int_{\mathbb{R}^n}uu'\,dx\right]_S^T+\int_S^T\left(qE'E^{q-1}\phi'+E^q\phi''\right)\int_{\mathbb{R}^n}uu'\,dx\,dt\\ &+2\int_S^T E^q\phi'\int_{\mathbb{R}^n}u'^2\,dx\,dt+c(\varepsilon)\int_S^T E^q\phi'\int_{|u'|\le 1}g(u')^2\,dx\,dt\\ &+\varepsilon\int_S^T E^q\phi'\int_{\mathbb{R}^n}u^2\,dx\,dt+\int_S^T E^q\phi'\int_{|u'|\ge 1}\sigma(t)ug(u')\,dx\,dt\end{aligned}\tag{2.65}$$

for every $\varepsilon>0$. Also, applying Hölder's and Young's inequalities, we have

$$\begin{aligned}&\int_S^T E^q\phi'\int_{|u'|>1}\sigma(t)ug(u')\,dx\,dt\\ &\quad\le\int_S^T E^q\phi'\sigma(t)\left(\int_\Omega|u|^{r+1}\,dx\right)^{1/(r+1)}\left(\int_{|u'|>1}|g(u')|^{(r+1)/r}\,dx\right)^{r/(r+1)}dt\\ &\quad\le c\int_S^T E^{(2q+1)/2}\phi'\sigma^{1/(r+1)}(t)\left(\int_{|u'|>1}\sigma(t)u'g(u')\,dx\right)^{r/(r+1)}dt\\ &\quad\le\int_S^T\phi'\sigma^{1/(r+1)}(t)E^{(2q+1)/2}(-E')^{r/(r+1)}\,dt\\ &\quad\le c\int_S^T\phi'\left(\sigma^{1/(r+1)}(t)E^{(2q+1)/2-r/(r+1)}\right)\left((-E')^{r/(r+1)}E^{r/(r+1)}\right)dt\\ &\quad\le c(\varepsilon')\int_S^T\phi'(-E'E)\,dt+\varepsilon'\int_S^T\phi'\sigma(t)E^{(r+1)((2q+1)/2-r/(r+1))}\,dt\\ &\quad\le c(\varepsilon')E(S)^2+\varepsilon'\sigma(0)E(0)^{(2rq-r-1)/2}\int_S^T\phi'E^{q+1}\,dt\end{aligned}\tag{2.66}$$

for every $\varepsilon' > 0$. Choosing ε and ε' small enough, we obtain

$$\begin{aligned}\int_S^T E^{q+1}\phi'\,dt &\le -\left[E^q\phi'\int_{\mathbb{R}^n} uu'\,dx\right]_S^T + \int_S^T \left(qE'E^{q-1}\phi' + E^q\phi''\right)\int_{\mathbb{R}^n} uu'\,dx\,dt\\ &\quad + \int_{|u'|\ge 1}\sigma(t)ug(u')\,dx\,dt + c\int_S^T E\phi'\int_{\mathbb{R}^n} u'^2\,dx\,dt\\ &\le cE(S) + c\int_S^T E\phi'\int_{\mathbb{R}^n} u'^2\,dx\,dt.\end{aligned} \tag{2.67}$$

Since $xg(x) \ge 0$ for all $x \in \mathbb{R}$, it follows that the energy is nonincreasing, locally absolutely continuous and $E'(t) = -\int_{\mathbb{R}^n}\sigma(t)u'g(u')\,dx$ a.c. in $\mathbb{R}_+$. □

Proof of (2.26). We consider the case $m = 1$, that is,

$$c_3|v| \le \left|g(v)\right| \le c_4|v| \quad \text{for all } |v| \le 1. \tag{2.68}$$

