EXPONENTIAL DICHOTOMY FOR EVOLUTION FAMILIES ON THE REAL LINE

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We give necessary and sufficient conditions for uniform exponential dichotomy of evolution families in terms of the admissibility of the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$. We show that the admissibility of the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is equivalent to the uniform exponential dichotomy of an evolution family if and only if $p \geq q$. As applications we obtain characterizations for uniform exponential dichotomy of semigroups.

1. Introduction

Exponential dichotomy is one of the most important asymptotic properties of evolution equations (see [1–5, 7–10, 12, 15, 19–25]). In the last few years new concepts of exponential dichotomy have been introduced and characterized, using discrete and continuous-time methods.

Integral equations have proved to be significant tools in the study of the asymptotic behaviour of C_0-semigroups, evolution families, and linear skew-product flows, respectively (see [7–10, 19–21, 23, 24]). For an evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in J, t\geq s}$, one considered the integral equation

$$f(t) = U(t,s)f(s) + \int_s^t U(t,\tau)v(\tau)\,d\tau, \quad t \geq s,\ t,s \in J, \qquad (\widetilde{E}_{\mathcal{U}})$$

where $J \in \{\mathbb{R}_+, \mathbb{R}\}$. In case $J = \mathbb{R}_+$, an important result has been proved by Van Minh et al. [24] and it is given by the following.

Theorem 1.1. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family such that for every $x \in X$ the mapping $(t,s) \mapsto U(t,s)x$ is continuous. Then, $\mathcal{U}$ is uniformly exponentially dichotomic if and only if for every $v \in C_0(\mathbb{R}_+,X)$ there is $f \in C_0(\mathbb{R}_+,X)$ such that the pair (f,v) verifies $(\widetilde{E}_{\mathcal{U}})$ and the subspace $Y_1 = \{x \in X : \sup_{t\geq 0} \|U(t,0)x\| < \infty\}$ is closed and complemented in X.*

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Theorem 1.1 has been generalized for the case of evolution families with nonuniform exponential growth in [8]. There we have proved that in the nonuniform case, the solvability in $C_0(\mathbb{R}_+,X)$ of $(\widetilde{E}_{\mathcal{U}})$ implies the nonuniform exponential dichotomy of the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$. The discrete-time version of Theorem 1.1 has been obtained in [9] for the case of discrete and continuous evolution families. Characterizations for uniform exponential dichotomy of evolution families on the half-line with L^p-spaces were obtained in [19, 23].

For the case $J = \mathbb{R}$, a significant result has been obtained by Latushkin et al. [7], as shown in the following.

Theorem 1.2. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family such that for every $x \in X$ the mapping $(t,s) \mapsto U(t,s)x$ is continuous, and let $\mathcal{F}(\mathbb{R},X)$ be one of the spaces $C_b(\mathbb{R},X)$, $C_0(\mathbb{R},X)$ or $L^p(\mathbb{R},X)$, $(p \in [1,\infty))$. Then, $\mathcal{U}$ is uniformly exponentially dichotomic if and only if for every $v \in \mathcal{F}(\mathbb{R},X)$ there is a unique $f \in \mathcal{F}(\mathbb{R},X)$ such that the pair (f,v) verifies $(\widetilde{E}_{\mathcal{U}})$.*

The main tool in [7] was the use of the evolution semigroup associated to $\mathcal{U}$. Theorem 1.2 has been generalized in [10], where pointwise and global exponential dichotomy of a linear skew-product flow $\pi = (\Phi,\sigma)$ is expressed in terms of the unique solvability in $C_0(\mathbb{R},X)$ of an associated integral equation:

$$f(t) = \Phi(\sigma(\theta,s),t-s)f(s) + \int_s^t \Phi(\sigma(\theta,\tau),t-\tau)v(\tau)d\tau, \quad t \geq s. \tag{E_π}$$

The purpose of the present paper is to give general characterizations for uniform exponential dichotomy of evolution families on the real line. The proofs are direct, the methods being based on input-output techniques, on the use of some specific operators associated to the integral equation $(\widetilde{E}_{\mathcal{U}})$, and on the properties of certain subspaces related to the evolution family. We will obtain that the admissibility of the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$, with $p,q \in [1,\infty)$, is a sufficient condition for uniform exponential dichotomy of evolution families, and it becomes necessary for $p \geq q$.

Finally, we apply our results in order to obtain necessary and sufficient conditions for uniform exponential dichotomy of a C_0-semigroup in terms of the unique solvability of an integral equation associated to it.

2. Evolution families

Let X be a real or complex Banach space. The norm on X and on $\mathcal{B}(X)$, the Banach algebra of all bounded linear operators on X, will be denoted by $\|\cdot\|$.

Definition 2.1. A family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ of bounded linear operators on X is called an *evolution family* if the following properties hold:

(i) $U(t,t) = I$, for all $t \in \mathbb{R}$;
(ii) $U(t,s)U(s,t_0) = U(t,t_0)$, for all $t \geq s \geq t_0$;
(iii) for every $x \in X$ and every t,t_0, the mapping $s \mapsto U(s,t_0)x$ is continuous on $[t_0,\infty)$ and the mapping $s \mapsto U(t,s)x$ is continuous on $(-\infty,t]$;

(iv) there exist $M \geq 1$ and $\omega > 0$ such that

$$||U(t,t_0)|| \leq Me^{\omega(t-t_0)}, \quad \forall t \geq t_0. \tag{2.1}$$

Definition 2.2. An evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is said to be *uniformly exponentially dichotomic* if there are a family of projections $\{P(t)\}_{t\in\mathbb{R}}$ and two constants $K \geq 1$ and $\nu > 0$ such that

(i) $U(t,t_0)P(t_0) = P(t)U(t,t_0)$, for all $t \geq t_0$;
(ii) $\|U(t,t_0)x\| \leq Ke^{-\nu(t-t_0)}\|x\|$, for all $x \in \operatorname{Im} P(t_0)$ and all $t \geq t_0$;
(iii) $\|U(t,t_0)y\| \geq (1/K)e^{\nu(t-t_0)}\|y\|$, for all $y \in \operatorname{Ker} P(t_0)$ and all $t \geq t_0$;
(iv) the restriction $U(t,t_0)_| : \operatorname{Ker} P(t_0) \to \operatorname{Ker} P(t)$ is an isomorphism, for all $t \geq t_0$.

