# AN EXISTENCE RESULT FOR A SEMIPOSITONE PROBLEM WITH A SIGN CHANGING WEIGHT

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We establish an existence result on positive solution for a class of reaction-diffusion equation with semipositone structure. In particular, our results apply to the diffusive logistic equation with a class of sign changing weight and constant yield harvesting. We establish the result via the method of subsuper solutions.



## 1. Introduction

In this paper we discuss the existence of positive classical solutions ($u \in C^{2,\alpha}(\overline{\Omega})$) of the boundary value problem

$$\begin{gathered} -\Delta u = \lambda\big(g(x)\big[u(1-u^p)\big] - \mathrm{ch}(x)\big), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $p > 0$, $c > 0$, and $\lambda > 0$ are parameters and $\Omega$ is an open bounded region with boundary $\partial\Omega$ in class $C^2$ in $\mathbb{R}^n$ for $n \geq 1$. Here $g : \overline{\Omega} \to \mathbb{R}$ is a $C^\alpha$ function while $h : \Omega \to \mathbb{R}$ is a nonnegative $C^\alpha$ function with $\|h\|_\infty = 1$. When $p = 1$, (1.1) arises in population dynamics where $1/\lambda$ is the diffusion coefficient and $\mathrm{ch}(x)$ represents the constant yield harvesting. In this case ($p = 1$), when $g(x)$ is a positive constant, various results have been established in [4]. Here we focus on sign changing weight functions $g$.

To precisely define our classes of weight functions, we first let $\lambda_1 > 0$ be the principal eigenvalue and $\phi > 0$ with $\|\phi\|_\infty = 1$ the corresponding eigenfunction of $-\Delta$ with the Dirichlet boundary conditions. It is well known that $\partial\phi/\partial\eta < 0$ on $\partial\Omega$ where $\eta$ is the unit outward normal. Hence there exists $\delta > 0$, $\sigma > 0$, and $m > 0$ such that

$$|\nabla\phi|^2 - \lambda_1\phi^2 \geq m \quad \text{on } \overline{\Omega}_\delta, \tag{1.2}$$

$$\phi \geq \sigma \quad \text{on } \Omega - \overline{\Omega}_\delta, \tag{1.3}$$

where $\Omega_\delta := \{x \in \Omega \mid d(x,\partial\Omega) < \delta\}$.

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In this paper we assume that the weight $g$ takes negative values in $\Omega_\delta$ but requires $g$ to be strictly positive in $\Omega - \Omega_\delta$. Define $\gamma := \min_{\Omega-\Omega_\delta} g(x)$, $\mu := \min_{\overline{\Omega}_\delta} g(x)$, and we assume that

$$|\mu| < \frac{m\gamma}{\lambda_1}\left(\frac{1}{p+1}\right)^{1/p}. \tag{1.4}$$

Further let $0 < x_1 < x_2 < \gamma/2\lambda_1$ be the positive roots of $q(x) = -\mu$ (see Figure 1.1), where

$$q(x) := x\left[1 - \frac{2\lambda_1}{\gamma}x\right]^{1/p}\left(\frac{p+1}{p}\right)2m. \tag{1.5}$$

Then we establish the following.

THEOREM 1.1. *Suppose (1.4) holds,* $1/x_2 < \lambda < 1/x_1$ *and* $c \le c_0(\lambda)$, *where*

$$c_0(\lambda) := \min\left\{\left(\frac{1}{p+1}\right)^{1/p}\left[\frac{2m}{\lambda}\left(1 - \frac{2\lambda_1}{\lambda\gamma}\right)^{1/p} + \frac{\mu p}{(p+1)}\right], \frac{p\gamma\sigma^2}{(p+1)^{(p+1)/p}}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]^{(p+1)/p}\right\}. \tag{1.6}$$

*Then (1.1) has at least one positive solution u such that* $\|u\|_\infty < 1$.

Note that when $c > 0$, (1.1) is a semipositone problem and it is well known in the literature that the study of positive solutions is mathematically challenging (see [2–4]). Here we also include the additional challenge of dealing with a sign changing weight function $g$.