Then we have

$$u'^2 \le \frac{c_{13}}{\sigma(t)}u'\rho(t,u') \quad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^n, \tag{2.69}$$

where $\rho(t,s) = \sigma(t)g(s)$ for all $s \in \mathbb{R}$. Therefore we deduce from (2.67) (applied with $q = 0$) that

$$\int_S^T E(t)\phi'(t)\,dt \le CE(S) + 2C\int_S^T \phi'(t)\int_{\mathbb{R}^n}\frac{1}{\sigma(t)}u'\rho(t,u')\,dx\,dt. \tag{2.70}$$

Define

$$\phi(t) = \int_0^t \sigma(\tau)\,d\tau. \tag{2.71}$$

It is clear that ϕ is a nondecreasing function of class C^2 on $\mathbb{R}_+$. The hypothesis (2.23) ensures that

$$\phi(t) \longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty. \tag{2.72}$$

Then we deduce from (2.70) that

$$\int_S^T E(t)\phi'(t)\,dt \le CE(S) + 2C\int_S^T\int_{\mathbb{R}^n} u'\rho(t,u')\,dx\,dt \le 3CE(S), \tag{2.73}$$

and thanks to Lemma 2.4 we obtain

$$E(t) \le E(0)e^{(1-\phi(t))/(3C)}. \tag{2.74}$$

□

Proof of (2.27). Now we assume that $m > 1$ in (2.2). Define ϕ by (2.71). We apply Lemma 2.4 with $q = (m-1)/2$.

We need to estimate

$$\int_S^T E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt. \tag{2.75}$$

For $t \geq 0$, consider

$$\Omega_1 = \{x \in \mathbb{R}^n, \ |u'| \leq 1\}, \qquad \Omega_2 = \{x \in \mathbb{R}^n, \ |u'| > 1\}. \tag{2.76}$$

First we note that for every $t \geq 0$,

$$\Omega_1 \cup \Omega_2 = \mathbb{R}^n. \tag{2.77}$$

Next we deduce from (2.2) and (2.3) that for every $t \geq 0$,

(i) if $x \in \Omega_1$, then $u'^2 \leq ((1/\sigma(t))u'\rho(t,u'))^{2/(m+1)}$,

(ii) if $x \in \Omega_2$, then $u'^2 \leq (1/\sigma(t))u'\rho(t,u')$.

Hence, using Hölder's inequality, we get that

$$\begin{aligned}
&\int_S^T E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt \\
&\quad \leq 2 \int_S^T E^q \phi' \int_{\mathbb{R}^n} \frac{1}{\sigma(t)} u' \rho(t,u') \, dx \, dt + 2 \int_S^T E^q \phi' \int_{\mathbb{R}^n} \left(\frac{1}{\sigma(t)} u' \rho(t,u') \right)^{2/(m+1)} dx \, dt \\
&\quad \leq 2 \int_S^T E^q \phi' \int_{\mathbb{R}^n} \frac{1}{\sigma(t)} u' \rho(t,u') \, dx \, dt + 2 \int_S^T E^q \phi' \int_{\{|x| \leq (L+t)\}} (u' g(u'))^{2/(m+1)} dx \, dt \\
&\quad \leq 2 \int_S^T E^q \phi' \int_{\mathbb{R}^n} \frac{1}{\sigma(t)} u' \rho(t,u') \, dx \, dt \\
&\qquad + 2 \int_S^T E^q \phi' (1+t)^{n(m-1)/(m+1)} \left(\int_{\mathbb{R}^n} u' g(u') \, dx \right)^{2/(m+1)} dt \\
&\quad \leq c E(S)^{1+q} + c' \int_S^T E^q \phi' (1+t)^{n(m-1)/(m+1)} \left(\frac{-E'}{\sigma(t)} \right)^{2/(m+1)} dt \\
&\quad \leq c E(S)^{1+q} + c' \int_S^T E^q \phi' (1+t)^{n(m-1)/(m+1)} \sigma^{-2/(m+1)}(t) (-E')^{2/(m+1)} \, dt.
\end{aligned} \tag{2.78}$$