Lemma 2.3. *If the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is uniformly exponentially dichotomic relative to the family of projections $\{P(t)\}_{t\in\mathbb{R}}$, then $\sup_{t\in\mathbb{R}} \|P(t)\| < \infty$ and for every $x \in X$, the mapping $t \mapsto P(t)x$ is continuous.*

Proof. This is a simple exercise. □

Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on X and let $p \in [1,\infty)$. For every $t_0 \in \mathbb{R}$, we consider the linear subspace

$$X_1(t_0) = \left\{x \in X : \int_{t_0}^{\infty} ||U(t,t_0)x||^p dt < \infty\right\}. \tag{2.2}$$

We denote by $\mathcal{F}_{\mathcal{U}}(t_0)$ the set of all functions $\varphi : \mathbb{R}_- \to X$ with the property that $\varphi(t) = U(t+t_0,s+t_0)\varphi(s)$, for all $s \leq t \leq 0$.

Remark 2.4. If $\varphi \in \mathcal{F}_{\mathcal{U}}(t_0)$, then φ is continuous on $\mathbb{R}_-$.

For every $t_0 \in \mathbb{R}$, we denote by $X_2(t_0)$ the linear space of all $x \in X$ with the property that there is a function $\varphi_x \in \mathcal{F}_{\mathcal{U}}(t_0)$ such that $\varphi_x(0) = x$ and $\int_{-\infty}^{0} \|\varphi_x(t)\|^p dt < \infty$.

Lemma 2.5. *If $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is an evolution family, then $U(t,t_0)X_k(t_0) \subset X_k(t)$, for all $t \geq t_0$ and all $k \in \{1,2\}$.*

Proof. This is immediate. □

Proposition 2.6. *If the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is uniformly exponentially dichotomic relative to the family of projections $\{P(t)\}_{t\in\mathbb{R}}$, then $X_1(t_0) = \operatorname{Im} P(t_0)$ and $X_2(t_0) = \operatorname{Ker} P(t_0)$, for every $t_0 \in \mathbb{R}$.*

Proof. Let $M \geq 1$, $\omega > 0$ be given by Definition 2.1 and let $K \geq 1$, $\nu > 0$ be given by Definition 2.2. Let $t_0 \in \mathbb{R}$.

It is easy to see that $\operatorname{Im} P(t_0) \subset X_1(t_0)$. If $x \in X_1(t_0)$, let $\alpha_x := (\int_{t_0}^{\infty} \|U(t,t_0)x\|^p dt)^{1/p}$. For $\tau \geq t_0 + 1$, from

$$||U(\tau,t_0)x|| \leq Me^{\omega}||U(t,t_0)x||, \quad \forall t \in [\tau-1,\tau], \tag{2.3}$$

it follows that

$$||U(\tau,t_0)x|| \le Me^{\omega}\alpha_x, \quad \forall \tau \ge t_0+1. \tag{2.4}$$

This implies that $q_x := \sup_{t\ge t_0} \|U(t,t_0)x\| < \infty$. Then from

$$\begin{aligned} ||x-P(t_0)x|| &\le Ke^{-\nu(t-t_0)}||U(t,t_0)(I-P(t_0))x|| \\ &\le Ke^{-\nu(t-t_0)}(q_x+K||P(t_0)x||), \quad \forall t \ge t_0, \end{aligned} \tag{2.5}$$

we obtain that $x \in \operatorname{Im} P(t_0)$.

If $x \in \operatorname{Ker} P(t_0)$, we define $\psi_x : \mathbb{R}_- \to X$, $\psi_x(t) = U(t_0,t_0+t)_{|}^{-1}x$, where for every $t \le 0$, $U(t_0,t_0+t)_{|}^{-1}$ denotes the inverse of the operator $U(t_0,t_0+t)_{|} : \operatorname{Ker} P(t_0+t) \to \operatorname{Ker} P(t_0)$. Then, $\psi_x(0) = x$, $\psi_x \in \mathcal{F}_{\mathcal{U}}(t_0)$, and

$$||\psi_x(t)|| \le Ke^{\nu t}\|x\|, \quad \forall t \le 0, \tag{2.6}$$

so $x \in X_2(t_0)$.

Let $x \in X_2(t_0)$. Then there is $\varphi_x \in \mathcal{F}_{\mathcal{U}}(t_0)$ such that

$$\varphi_x(0) = x, \qquad \lambda_x := \left(\int_{-\infty}^{0} ||\varphi_x(t)||^p dt\right)^{1/p} < \infty. \tag{2.7}$$

Let $t \le 0$. From

$$\varphi_x(t) = U(t+t_0,s+t_0)\varphi_x(s), \quad \forall s \in [t-1,t], \tag{2.8}$$

it follows that

$$||\varphi_x(t)|| \le Me^{\omega}\lambda_x, \quad \forall t \le 0. \tag{2.9}$$

Then from

$$\begin{aligned} ||P(t_0)x|| &= ||U(t_0,t_0+t)P(t_0+t)\varphi_x(t)|| \le Ke^{\nu t}||P(t_0+t)\varphi_x(t)|| \\ &\le KMe^{\omega}\lambda_x \sup_{s\in\mathbb{R}}||P(s)||e^{\nu t}, \quad \forall t \le 0, \end{aligned} \tag{2.10}$$

it follows that $P(t_0)x = 0$, so $x \in \operatorname{Ker} P(t_0)$. □

Remark 2.7. If an evolution family $\mathcal{U} = \{U(t,s)\}_{t\ge s}$ is uniformly exponentially dichotomic with respect to a family of projections, then according to the above result this family of projections is uniquely determined.