Finally, we also deduce a result for the case when $g(x) \ge 0$ on $\overline{\Omega}_\delta$. In particular we prove the following.

COROLLARY 1.2. *If* $g(x) \ge 0$ *on* $\overline{\Omega}_\delta$ *and* $c = 0$, *then for any* $\lambda \ge 2\lambda_1/\gamma$ *(1.1) has a positive solution.*

We establish our results by the method of subsuper solutions. By a subsolution we mean a function $w \in C^2(\overline{\Omega})$ such that

$$\begin{aligned} -\Delta w &\le \lambda\big(g(x)[w(1-w^p)] - \mathrm{ch}(x)\big), \quad x \in \Omega, \\ w &\le 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.7}$$

and by a supersolution a function $v \in C^2(\overline{\Omega})$ such that

$$\begin{aligned} -\Delta v &\ge \lambda\Big(g(x)[v(1-v^p)] - \mathrm{ch}(x)\Big), \quad x \in \Omega, \\ v &\ge 0, \quad x \in \partial\Omega. \end{aligned} \tag{1.8}$$

Then it is well known (see [1, 5]) that if there exists a subsolution $w$ and a supersolution $v$ such that $w < v$, then there exists a solution $u \in C^2(\overline{\Omega})$ such that $w \le u \le v$.

We will prove Theorem 1.1 in Section 2 and Corollary 1.2 in Section 3.

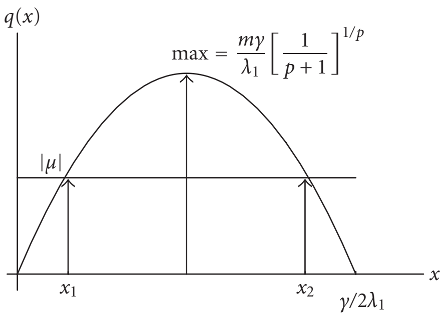


Figure 1.1

## 2. Proof of Theorem 1.1

*Proof.* Let $w = k_0\phi^2$, where

$$k_0 = \left(\frac{1}{p+1}\right)^{1/p}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]^{1/p}. \tag{2.1}$$

We will prove that $w$ is a subsolution. Now

$$-\Delta w = -\nabla\cdot\nabla\left(k_0\phi^2\right) = -\nabla\cdot\left(2k_0\phi\nabla\phi\right) = -2k_0\left(\nabla\phi\cdot\nabla\phi + \phi\Delta\phi\right) = 2k_0\left(\lambda_1\phi^2 - |\nabla\phi|^2\right). \tag{2.2}$$

First we consider the case when $x \in \overline{\Omega}_\delta$. Since the maximum of $s(1-s^p)$ is $p/(p+1)^{(p+1)/p}$, we have

$$\lambda\left(g(x)\left[w\left(1-w^p\right)\right] - ch(x)\right) \geq \lambda\left(\mu\left[\frac{p}{(p+1)^{(p+1)/p}}\right] - c\right). \tag{2.3}$$

Since

$$c < c_0 \leq \left(\frac{1}{p+1}\right)^{1/p}\left[\frac{2m}{\lambda}\left(1 - \frac{2\lambda_1}{\lambda\gamma}\right)^{1/p} + \frac{\mu p}{(p+1)}\right] = \frac{2k_0 m}{\lambda} + \frac{\mu p}{(p+1)^{(p+1)/p}}, \tag{2.4}$$

combining (2.3)-(2.4) and using (1.2)-(2.2), we have

$$\lambda\left(\mu\left[\frac{p}{(p+1)^{(p+1)/p}}\right] - c\right) \geq -\Delta w. \tag{2.5}$$

Hence

$$-\Delta w \leq \left(g(x)\left[w\left(1-w^p\right)\right] - \mathrm{ch}(x)\right) \quad \text{on } \overline{\Omega}_\delta. \tag{2.6}$$