Set $\varepsilon > 0$; thanks to Young's inequality and to our definitions of p and ϕ, we obtain

$$\begin{aligned}
&\int_S^T E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt \\
&\quad \leq c E(S)^{1+q} + 2 \frac{m-1}{m+1} \varepsilon^{(m+1)/(m-1)} \int_S^T E^{1+q} (\phi')^{(m+1)/(m-1)} (1+t)^n \sigma^{-2/(m-1)} \, dt \\
&\qquad + \frac{4}{m+1} \frac{1}{\varepsilon^{(m+1)/2}} E(S).
\end{aligned} \tag{2.79}$$

We choose ϕ' such that

$$\phi'^{2/(m-1)} (1+t)^n \sigma^{-2/(m-1)} = 1, \tag{2.80}$$

so

$$\phi(t) = \int_0^t (1+s)^{-n(m-1)/2}\sigma(s)ds. \tag{2.81}$$

Then we deduce from (2.79) that

$$\int_S^T E^{1+q}\phi'\,dt \le 2CE(S), \tag{2.82}$$

and thanks to Lemma 2.4 (applied with $c = 0$) we obtain

$$E(t) \le \frac{C}{\phi(t)^{2/(m-1)}}. \tag{2.83}$$

□

Proof of Theorem 2.8. First, we see that if $u \in \widetilde{\mathcal{W}}^{**}$, then

$$\|\nabla_x u\|_2^2 + 2\int_{\mathbb{R}^n} F(u(t))\,dx \ge \|\nabla_x u\|_2^2 - \frac{2}{p+1}\|u(t)\|_{p+1}^{p+1} \ge \frac{p-1}{p+1}\|\nabla_x u\|_2^2. \tag{2.84}$$

In the proof, we often use the following inequality:

$$\|u(t)\|_2 \le \frac{1}{\tilde{d}(t)}\|\lambda(x)u(t)\|_2. \tag{2.85}$$

Now, we assume that $I(u_0) > (1/2)\|\nabla_x u_0\|_2^2$. Then

$$I(u(t)) \ge \frac{1}{2}\|\nabla_x u(t)\|_2^2 \tag{2.86}$$

for some interval near $t = 0$. As long as (2.86) holds, we have $J(t) \equiv I(u(t))$. Thus

$$\begin{aligned}
2\eta_0 \int_S^T E^{q+1}\phi'\,dt &\le -\left[E^q\phi' \int_{\mathbb{R}^n} uu'\,dx\right]_S^T + \int_S^T \left(qE'E^{q-1}\phi' + E^q\phi''\right)\int_{\mathbb{R}^n} uu'\,dx\,dt \\
&\quad + 2\int_S^T E^q\phi' \int_{\mathbb{R}^n} u'^2\,dx\,dt - \int_S^T E^q\phi' \int_{\mathbb{R}^n} \sigma(t)ug(u')\,dx\,dt \\
&\le -\left[E^q\phi' \int_{\mathbb{R}^n} uu'\,dx\right]_S^T + \int_S^T \left(qE'E^{q-1}\phi' + E^q\phi''\right)\int_{\mathbb{R}^n} uu'\,dx\,dt \\
&\quad + 2\int_S^T E^q\phi' \int_{\mathbb{R}^n} u'^2\,dx\,dt + c(\varepsilon)\int_S^T E^q\phi' \int_{|u'|\le 1} \left(\frac{\sigma(t)}{\lambda(x)}\right)^2 g(u')^2\,dx\,dt \\
&\quad + \varepsilon \int_S^T E^q\phi' \int_{\mathbb{R}^n} \lambda^2(x)u^2\,dx\,dt + \int_S^T E^q\phi' \int_{|u'|\ge 1} \sigma(t)ug(u')\,dx\,dt.
\end{aligned} \tag{2.87}$$

If $\sigma(t) = \mathbb{O}(\tilde{d}(t))$, that is, $\sigma(t) \to 0$ as $t \to \infty$ more rapidly than $\tilde{d}(t)$, we find the same results of asymptotic behaviour as in Theorem 2.7.