3. Exponential dichotomy and admissibility of the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ for evolution families

Let X be a Banach space and let $\mathcal{H}(\mathbb{R},X)$ be the space of all Bochner measurable functions $v : \mathbb{R} \to X$, identifying the functions which are equal almost everywhere. For every

$p \in [1,\infty)$, the linear space

$$L^p(\mathbb{R},X) = \left\{ v \in \mathcal{H}(\mathbb{R},X) : \int_{-\infty}^{\infty} ||v(\tau)||^p d\tau < \infty \right\} \tag{3.1}$$

is a Banach space with respect to the norm

$$\|v\|_p := \left(\int_{-\infty}^{\infty} ||v(\tau)||^p d\tau \right)^{1/p}. \tag{3.2}$$

Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on X and let $p,q \in [1,\infty)$. We consider the integral equation

$$f(t) = U(t,s)f(s) + \int_s^t U(t,\tau)v(\tau)d\tau, \quad \forall t \geq s, \tag{$E_{\mathcal{U}}$}$$

with $f \in L^p(\mathbb{R},X)$ and $v \in L^q(\mathbb{R},X)$.

Definition 3.1. The pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is said to be *admissible* for the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ if for every $v \in L^q(\mathbb{R},X)$ there is a unique $f \in L^p(\mathbb{R},X)$ such that the pair (f,v) verifies $(E_{\mathcal{U}})$.

If the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is admissible for the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$, then it makes sense to define the operator

$$\Gamma : L^q(\mathbb{R},X) \longrightarrow L^p(\mathbb{R},X), \quad \Gamma v = f. \tag{3.3}$$

It is easy to see that Γ is linear and it is closed. It follows that Γ is bounded, so there is $\gamma > 0$ such that $\|\Gamma v\|_p \leq \gamma \|v\|_q$, for all $v \in L^q(\mathbb{R},X)$.

Proposition 3.2. *If the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is admissible for the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$, then*

(i) $X_1(t_0) \cap X_2(t_0) = \{0\}$*, for all* $t_0 \in \mathbb{R}$*;*

(ii) $X_1(t_0) + X_2(t_0) = X$*, for all* $t_0 \in \mathbb{R}$*;*

(iii) *the restriction* $U(t,t_0)_| : X_2(t_0) \to X_2(t)$ *is an isomorphism, for all* $t \geq t_0$.

Proof. (i) Let $t_0 \in \mathbb{R}$ and let $x \in X_1(t_0) \cap X_2(t_0)$. Then, there is a function $\varphi_x \in \mathcal{F}_{\mathcal{U}}(t_0)$ such that $\varphi_x(0) = x$ and $\int_{-\infty}^0 \|\varphi_x(t)\|^p dt < \infty$. We define

$$f : \mathbb{R} \longrightarrow X, \qquad f(t) = \begin{cases} U(t,t_0)x, & t > t_0, \\ \varphi_x(t-t_0), & t \leq t_0. \end{cases} \tag{3.4}$$

Then, it is easy to see that $f(t) = U(t,s)f(s)$, for all $t \geq s$. Since $x \in X_1(t_0)$, we obtain that $f \in L^p(\mathbb{R},X)$. It follows that $f = 0$, so $x = f(t_0) = 0$.

(ii) Let $x \in X$ and let $t_0 \in \mathbb{R}$. We consider the function

$$v : \mathbb{R} \longrightarrow X, \qquad v(\tau) = \chi_{[t_0,t_0+1]}(\tau)U(\tau,t_0)x, \tag{3.5}$$

where $\chi_{[t_0,t_0+1]}$ denotes the characteristic function of the interval $[t_0,t_0+1]$. From hypothesis, there is $f \in L^p(\mathbb{R},X)$ such that the pair (f,v) verifies $(E_{\mathcal{U}})$. Then

$$f(t) = U(t,t_0)f(t_0) + \int_{t_0}^{t} U(t,\tau)v(\tau)d\tau = U(t,t_0)(f(t_0)+x), \quad \forall t \geq t_0+1. \tag{3.6}$$

Since $f \in L^p(\mathbb{R},X)$, it follows that $f(t_0)+x \in X_1(t_0)$. Let $\varphi : \mathbb{R}_- \to X, \varphi(t) = f(t+t_0)$. From the fact that the pair (f,v) verifies $(E_{\mathcal{U}})$, it follows that

$$\varphi(t) = U(t_0+t,t_0+s)\varphi(s), \quad \forall s \leq t \leq 0 \tag{3.7}$$

which shows that $\varphi \in \mathcal{F}_{\mathcal{U}}(t_0)$. Since $f \in L^p(\mathbb{R},X)$, it follows that $f(t_0) \in X_2(t_0)$. Finally, we obtain that $x = (x+f(t_0)) - f(t_0) \in X_1(t_0)+X_2(t_0)$.

(iii) Let $t > t_0$. Let $y \in \operatorname{Ker} U(t,t_0) \cap X_2(t_0)$, and let $\varphi_y \in \mathcal{F}_{\mathcal{U}}(t_0)$ with $\varphi_y(0) = y$ and $\int_{-\infty}^{0} \|\varphi_y(s)\|^p ds < \infty$. Considering the function

$$h : \mathbb{R} \longrightarrow X, \qquad h(\tau) = \begin{cases} U(\tau,t_0)y, & \tau > t_0, \\ \varphi_y(\tau - t_0), & \tau \leq t_0, \end{cases} \tag{3.8}$$

we have that $h \in L^p(\mathbb{R},X)$. It is easy to observe that the pair $(h,0)$ verifies $(E_{\mathcal{U}})$. This implies that $h = 0$. In particular, it follows that $y = h(t_0) = 0$, so, the operator $U(t,t_0)_| : X_2(t_0) \to X_2(t)$ is injective.

To prove the surjectivity, let $x \in X_2(t)$. We consider the functions

$$v : \mathbb{R} \longrightarrow X, \qquad v(\tau) = -\chi_{[t,t+1]}(\tau)U(\tau,t)x,$$
$$f : [t,\infty) \longrightarrow X, \qquad f(\tau) = \begin{cases} (t+1-\tau)U(\tau,t)x, & \tau \in [t,t+1], \\ 0, & \tau > t+1. \end{cases} \tag{3.9}$$

We observe that $v \in L^q(\mathbb{R},X)$ and

$$f(r) = U(r,s)f(s) + \int_{s}^{r} U(r,\tau)v(\tau)d\tau, \quad \forall r \geq s \geq t. \tag{3.10}$$

From hypothesis there is $g \in L^p(\mathbb{R},X)$ such that the pair (g,v) verifies $(E_{\mathcal{U}})$. It follows that $f(r) - g(r) = U(r,t)(f(t) - g(t))$, for all $r \geq t$ which implies that $x - g(t) = f(t) - g(t) \in X_1(t)$.