Next consider the case when $x \in \Omega - \overline{\Omega}_\delta$. By the definition of $\gamma$, we have

$$\begin{aligned}
&\lambda(g(x)[w(1-w^p)] - \mathrm{ch}(x)) \\
&\quad \geq \lambda(\gamma[k_0\phi^2(1-k_0^p\phi^{2p})] - c) \geq \lambda(\gamma[k_0\phi^2(1-k_0^p)] - c) \\
&\quad \geq \lambda\left(\gamma[k_0\phi^2(1-k_0^p)] - \frac{p\gamma}{(p+1)^{(p+1)/p}}\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]^{(p+1)/p}\sigma^2\right) \quad \text{since } c \leq c_0 \\
&\quad \geq \lambda\left(\gamma[k_0\phi^2(1-k_0^p)] - \frac{p\gamma}{(p+1)}\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]k_0\phi^2\right) \quad \text{using (1.3), (2.1)} \\
&\quad = \lambda\gamma k_0\phi^2\left\{1-k_0^p - \frac{p}{(p+1)}\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]\right\} \\
&\quad = \lambda\gamma k_0\phi^2\{1-k_0^p - pk_0^p\} \quad \text{by (2.1)} \\
&\quad = \lambda\gamma k_0\phi^2\{1-[p+1]k_0^p\} \\
&\quad = \lambda\gamma k_0\phi^2\left\{1-\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]\right\} \quad \text{by (2.1)} \\
&\quad = 2k_0\lambda_1\phi^2 \geq 2k_0[\lambda_1\phi^2 - |\nabla\phi|^2] \\
&\quad = -\Delta w \quad \text{using (2.2).}
\end{aligned} \tag{2.7}$$

Hence

$$-\Delta w \leq (g(x)[w(1-w^p)] - \mathrm{ch}(x)) \quad \text{on } \Omega - \overline{\Omega}_\delta. \tag{2.8}$$

From (2.6) and (2.8) we have

$$-\Delta w \leq (g(x)[w(1-w^p)] - \mathrm{ch}(x)) \quad \text{on } \Omega. \tag{2.9}$$

Thus $w = k_0\phi^2$ is a subsolution of (1.1).

Next it is easy to see that $v \equiv 1$ is a supersolution of (1.1) and $v > w$ on $\overline{\Omega}$. Thus we have a positive solution $u$ such that $\|u\|_\infty < 1$. □

## 3. Proof of Corollary 1.2

*Proof.* Since $g(x) \geq 0$ and $c = 0$, on $\overline{\Omega}_\delta$, $\lambda(g(x)[w(1-w^p)]) \geq 0$. But $-\Delta w \leq -2k_0 m$ and is negative; hence, on $\overline{\Omega}_\delta$, we have

$$-\Delta w \leq g(x)[w(1-w^p)] \quad \text{on } \overline{\Omega}_\delta, \tag{3.1}$$

and on $\Omega - \overline{\Omega}_\delta$, we have

$$\begin{aligned}
&\lambda g(x)[w(1-w^p)] \\
&\quad \geq \lambda\gamma[k_0\phi^2(1-k_0^p\phi^{2p})] \geq \lambda\gamma[k_0\phi^2(1-k_0^p)] \\
&\quad \geq \lambda\gamma k_0\phi^2\left[1-\frac{1}{p+1}\left[1-\frac{2\lambda_1}{\lambda\gamma}\right]\right] \quad \text{by (2.1)} \\
&\quad = \frac{k_0\phi^2}{p+1}[p\lambda\gamma+2\lambda_1] \\
&\quad \geq \frac{k_0\phi^2}{p+1}[2\lambda_1(p+1)] \quad \text{since } \lambda \geq \frac{2\lambda_1}{\gamma} \\
&\quad = 2\lambda_1 k_0\phi^2 \\
&\quad \geq 2k_0[\lambda_1\phi^2 - |\nabla\phi|^2] = -\Delta w.
\end{aligned} \tag{3.2}$$

Hence we have

$$-\Delta w \leq g(x)[w(1-w^p)] \quad \text{on } \Omega - \overline{\Omega}_\delta. \tag{3.3}$$

Using (3.1)–(3.3) we have that $w = k_0\phi^2$ is a subsolution. Again we note that $v \equiv 1$ is a supersolution. Hence the result holds. □

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