If $\tilde{d}(t) = \mathbb{O}(\sigma(t))$, so, we obtain

$$\begin{aligned}
&\int_S^T E^{q+1}\phi'\,dt \le -\left[E^q\phi'\int_{\mathbb{R}^n} uu'\,dx\right]_S^T + \int_S^T \left(qE'E^{q-1}\phi' + E^q\phi''\right)\int_{\mathbb{R}^n} uu'\,dx\,dt \\
&\int_S^T E^q\phi'\int_{|u'|\ge 1} u'^2\,dx\,dt \int_S^T E^q\phi'\int_{|u'|\ge 1}\sigma(t)ug(u')\,dx\,dt \\
&\qquad + c(\varepsilon)\int_S^T E^q\phi'\int_{|u'|\le 1}\left(\frac{\sigma(t)}{\lambda(x)}\right)^2 |u'|^2\,dx\,dt.
\end{aligned} \tag{2.88}$$

We consider the case $m = 1$. Thus under (2.2) and (2.3), we have

$$\int_S^T E^q\phi'\int_{\mathbb{R}^n}\left(\frac{\sigma(t)}{\lambda(x)}\right)^2 u'^2\,dx\,dt \le C\int_S^T E^q\phi'\int_{\mathbb{R}^n}\left(\frac{\sigma^\alpha(t)}{\tilde{d}^2(t)}\right)\sigma(t)u'g(u')\,dx\,dt \tag{2.89}$$

for all $0 \le \alpha < 1$. We choose

$$\phi(t) = \int_0^t \frac{\tilde{d}^2(s)}{\sigma^\alpha(s)}\,ds. \tag{2.90}$$

It is clear that ϕ is a nondecreasing function of class C^2 on $\mathbb{R}_+$. Hypothesis (2.33) ensures that

$$\phi(t) \longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty. \tag{2.91}$$

By (2.85), the definition of E, and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
-\left[E^q\phi'\int_{\mathbb{R}^n} uu'\,dx\right]_S^T &= E^q(S)\phi'(S)\int_{\mathbb{R}^n} u(S)u'(S)\,dx - E^q(T)\phi'(T)\int_{\mathbb{R}^n} u(T)u'(T)\,dx \\
&\le CE^{q+1}(S)\left[\frac{\phi'(S)}{\tilde{d}(S)} + \frac{\phi'(T)}{\tilde{d}(T)}\right] \le CE^{q+1}(S), \\
\int_S^T \left(qE'E^{q-1}\phi' + E^q\phi''\right)\int_{\mathbb{R}^n} uu'\,dx\,dt &\le \int_S^T q|E'|E^q\frac{\phi'(t)}{\tilde{d}(t)}\,dt + \int_S^T E^{q+1}\frac{|\phi''(t)|}{\tilde{d}(t)}\,dt
\end{aligned} \tag{2.92}$$

when we have (in the case $m = 1$)

$$\phi' = \frac{\tilde{d}^2(t)}{\sigma^\alpha(t)}, \qquad \phi''(t) = \frac{2\tilde{d}(t)\tilde{d}'(t)}{\sigma^\alpha(t)} - \alpha\frac{\sigma'(t)\tilde{d}^2(t)}{\sigma^{\alpha+1}(t)}. \tag{2.93}$$

So

$$\left|\frac{\phi''(t)}{\tilde{d}(t)}\right| \le -\frac{2\tilde{d}'(t)}{\tilde{d}^\alpha(t)}\frac{\tilde{d}^\alpha(t)}{\sigma^\alpha(t)} - \alpha\frac{\tilde{d}(t)}{\sigma(t)}\frac{\sigma'(t)}{\sigma^\alpha(t)}, \tag{2.94}$$

$\tilde{d}(t)/\sigma(t)$ is bounded, so we obtain

$$\begin{aligned}\int_S^T E^{q+1}\frac{|\phi''(t)|}{\tilde{d}(t)}\,dt &\le -E^{q+1}(S)\Big[\tilde{d}^{1-\alpha}(t)+\sigma^{1-\alpha}(t)\Big]_S^T \\ &\le E^{q+1}(S)\big(\tilde{d}^{1-\alpha}(t)+\sigma^{1-\alpha}(t)\big) \\ &\le CE^{q+1}(S).\end{aligned} \tag{2.95}$$