From (ii) there is $y_1 \in X_1(t_0)$ and $y_2 \in X_2(t_0)$ such that $g(t_0) = y_1 + y_2$. Since $g(t) = U(t,t_0)g(t_0)$, we obtain that $g(t) = U(t,t_0)y_1 + U(t,t_0)y_2$, then $x - U(t,t_0)y_2 = (x - g(t)) + U(t,t_0)y_1$. From Lemma 2.5 and from (i), we deduce that $x - U(t,t_0)y_2 = 0$, so $x \in U(t,t_0)X_2(t_0)$.

This shows that the operator $U(t,t_0)_| : X_2(t_0) \to X_2(t)$ is surjective and completes the proof. $\square$

Lemma 3.3. *Let $t_0 < t_1 \leq \infty$ and let $\alpha : [t_0, t_1) \to \mathbb{R}_+$ be a continuous function with the property that there are $M \geq 1$ and $\omega, h \in (0, \infty)$ such that*

$$\alpha(t) \leq M e^{\omega(t-s)} \alpha(s), \quad \forall s, t \in [t_0, t_1),\ s \leq t, \tag{3.11}$$

$$\int_{t+h}^{t+2h} \alpha(\tau)d\tau \leq \frac{1}{e} \int_{t}^{t+h} \alpha(\tau)d\tau, \tag{3.12}$$

for every $t \in [t_0, t_1)$ with $t + 2h < t_1$. Then

$$\alpha(t) \leq K e^{-\nu(t-t_0)} \alpha(t_0), \quad \forall t \in [t_0, t_1), \tag{3.13}$$

where $K = (Me)^2 e^{3\omega h}$ and $\nu = 1/h$.

Proof. Let $t \in [t_0, t_1)$, $n \in \mathbb{N}$, and $r \in [0, h)$ such that $t = t_0 + nh + r$. If $n \geq 2$, then

$$\int_{t_0+(n-1)h}^{t_0+nh} \alpha(\tau)d\tau \leq e^{-(n-1)} \int_{t_0}^{t_0+h} \alpha(\tau)d\tau. \tag{3.14}$$

Using the relation (3.11), we have that

$$\int_{t_0}^{t_0+h} \alpha(\tau)d\tau \leq Mhe^{\omega h}\alpha(t_0), \qquad \alpha(t) \leq \frac{M}{h} e^{2\omega h} \int_{t_0+(n-1)h}^{t_0+nh} \alpha(\tau)d\tau. \tag{3.15}$$

From relations (3.14)–(3.15), it follows that

$$\alpha(t) \leq M^2 e^{3\omega h} e^{-(n-1)} \alpha(t_0). \tag{3.16}$$

Denoting $\nu = 1/h$ and taking $K = (Me)^2 e^{3\omega h}$, we obtain that

$$\alpha(t) \leq K e^{-\nu(t-t_0)} \alpha(t_0). \tag{3.17}$$

If $n \in \{0, 1\}$, then $t - t_0 < 2h$. It follows that

$$\alpha(t) \leq M e^{2\omega h} \alpha(t_0) \leq K e^{-\nu(t-t_0)} \alpha(t_0). \tag{3.18}$$

□

Theorem 3.4. *If the pair $(L^p(\mathbb{R}, X), L^q(\mathbb{R}, X))$ is admissible for the evolution family $\mathcal{U} = \{U(t,s)\}_{t \geq s}$, then there exist $K \geq 1$ and $\nu > 0$ such that*

$$\left|\left|U\left(t, t_0\right)x\right|\right| \leq K e^{-\nu(t-t_0)} \|x\|, \quad \forall x \in X_1(t_0),\ \forall t \geq t_0. \tag{3.19}$$

Proof. From hypothesis there is $\gamma \geq 1$ such that

$$\|\Gamma v\|_p \leq \gamma \|v\|_q, \quad \forall v \in L^q(\mathbb{R}, X). \tag{3.20}$$

We denote $h = (\gamma e)^p$.

Let $t_0 \in \mathbb{R}$, let $x \in X_1(t_0) \setminus \{0\}$, and let $t_1 = \sup\{t \geq t_0 : U(t, t_0)x \neq 0\}$. We consider the function $\varphi : [t_0, t_1) \to X$, $\varphi(t) = U(t, t_0)x$.

If $t_1 > t_0 + 2h$, for every $t \geq t_0$ with $t + 2h < t_1$, we consider the functions

$$\begin{aligned} v : \mathbb{R} \longrightarrow X, \qquad & v(\tau) = \chi_{[t,t+h]}(\tau)\frac{\varphi(\tau)}{||\varphi(\tau)||}, \\ f : \mathbb{R} \longrightarrow X, \qquad & f(\tau) = \int_{-\infty}^{\tau} \frac{\chi_{[t,t+h]}(s)}{||\varphi(s)||} ds\varphi(\tau). \end{aligned} \tag{3.21}$$

We have that $v \in L^q(\mathbb{R},X)$ and since $x \in X_1(t_0)$, it follows that $f \in L^p(\mathbb{R},X)$. It is easy to see that the pair (f,v) verifies $(E_{\mathcal{U}})$, so $\Gamma v = f$. From (3.20) it follows that $\|f\|_p \leq \gamma\|v\|_q = \gamma h^{1/q}$. In particular, this inequality shows that

$$\left(\int_{t+h}^{t+2h} ||f(\tau)||^p d\tau\right)^{1/p} \leq \gamma h^{1/q}. \tag{3.22}$$

We denote $\delta = \int_t^{t+h}(1/\|\varphi(s)\|)ds$. Then, from (3.22) we deduce that

$$\left(\int_{t+h}^{t+2h} ||\varphi(\tau)||^p d\tau\right)^{1/p} \leq \frac{\gamma}{\delta} h^{1/q}. \tag{3.23}$$

Let

$$h' = \begin{cases} 1, & \text{for } p = 1, \\ h^{1/p'}, & \text{for } p \in (1,\infty),\ p' = \dfrac{p}{p-1}. \end{cases} \tag{3.24}$$