Then we deduce from (2.88) that

$$\int_S^T E\phi'\,dt \le CE(S)+2C\int_S^T\int_{\mathbb{R}^n}\sigma(t)u'g(u')\,dx\,dt \le 3CE(S), \tag{2.96}$$

and thanks to Lemma 2.4 we obtain

$$E(t)\le E(0)e^{(1-\phi(t))/3C}. \tag{2.97}$$

Using the condition that $\tilde{d}(t)=\mathbb{O}(\sigma(t))$ and using Hölder's inequality, we get that

$$\begin{aligned}&\int_S^T E^q\phi'\int_{|u'|\le 1}\frac{\sigma^2(t)}{\lambda^2(x)}u'^2\,dx\,dt \\ &\quad\le 2\int_S^T E^q\phi'\int_{\mathbb{R}^n}\frac{\sigma^2(t)}{\lambda^2(x)}\left(\frac{1}{\sigma(t)}\sigma(t)u'g(u')\right)^{2/(m+1)}dx\,dt \\ &\quad\le 2\int_S^T E^q\phi'\int_{\{|x|\le L+t\}}\frac{\sigma^2(t)}{\lambda^2(x)}\left(\frac{1}{\sigma(t)}\sigma(t)u'g(u')\right)^{2/(m+1)}dx\,dt \\ &\quad\le 2\int_S^T E^q\phi'\left(\frac{\sigma(t)}{\tilde{d}(t)}\right)^2(1+t)^{n(m-1)/(m+1)}\left(\int_{\mathbb{R}^n}u'g(u')\,dx\right)^{2/(m+1)}dt \\ &\quad\le c'\int_S^T E^q\phi'\left(\frac{\sigma(t)}{\tilde{d}(t)}\right)^2(1+t)^{n(m-1)/(m+1)}\left(\frac{-E'}{\sigma(t)}\right)^{2/(m+1)}dt \\ &\quad\le c'\int_S^T E^q\phi'\left(\frac{\sigma(t)}{\tilde{d}(t)}\right)^2(1+t)^{n(m-1)/(m+1)}\sigma^{-2/(m+1)}(t)(-E')^{2/(m+1)}\,dt \\ &\quad\le c'\int_S^T E^q\phi'\frac{\sigma^{\alpha+1}(t)}{\tilde{d}^2(t)}(1+t)^{n(m-1)/(m+1)}\sigma^{-2/(m+1)}(t)(-E')^{2/(m+1)}\,dt.\end{aligned} \tag{2.98}$$

Set $\varepsilon>0$; thanks to Young's inequality and to our definitions of p and ϕ, we obtain

$$\begin{aligned}&\int_S^T E^q\phi'\int_{\mathbb{R}^n}\left(\frac{\sigma(t)}{\lambda(x)}\right)^2u'^2\,dx\,dt \\ &\quad\le cE(S)^{1+q}+2\frac{m-1}{m+1}\varepsilon^{(m+1)/(m-1)} \\ &\qquad\times\int_S^T E^{1+q}(\phi')^{(m+1)/(m-1)}\frac{\sigma^{(1+\alpha)(m+1)/(m-1)}(t)}{\tilde{d}^{2(m+1)/(m-1)}(t)}(1+t)^n\sigma^{-2/(m-1)}\,dt+\frac{4}{m+1}\frac{1}{\varepsilon^{(m+1)/2}}E(S).\end{aligned} \tag{2.99}$$

We choose ϕ such that

$$\phi(t) = \int_0^t (1+\tau)^{-n(m-1)/2}\sigma^{-((1+\alpha)(m+1)-2)/2}(\tau)\tilde{d}^{m+1}(\tau)\,d\tau. \tag{2.100}$$

It is clear that ϕ is a nondecreasing function of class C^2 on $\mathbb{R}_+$. The hypothesis (2.35) ensures that $\phi(t) \to +\infty$ as $t \to +\infty$. By (2.85), the definition of E, and the Cauchy-Schwartz inequality we have (2.92) when we have (the case $m > 1$)

$$\begin{aligned}
\phi' &= (1+t)^{-n(m-1)/2}\sigma^{-((1+\alpha)(m+1)-2)/2}(t)\tilde{d}^{m+1}(t),\\
\phi''(t) &= -\frac{n(m-1)}{2}(1+t)^{-n(m-1)/2-1}\sigma^{-((1+\alpha)(m+1)-2)/2}(t)\tilde{d}^{m+1}(t)\\
&\quad + (1+t)^{-n(m-1)/2}\bigg(-\frac{(1+\alpha)(m+1)-2}{2}\sigma^{-(1+\alpha)(m+1)/2}(t)\sigma'(t)\tilde{d}^{m+1}(t)\\
&\quad + (m+1)\tilde{d}^{m}(t)\tilde{d}'(t)\sigma^{-((1+\alpha)(m+1)-2)/2}(t)\bigg).
\end{aligned} \tag{2.101}$$

Thus

$$\begin{aligned}
\left|\frac{\phi''(t)}{\tilde{d}(t)}\right| &\le C\frac{\tilde{d}^m(t)}{\sigma^m(t)}\sigma^{(1-\alpha)(m+1)/2}(t) - C'\frac{\tilde{d}^m(t)}{\sigma^m(t)}\frac{\sigma'(t)}{\sigma^{((1+\alpha)(m+1)-2m)/2}(t)}\\
&\quad - C''\frac{\tilde{d}^{((1+\alpha)(m+1)-2)/2}(t)}{\sigma^{((1+\alpha)(m+1)-2)/2}(t)}\frac{\tilde{d}'(t)}{\tilde{d}^{((1+\alpha)(m+1)-2m)/2}(t)},
\end{aligned} \tag{2.102}$$

$\tilde{d}(t)/\sigma(t)$ is bounded, so we obtain

$$\begin{aligned}
\int_S^T E^{q+1}\frac{|\phi''(t)|}{\tilde{d}(t)}\,dt &\le -E^{q+1}(S)\Big[\tilde{d}^{(1-\alpha)(m+1)/2}(t) + \sigma^{(1-\alpha)(m+1)/2}(t)\Big]_S^T\\
&\le E^{q+1}(S)\big(\tilde{d}^{(1-\alpha)(m+1)/2(S)+\sigma^{(1-\alpha)(m+1)/2}(S)}\big)\\
&\le CE^{q+1}(S).
\end{aligned} \tag{2.103}$$

We deduce from this choice

$$\int_S^T E^{1+q}\phi'\,dt \le 2CE(S), \tag{2.104}$$

and thanks to Lemma 2.4 (applied with $c = 0$), we obtain

$$E(t) \le \frac{C(E(0))}{\phi^{2/(m-1)}}. \tag{2.105}$$

Since $u_0 \in \widetilde{\mathcal{W}}^{**}$ and $\widetilde{\mathcal{W}}^{**}$ is an open set, putting

$$T_1 = \sup\{t \in [0,+\infty) : u(s) \in \widetilde{\mathcal{W}}^{**} \text{ for } 0 \le s \le t\}, \tag{2.106}$$

we see that $T_1 > 0$ and $u(t) \in \widetilde{\mathcal{W}}^{**}$ for $0 \le t < T_1$. If $T_1 < T_{\max} < \infty$, where $T_{\max}$ is the lifespan of the solution, then $u(T_1) \in \partial\widetilde{\mathcal{W}}^{**}$; that is,