Then, we have

$$\int_{t+h}^{t+2h} ||\varphi(\tau)||d\tau \leq h'\left(\int_{t+h}^{t+2h} ||\varphi(\tau)||^p d\tau\right)^{1/p}. \tag{3.25}$$

Using (3.23), we deduce that

$$\int_{t+h}^{t+2h} ||\varphi(\tau)||d\tau \leq \frac{\gamma}{\delta} h' h^{1/q}. \tag{3.26}$$

Since

$$h^2 \leq \left(\int_t^{t+h} \frac{1}{||\varphi(\tau)||} d\tau\right)\left(\int_t^{t+h} ||\varphi(\tau)||d\tau\right) = \delta\left(\int_t^{t+h} ||\varphi(\tau)||d\tau\right) \tag{3.27}$$

from (3.26) we obtain that

$$\int_{t+h}^{t+2h} ||\varphi(\tau)||d\tau \leq \gamma\frac{h' h^{1/q}}{h^2} \int_t^{t+h} ||\varphi(\tau)||d\tau \leq \gamma\frac{h'}{h} \int_t^{t+h} ||\varphi(\tau)||d\tau. \tag{3.28}$$

By the definition of h, from (3.28) it follows that

$$\int_{t+h}^{t+2h} ||\varphi(\tau)||d\tau \leq \frac{1}{e} \int_t^{t+h} ||\varphi(\tau)||d\tau. \tag{3.29}$$

Let M, ω be given by Definition 2.1. Applying Lemma 3.3 for $\alpha = \|\varphi\|$, it follows that

$$||\varphi(t)|| \leq Ke^{-\nu(t-t_0)}||\varphi(t_0)||, \quad \forall t \in [t_0, t_1), \tag{3.30}$$

where $K = (Me)^2 e^{3\omega h}$ and $\nu = 1/h$.

Because K and ν do not depend on t_0 or x, the proof is complete. □

Corollary 3.5. *If the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is admissible for the evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$, then $X_1(t_0)$ is a closed linear subspace, for all $t_0 \in \mathbb{R}$.*

Proof. Let $t_0 \in \mathbb{R}$ be fixed and let $(x_n) \subset X_1(t_0)$ be a sequence convergent to $x \in X$. It follows that there is $L > 0$ such that $\|x_n\| \leq L$, for all $n \in \mathbb{N}$. If K, ν are given by Theorem 3.4, we deduce that $\|U(t,t_0)x_n\| \leq KLe^{-\nu(t-t_0)}$, for all $t \geq t_0$ and all $n \in \mathbb{N}$. Hence, we obtain that $\|U(t,t_0)x\| \leq KLe^{-\nu(t-t_0)}$, for all $t \geq t_0$, so $x \in X_1(t_0)$. It follows that $X_1(t_0)$ is closed. □

Lemma 3.6. *Let $\alpha : [t_0,\infty) \to \mathbb{R}_+$ be a continuous function with the property that there are $M \geq 1$ and $\omega, h \in (0,\infty)$ such that*

$$\alpha(t) \leq Me^{\omega(t-s)}\alpha(s), \quad \forall t \geq s \geq t_0, \tag{3.31}$$

$$\int_{t+h}^{t+2h} \alpha(\tau)d\tau \geq e\int_t^{t+h} \alpha(\tau)d\tau, \quad \forall t \geq t_0. \tag{3.32}$$

Then

$$\alpha(t) \geq \frac{1}{K}e^{\nu(t-s)}\alpha(s), \quad \forall t \geq s \geq t_0 + h, \tag{3.33}$$

where $K = M^2 e^{3\omega h}$ and $\nu = 1/h$.

Proof. Let $t > s \geq t_0 + h$, $n \in \mathbb{N}$, and $r \in [0,h)$ such that $t - s = nh + r$. From (3.32) it follows that

$$\int_{s+(n+1)h}^{s+(n+2)h} \alpha(\tau)d\tau \geq e^{n+2}\int_{s-h}^{s} \alpha(\tau)d\tau. \tag{3.34}$$

Using the relation (3.31), we have that

$$\int_{s+(n+1)h}^{s+(n+2)h} \alpha(\tau)d\tau \leq Mhe^{2\omega h}\alpha(t), \qquad h\alpha(s) \leq Me^{\omega h}\int_{s-h}^{s} \alpha(\tau)d\tau. \tag{3.35}$$

From (3.34)–(3.35), it follows that

$$\alpha(t) \geq \frac{e^{n+2}}{M^2 e^{3\omega h}}\alpha(s) \geq \frac{1}{K}e^{\nu(t-s)}\alpha(s), \tag{3.36}$$

where $\nu = 1/h$ and $K = M^2 e^{3\omega h}$. □

Theorem 3.7. *If the pair* $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ *is admissible for the evolution family* $\mathcal{U} = \{U(t,s)\}_{t\geq s}$*, then there exist* $K \geq 1$ *and* $\nu > 0$ *such that*

$$||U(t,t_0)y|| \geq \frac{1}{K}e^{\nu(t-t_0)}\|y\|, \quad \forall y \in X_2(t_0),\ \forall t \geq t_0. \tag{3.37}$$

Proof. From hypothesis there is $\gamma \geq 1$ such that

$$\|\Gamma v\|_p \leq \gamma\|v\|_q, \quad \forall v \in L^q(\mathbb{R},X). \tag{3.38}$$

We denote $h = (\gamma e)^p$.

Let $t_0 \in \mathbb{R}$ and let $y \in X_2(t_0) \setminus \{0\}$. From Proposition 3.2(iii) there is $z \in X_2(t_0 - h) \setminus \{0\}$ such that $U(t_0, t_0 - h)z = y$. Denoting by $\varphi : [t_0 - h, \infty) \to X$, $\varphi(t) = U(t, t_0 - h)z$, and using Proposition 3.2(iii), we have that $\varphi(t) \neq 0$, for all $t \geq t_0 - h$.