$$I(u(T_1)) = 0, \quad u(T_1) \neq 0. \tag{2.107}$$

We see from Lemma 2.2 and (2.85) that

$$\begin{aligned}\|u(t)\|_{p+1}^{p+1} &\le C\|u(t)\|_2^{(4-(n-2)(p-1))/2}\|\nabla_x u(t)\|_2^{n(p-1)/2}\\ &\le \big(\tilde{d}(t)\big)^{-(4-(n-2)(p-1))/2}E^{(p-1)/2}\|\nabla_x u(t)\|_2^2\\ &\le B(t)\|\nabla_x u(t)\|_2^2\end{aligned} \tag{2.108}$$

for $0 \le t \le T_1$, where we have used the assumption $p \ge (n+4)/n$ and

$$B(t) = \frac{C(E(0))\tilde{d}^{-(4-(n-2)(p-1))/2}(t)}{\Big(\int_0^t (1+\tau)^{-n(m-1)/2}\sigma^{-((1+\alpha)(m+1)-2)/2}(\tau)\tilde{d}^{m+1}(\tau)\,d\tau\Big)^{(p-1)/(m-1)}}. \tag{2.109}$$

Next, we put

$$T_2 \equiv \sup\Big\{t \in [0,+\infty) : B(s) < \frac{1}{2} \text{ for } 0 \le s < t\Big\}, \tag{2.110}$$

and then we see that $T_2 > 0$ and $T_2 = T_1$ because $B(t) < 1/2$ by the condition that $E(0)$ is small. Then

$$I(u(t)) \ge \|\nabla_x u(t)\|_2^2 - B(t)\|\nabla_x u(t)\|_2^2 \ge \frac{1}{2}\|\nabla_x u(t)\|_2^2 \tag{2.111}$$

for $0 \le t \le T_1$. Moreover, (2.107) and (2.111) imply

$$K(u(T_1)) \ge \frac{1}{2}\|\nabla_x u(T_1)\|_2^2 > 0, \tag{2.112}$$

which is a contradiction, and hence, it might be $T_1 = T_{\max}$. Therefore, (2.105) holds true for $0 \le T \le T_{\max}$. To prove global existence in $H^2 \cap H^1$, we need to derive the estimates for second derivatives of $u(t)$ on the basis of the energy estimate of $E(t)$, we utilize the differentiated equation

$$u_{ttt} - \Delta_x u' + \lambda^2(x)u' + \sigma'(t)g(u') + \sigma(t)g(u')u'' + f'(u)u' = 0, \tag{2.113}$$

where $f(u) = |u|^{p-1}u$. Multiplying (2.113) by u'', we have

$$\begin{aligned}&\frac{d}{dt}E_2(t) + 2\sigma(t)\int_{\mathbb{R}^n} g'(u')|u''(t)|^2\,dx\\ &\quad\le 2\int_{\mathbb{R}^n}|f'(u)||u'(t)||u''(t)|\,dx + 2|\sigma'(t)|\int_{\mathbb{R}^n}|g(u')||u''(t)|\,dx,\end{aligned} \tag{2.114}$$

where we set

$$E_2(t) = \|u''(t)\|_2^2 + \|\nabla_x u(t)\|_2^2 + \|\lambda u'(t)\|_2^2. \tag{2.115}$$

By (2.2) and (2.3), we have

$$\begin{aligned}\int_{\mathbb{R}^n}|g(u')|^2\,dx &\le C\int_{|u'|\le 1}|u'|^{2/m}\,dx + C'\int_{|u'|\ge 1}|u'|^{2r}\,dx\\ &\le C(L+t)^{n(m-1)/m}E^{1/m} + C'E^{(2-(n-2)(r-1))/2}E_2^{n(r-1)/2}\end{aligned} \tag{2.116}$$

and by Lemma 2.2

$$\begin{aligned} 2\int_{\mathbb{R}^n} |f'(u)|\,|u'(t)|\,|u''(t)|\,dx &\le C\|u(t)\|_{n(p-1)}^{p-1}\|u'(t)\|_{2n/(n-2)}\|u''(t)\|_2 \\ &\le c\|u(t)\|_{n(p-1)}^{p-1}\|\nabla_x u'(t)\|_2\|u''(t)\|_2 \\ &\le c\|u(t)\|_{n(p-1)}^{p-1}E_2(t). \end{aligned} \tag{2.117}$$