Let $t \geq t_0 - h$. We consider the function

$$v : \mathbb{R} \longrightarrow X, \qquad v(\tau) = -\chi_{[t+h,t+2h]}(\tau)\frac{\varphi(\tau)}{||\varphi(\tau)||}. \tag{3.39}$$

Since

$$z_1 = \left(\int_{t+h}^{t+2h} \frac{ds}{||\varphi(s)||}\right)z \in X_2(t_0 - h), \tag{3.40}$$

there is $\lambda \in \mathcal{F}_{\mathcal{U}}(t_0 - h)$ with $\lambda(0) = z_1$ and $\int_{-\infty}^{0} \|\lambda(s)\|^p ds < \infty$. Let

$$f : \mathbb{R} \longrightarrow X, \quad f(\tau) = \begin{cases} \displaystyle\int_\tau^\infty \frac{\chi_{[t+h,t+2h]}(s)}{||\varphi(s)||} ds\varphi(\tau), & \tau \geq t_0 - h, \\ \lambda(\tau - t_0 + h), & \tau < t_0 - h. \end{cases} \tag{3.41}$$

We have that $v \in L^q(\mathbb{R},X)$, $f \in L^p(\mathbb{R},X)$, and the pair (f,v) verifies $(E_{\mathcal{U}})$. So $\Gamma v = f$ and from (3.38) it follows that $\|f\|_p \leq \gamma\|v\|_q = \gamma h^{1/q}$. In particular, from this inequality, we deduce that

$$\left(\int_t^{t+h} ||f(\tau)||^p d\tau\right)^{1/p} \leq \gamma h^{1/q}. \tag{3.42}$$

We denote $\delta = \int_{t+h}^{t+2h}(1/\|\varphi(s)\|)ds$. Then, from (3.42) we obtain that

$$\left(\int_t^{t+h} ||\varphi(\tau)||^p d\tau\right)^{1/p} \leq \frac{\gamma}{\delta}h^{1/q}. \tag{3.43}$$

Let

$$h' = \begin{cases} 1, & \text{for } p = 1, \\ h^{1/p'}, & \text{for } p \in (1,\infty),\ p' = \dfrac{p}{p-1}. \end{cases} \tag{3.44}$$

Using analogous arguments as in the proof of Theorem 3.4, we immediately deduce that

$$\int_t^{t+h} ||\varphi(\tau)||d\tau \le \gamma \frac{h' h^{1/q}}{h^2} \int_{t+h}^{t+2h} ||\varphi(\tau)||d\tau \le \frac{1}{e} \int_{t+h}^{t+2h} ||\varphi(\tau)||d\tau. \tag{3.45}$$

Let M, ω be given by Definition 2.1. Applying Lemma 3.6 for $\alpha = \|\varphi\|$, it follows that

$$||\varphi(t)|| \ge \frac{1}{K} e^{\nu(t-s)} ||\varphi(s)||, \quad \forall t \ge s \ge t_0, \tag{3.46}$$

where $K = M^2 e^{3\omega h}$ and $\nu = 1/h$. This implies that

$$||U(t,t_0-h)z|| \ge \frac{1}{K} e^{\nu(t-t_0)} ||U(t_0,t_0-h)z||, \quad \forall t \ge t_0 \tag{3.47}$$

which means that

$$||U(t,t_0)y|| \ge \frac{1}{K} e^{\nu(t-t_0)} \|y\|, \quad \forall t \ge t_0. \tag{3.48}$$

Since K and ν do not depend on t_0 or y, we obtain the conclusion. □

Corollary 3.8. *If the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is admissible for the evolution family $\mathcal{U} = \{U(t,s)\}_{t\ge s}$, then $X_2(t_0)$ is a closed linear subspace, for all $t_0 \in \mathbb{R}$.*

Proof. Let $t_0 \in \mathbb{R}$. If $y \in X_2(t_0)$ and φ_y is a function given by the definition of the space $X_2(t_0)$ with $\varphi_y(0) = y$, then it is easy to see that $\varphi_y(s) \in X_2(t_0+s)$, for all $s \le 0$.

Let $(x_n) \subset X_2(t_0)$ be a sequence convergent to $x \in X$. For every $n \in \mathbb{N}$ there is a function $\varphi_n \in \mathcal{F}_{\mathcal{U}}(t_0)$ such that $\varphi_n(0) = x_n$ and $\int_{-\infty}^0 \|\varphi_n(\tau)\|^p d\tau < \infty$. Since

$$\varphi_n(0) = U(t_0,t_0+s)\varphi_n(s), \quad \forall s \le 0,\ \forall n \in \mathbb{N}, \tag{3.49}$$

for K, ν given by Theorem 3.7, it follows that

$$\begin{aligned} ||x_n - x_m|| &= ||U(t_0,t_0+s)(\varphi_n(s) - \varphi_m(s))|| \\ &\ge \frac{1}{K} e^{-\nu s} ||\varphi_n(s) - \varphi_m(s)||, \quad \forall s \le 0,\ \forall m,n \in \mathbb{N}. \end{aligned} \tag{3.50}$$

Using the fact that (x_n) is fundamental, from (3.50) we obtain that for every $s \le 0$ the sequence $(\varphi_n(s))$ is fundamental, so it is convergent. We denote $\varphi(s) := \lim_{n\to\infty} \varphi_n(s)$, for all $s \le 0$. Hence $\varphi(0) = x$ and $\varphi \in \mathcal{F}_{\mathcal{U}}(t_0)$. From (3.50) we deduce that

$$||\varphi(s)|| \le K e^{\nu s} ||x_n - x|| + ||\varphi_n(s)||, \quad \forall (s,n) \in \mathbb{R}_- \times \mathbb{N}. \tag{3.51}$$

This implies that $x \in X_2(t_0)$ and the proof is complete. □

The first main result of this section is given by the following.

Theorem 3.9. *If the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is admissible for the evolution family $\mathcal{U} = \{U(t,s)\}_{t\ge s}$, then $\mathcal{U}$ is uniformly exponentially dichotomic.*

Proof. From Proposition 3.2, Corollary 3.5, and Corollary 3.8, it follows that for every $t \in \mathbb{R}$, $X_1(t) \oplus X_2(t) = X$. Let $P(t)$ be the projection corresponding to $X_1(t)$, that is, $\operatorname{Im} P(t) = X_1(t)$ and $\operatorname{Ker} P(t) = X_2(t)$. Using Lemma 2.5, we have that $P(t)U(t,t_0) = U(t,t_0)P(t_0)$, for all $t \geq t_0$. From Proposition 3.2, the restriction $U(t,t_0)_| : \operatorname{Ker} P(t_0) \to \operatorname{Ker} P(t)$ is an isomorphism, for all $t \geq t_0$. Finally, using Theorem 3.4 and Theorem 3.7, we obtain that $\mathcal{U}$ is uniformly exponentially dichotomic. □

Theorem 3.9 gives a sufficient condition for the uniform exponential dichotomy of an evolution family. In what follows, we will establish when the uniform exponential dichotomy of an evolution family implies the admissibility of the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$.

Lemma 3.10. *Let $p,q \in [1,\infty)$ with $p \geq q$, let $\nu > 0$, and let $v \in L^q(\mathbb{R},\mathbb{R}_+)$. Then, the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}_+$ defined by*

$$f_1(t) = \int_{-\infty}^{t} e^{-\nu(t-s)} v(s)\,ds, \qquad f_2(t) = \int_{t}^{\infty} e^{-\nu(s-t)} v(s)\,ds \tag{3.52}$$

belong to $L^p(\mathbb{R},\mathbb{R}_+)$.

Proof. This follows using Hölder's inequality. □

Proposition 3.11. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family and let $p,q \in [1,\infty)$. If $X_1(t_0) \cap X_2(t_0) = \{0\}$, for all $t_0 \in \mathbb{R}$, then for every $v \in L^q(\mathbb{R},X)$ there exists at most one $f \in L^p(\mathbb{R},X)$ such that the pair (f,v) verifies $(E_{\mathcal{U}})$.*

Proof. Let $v \in L^q(\mathbb{R},X)$. Suppose that there are $f, f_1 \in L^p(\mathbb{R},X)$ such that the pairs (f,v) and (f_1,v) verify $(E_{\mathcal{U}})$. Then, we have

$$f_1(t) - f(t) = U(t,s)\big(f_1(s) - f(s)\big), \quad \forall t \geq s. \tag{3.53}$$

Let $t_0 \in \mathbb{R}$. From $f_1(t) - f(t) = U(t,t_0)(f_1(t_0) - f(t_0))$ and $f, f_1 \in L^p(\mathbb{R},X)$, it follows that $f_1(t_0) - f(t_0) \in X_1(t_0)$.

Let $\psi : \mathbb{R}_- \to X, \psi(s) = f_1(t_0+s) - f(t_0+s)$. From (3.53) we obtain that $\psi \in \mathcal{F}_{\mathcal{U}}(t_0)$. Because $f_1, f \in L^p(\mathbb{R},X)$, it follows that $\psi(0) = f_1(t_0) - f(t_0) \in X_2(t_0)$. Using the hypothesis, we obtain that $f_1(t_0) = f(t_0)$. Since $t_0 \in \mathbb{R}$ was arbitrary, we deduce that $f = f_1$. □

Theorem 3.12. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family and let $p,q \in [1,\infty)$ with $p \geq q$. Then $\mathcal{U}$ is uniformly exponentially dichotomic if and only if the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is admissible for $\mathcal{U}$.*

Proof (Necessity). Let $\{P(t)\}_{t\in\mathbb{R}}$ be the family of projections given by Definition 2.2. For $v \in L^q(\mathbb{R},X)$ we consider the function

$$f : \mathbb{R} \longrightarrow X, \qquad f(t) = \int_{-\infty}^{t} U(t,s)P(s)v(s)\,ds - \int_{t}^{\infty} U(s,t)_|^{-1}(I - P(s))v(s)\,ds, \tag{3.54}$$

where for every $s \geq t$, $U(s,t)_{|}^{-1}$ denotes the inverse of the operator $U(s,t) : X_2(t) \to X_2(s)$. Using Lemma 3.10, we obtain that $f \in L^p(\mathbb{R},X)$. An easy computation shows that the pair (f,v) verifies $(E_{\mathcal{U}})$.

From Proposition 2.6, we have that $X_1(t) = \operatorname{Im} P(t)$ and $X_2(t) = \operatorname{Ker} P(t)$, for all $t \in \mathbb{R}$. Using Proposition 3.11, we obtain the uniqueness of f. It follows that the pair $(L^p(\mathbb{R},X)$, $L^q(\mathbb{R},X))$ is admissible for $\mathcal{U}$.

Sufficiency. This follows from Theorem 3.9. □

Remark 3.13. For the particular case $p = q$ and for evolution families $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ with the property that for every $x \in X$, the mapping $(t,s) \mapsto U(t,s)x$ is continuous, the above theorem has been proved by Latushkin et al. [7]. The fact that $p = q$ and the strong continuity of $\mathcal{U}$ were essentially used in their approach, because their method was based on the use of the evolution semigroup associated to $\mathcal{U}$.

Remark 3.14. Generally, if $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is uniformly exponentially dichotomic and $p,q \in [1,\infty)$ with $p < q$, it does not result that the pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is admissible for $\mathcal{U}$. This fact is illustrated by the following example.

Example 3.15. Let $X = \mathbb{R}^2$ and

$$U(t,s)(x_1,x_2) = (e^{-(t-s)}x_1, e^{t-s}x_2), \quad \forall t \geq s,\ \forall (x_1,x_2) \in \mathbb{R}^2. \tag{3.55}$$

Then, $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ is uniformly exponentially dichotomic.

If $p,q \in [1,\infty)$ with $p < q$, let $\delta \in (p,q)$. We consider the function

$$v : \mathbb{R} \longrightarrow \mathbb{R}^2, \qquad v(t) = \left(\frac{1}{(1+|t|)^{1/\delta}}, 0\right). \tag{3.56}$$

We have that $v \in L^q(\mathbb{R},\mathbb{R}^2) \setminus L^p(\mathbb{R},\mathbb{R}^2)$.