Since $(n+2)/n \le p \le n/(n-2)$, then

$$\|u(t)\|_{n(p-1)}^{p-1} \le \tilde{d}^{-(2-(n-2)(p-1))/2}E^{(p-1)/2}. \tag{2.118}$$

Thus, we have

$$\begin{aligned} \frac{d}{dt}E_2 &\le \tilde{d}^{-(2-(n-2)(p-1))/2}E^{(p-1)/2}E_2(t) \\ &\quad + 2|\sigma'(t)|\left((L+t)^{n(m-1)/m}E^{1/m} + CE^{(2-(n-2)(r-1))/2}E_2^{n(r-1)/2}\right)^{1/2}E_2^{1/2}(t). \end{aligned} \tag{2.119}$$

We have also, applying Young inequality,

$$E^{(2-(n-2)(r-1))/2}E_2^{n(r-1)/2} \le CE^{(2-(n-2)(r-1))/(2-n(r-1))}(t) + E_2(t), \tag{2.120}$$

hence, we deduce that

$$\frac{d}{dt}E_2(t) \le C(t)\left(1+E_2(t)\right). \tag{2.121}$$

So, we obtain

$$E_2(t) \le C' e^{\int_0^t C(s)\,ds}. \tag{2.122}$$

From (2.122) and the first equation of problem (1.1), we also prove easily that

$$\|\Delta_x u(t)\|_2 \le C'(t) < \infty \tag{2.123}$$

for all $t \ge 0$. Indeed, we have

$$\|\Delta_x u(t)\|_2 \le \|u''(t)\|_2 + \|\lambda^2 u(t)\|_2 + \sigma(t)\|g(u'(t))\|_2 + \|f(u)\|_2 \tag{2.124}$$

and also we have

$$\|f(u)\|_2^2 \le C\|u(t)\|_{2p}^{2p} \le C\|u(t)\|_{n(p-1)/2}^{2(p-1)}\|\Delta_x u(t)\|_2^2. \tag{2.125}$$

Here, to check the last inequality of (2.125), we note that if $(n+4)/n \le p \le (n+2)/(n-2)$, then

$$\begin{aligned} C\|u(t)\|_{n(p-1)/2}^{2(p-1)} &\le C\|u(t)\|_2^{4-(n-2)(p-1)}\|\nabla_x u(t)\|_2^{n(p-1)-4} \\ &\le \left(\tilde{d}(t)\right)^{4-(n-2)(p-1)}E^{p-1}(t) < \frac{1}{2}. \end{aligned} \tag{2.126}$$

Thus the solution in the sense of Theorem 2.6 exists globally in time t under the assumption $(u_0, u_1) \in H^2 \times H^1$.

When $(u_0, u_1) \in H^1 \times L^2$ we approximate (u_0, u_1) by $(u_0^k, u_1^k) \in H^2 \times H^1$, $k = 1, 2, \ldots$, in the topologies of $H^1 \times L^2$, which satisfy

$$\operatorname{supp} u_0^k \cap \operatorname{supp} u_1^k \subset \{x \in \mathbb{R}^n \mid |x| \le L\}. \tag{2.127}$$

Since $\lim_{k\to\infty}(u_0^k, u_1^k) = (u_0, u_1)$ in $H^1 \times L^2$, then for these initial data problem (1.1) has global solutions $u^k \in W^{2,\infty}_{\mathrm{loc}}([0,\infty); L^2) \cap W^{1,\infty}_{\mathrm{loc}}([0,\infty); H^1) \cap L^{\infty}_{\mathrm{loc}}([0,\infty); H^2)$, which satisfy (2.122) and (2.123). We can easily see that $\{u^k(t)\}$ converges uniformly on each compact interval $[0, T]$, $T > 0$. Uniqueness follow from a standard argument. The proof of Theorem 2.8 is now completed. □

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