Suppose that the pair $(L^p(\mathbb{R},\mathbb{R}^2), L^q(\mathbb{R},\mathbb{R}^2))$ is admissible for $\mathcal{U}$. Then, there is $f \in L^p(\mathbb{R},\mathbb{R}^2)$ such that the pair (f,v) verifies $(\widetilde{E}_{\mathcal{U}})$. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$, $P(x_1,x_2) = (x_1,0)$. Denoting $f_1 = Pf$ and $v_1 = Pv$, in particular, we obtain that

$$f_1(t) = e^{-t} f_1(0) + \int_0^t e^{-(t-\tau)} v_1(\tau)\, d\tau, \quad \forall t \geq 0. \tag{3.57}$$

Denoting

$$\varphi(t) = e^{-t} \int_0^t e^{\tau} v_1(\tau)\, d\tau, \quad \forall t \geq 0 \tag{3.58}$$

from (3.57) we deduce that $\varphi \in L^p(\mathbb{R}_+,\mathbb{R})$. But, since

$$\lim_{t\to\infty} \frac{\varphi(t)}{v_1(t)} = \lim_{t\to\infty} \frac{e^t v_1(t)}{e^t v_1(t) - (1/\delta(t+1)) e^t v_1(t)} = 1 \tag{3.59}$$

and $v_1 \notin L^p(\mathbb{R}_+,\mathbb{R})$, we obtain that $\varphi \notin L^p(\mathbb{R}_+,\mathbb{R})$, which is a contradiction. In conclusion, the pair $(L^p(\mathbb{R},\mathbb{R}^2), L^q(\mathbb{R},\mathbb{R}^2))$ is not admissible for $\mathcal{U}$.

4. An application for the case of C_0-semigroups

Let X be a real or complex Banach space.

Definition 4.1. A family $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators on X is said to be a *C_0-semigroup* if the following properties are satisfied:

(i) $T(0) = I$, the identity operator on X;
(ii) $T(t+s) = T(t)T(s)$, for all $t,s \geq 0$;
(iii) $\lim_{t\searrow 0} T(t)x = x$, for every $x \in X$.

Definition 4.2. A C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be *uniformly exponentially dichotomic* if there exist a projection $P \in \mathcal{B}(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that

(i) $PT(t) = T(t)P$, for all $t \geq 0$;
(ii) $\|T(t)x\| \leq Ke^{-\nu t}\|x\|$, for all $x \in \operatorname{Im} P$ and all $t \geq 0$;
(iii) $\|T(t)x\| \geq 1/Ke^{\nu t}\|x\|$, for all $x \in \operatorname{Ker} P$ and all $t \geq 0$;
(iv) the restriction $T(t)_{|} : \operatorname{Ker} P \to \operatorname{Ker} P$ is an isomorphism, for every $t \geq 0$.

Remark 4.3. If $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a C_0-semigroup, we can associate to it an evolution family $\mathcal{U}_T = \{U_T(t,s)\}_{t\geq s}$, by $U_T(t,s) = T(t-s)$, for every $t \geq s$.

Proposition 4.4. *The C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is uniformly exponentially dichotomic if and only if the evolution family $\mathcal{U}_T = \{U_T(t,s)\}_{t\geq s}$ associated to $\mathbf{T}$ is uniformly exponentially dichotomic.*

Proof (Necessity). If the semigroup $\mathbf{T}$ is uniformly exponentially dichotomic, then it is easy to see that the evolution family $\mathcal{U}_T$ is uniformly exponentially dichotomic relative to the family of projections $\{P(t)\}_{t\in\mathbb{R}}$, where $P(t) = P$, for every $t \in \mathbb{R}$, and with the same constants.

Sufficiency. Suppose that the evolution family $\mathcal{U}_T$ is uniformly exponentially dichotomic relative to the family of projections $\{P(t)\}_{t\in\mathbb{R}}$ and the constants K and ν. For every $t_0 \in \mathbb{R}$, we denote

$$X_1(t_0) = \left\{x \in X : \int_{t_0}^{\infty} ||U_T(t,t_0)x||dt < \infty\right\} \tag{4.1}$$

and by $X_2(t_0)$ the linear subspace of all $x \in X$ with the property that there exists $\varphi_x : \mathbb{R}_- \to X$ with $\varphi_x \in \mathcal{F}_{\mathcal{U}}(t_0)$, $\varphi_x(0) = x$, and $\int_{-\infty}^{0} \|\varphi_x(t)\|dt < \infty$.

From Proposition 2.6, it follows that $\operatorname{Im} P(t) = X_1(t)$ and $\operatorname{Ker} P(t) = X_2(t)$, for all $t \in \mathbb{R}$. We observe that $X_1(t) = X_1(0)$ and $X_2(t) = X_2(0)$, for all $t \in \mathbb{R}$. This shows that $P(t) = P(0)$, for all $t \in \mathbb{R}$. Denoting $P = P(0)$, it is a simple exercise to verify that the C_0-semigroup $\mathbf{T}$ is uniformly exponentially dichotomic relative to the projection P and the constants K and ν. □

We denote by $L^1_{\text{loc}}(\mathbb{R},X)$ the linear space of all measurable functions $v : \mathbb{R} \to X$, which are Bochner integrable on every segment $[a,b]$, with $a,b \in \mathbb{R}$, $a < b$.

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X. We consider the equation

$$f(t) = T(t-s)f(s) + \int_s^t T(t-\tau)v(\tau)d\tau, \quad \forall t \geq s \qquad (E_T)$$

with $f, v \in L^1_{\text{loc}}(\mathbb{R}, X)$.

Definition 4.5. Let $p, q \in [1,\infty)$. The pair $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ is said to be *admissible* for the C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ if for every $v \in L^q(\mathbb{R},X)$ there is a unique $f \in L^p(\mathbb{R},X)$ such that the pair (f,v) verifies (E_T).

THEOREM 4.6. *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a* C_0*-semigroup on* X *and let* $p, q \in [1,\infty)$. *Then,*

(i) *if the pair* $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ *is admissible for* $\mathbf{T}$, *then* $\mathbf{T}$ *is uniformly exponentially dichotomic;*

(ii) *if* $p \geq q$, *then* $\mathbf{T}$ *is uniformly exponentially dichotomic if and only if the pair* $(L^p(\mathbb{R},X), L^q(\mathbb{R},X))$ *is admissible for* $\mathbf{T}$.

Proof. (i) This follows from Proposition 4.4 and Theorem 3.9.

(ii) This follows from Proposition 4.4 and Theorem 3.12. □

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