

# ON A CERTAIN FUNCTIONAL EQUATION IN THE ALGEBRA OF POLYNOMIALS WITH COMPLEX COEFFICIENTS

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Many analytical problems can be reduced to determining the number of roots of a polynomial in a given disc. In turn, the latter problem admits further reduction to the generalized Rauss-Hurwitz problem of determining the number of roots of a polynomial in a semiplane. However, this procedure requires complicated coefficient transformations. In the present paper we suggest a *direct* method to evaluate the number of roots of a polynomial with complex coefficients in a disc, based on studying a certain equation in the algebra of polynomials. An application for computing the rotation of plane polynomial vector fields is also given.

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## 1. Functional equations: basic properties of solutions

Let

$$f(z) = a_0 + a_1z + \cdots + a_nz^n, \quad a_n \neq 0, \quad (1.1)$$

$$F(z) = b_0 + b_1z + \cdots + b_nz^n + b_{n+1}z^{n+1}, \quad b_0 \neq 0, b_{n+1} \neq 0, \quad (1.2)$$

be polynomials with complex coefficients  $a_0, a_1, \dots, a_n$  and  $b_0, b_1, \dots, b_{n+1}$  of degree  $n$  and  $n + 1$ , respectively. Assume the polynomials  $f$  and  $F$  to satisfy the functional equation

$$(a + bz)f(z) + (c + dz)f^*(z) = F(z), \quad (1.3)$$

where  $a, b, c, d$  are certain complex numbers and the polynomial  $f^*$  is defined by

$$f^*(z) = \bar{a}_0z^n + \bar{a}_1z^{n-1} + \cdots + \bar{a}_n = z^n \overline{f\left(\frac{1}{\bar{z}}\right)}. \quad (1.4)$$

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Consider along with (1.3) the following functional equation:

$$g(z) \cdot f(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z), \quad (1.5)$$

where

$$g(z) = \alpha z^2 + \beta z + \bar{\alpha}, \quad \alpha = \bar{a}b - \bar{c}d, \quad \beta = |a|^2 + |b|^2 - |c|^2 - |d|^2, \quad (1.6)$$

$$F^*(z) = \bar{b}_0 z^{n+1} + \bar{b}_1 z^n + \dots + \bar{b}_{n+1}. \quad (1.7)$$

LEMMA 1.1. *If the polynomials  $f$  and  $F$  satisfy the functional equation (1.3), then they also satisfy the functional equation (1.5).*

*Conversely, if at least one of the numbers  $\alpha, \beta$  is different from zero and  $f, F$  satisfy (1.5), then they satisfy (1.3) as well.*

*Proof.* Let polynomials (1.1) and (1.2) satisfy (1.3). By the definition of  $f^*$  and  $F^*$ , one has

$$(\bar{a}z + \bar{b})f^*(z) + (\bar{c}z + \bar{d})f(z) = F^*(z). \quad (1.8)$$

Multiplying (1.3) (resp., (1.8)) by  $(\bar{a}z + \bar{b})$  (resp., by  $(c + dz)$ ) and taking the difference of the obtained expressions, one obtains (1.5), where the coefficients  $\alpha$  and  $\beta$  are defined by (1.6). Thus the first implication is established.

Conversely, assume that  $f$  and  $F$  satisfy (1.5), and at least one of  $\alpha, \beta$  is different from zero. Since the coefficient  $\beta$  of  $g$  is real (cf. [4]), one has

$$z^2 \overline{g\left(\frac{1}{z}\right)} = g(z). \quad (1.9)$$

Therefore, it follows from (1.5) that

$$g(z)f^*(z) = (a + bz)F^*(z) - (\bar{c}z + \bar{d})F(z). \quad (1.10)$$

Multiplying (1.5) (resp., (1.10)) by  $(a + bz)$  (resp., by  $(c + dz)$ ) and summing up the obtained expressions, one arrives at the following equality:

$$g(z)[(a + bz)f(z) + (c + dz)f^*(z)] = g(z)F(z). \quad (1.11)$$

Since  $g(z) \not\equiv 0$  and the algebra of polynomials does not contain zero divisors, it follows that  $f$  and  $F$  satisfy (1.3).

The lemma is completely proved.  $\square$

Assume that the polynomials  $f$  and  $F$  satisfy (1.3). It follows from (1.1)–(1.3) that (1.3) is equivalent to the following system:

$$aa_k + c\bar{a}_{n-k} + ba_{k-1} + d\bar{a}_{n-k+1} = b_k, \quad k = 0, 1, \dots, n+1, \quad (1.12)$$

where we put  $a_k = 0$  for  $k < 0$  and  $k > n$ .

Similarly, (1.5) is equivalent to the following system:

$$\bar{\alpha}a_k + \beta a_{k-1} + \alpha a_{k-2} = \bar{a}b_{k-1} - d\bar{b}_{n-k+2} + \bar{b}b_k - c\bar{b}_{n-k+1}, \quad k = 0, 1, \dots, n+2, \quad (1.13)$$

where  $a_k = b_k = 0$  for  $k < 0$  and  $a_k = b_{k+1} = 0$  for  $k > n$ .

Thus, under the assumption that  $a, b, c$ , and  $d$  satisfy the condition  $|\alpha| + |\beta| > 0$ , system (1.3) is equivalent to (1.12) as well as to (1.13).

Below we will list some properties of solutions to (1.3).

- (1) (a) The coefficients  $a, b, c, d$  along with the polynomial  $f$  determine the polynomial  $F$  uniquely.
- (b) If the polynomials  $f$  and  $F$  are defined by (1.1) and (1.2) and satisfy (1.3), then the coefficients  $a, b, c, d$  satisfy the conditions

$$|a| + |c| > 0, \quad |b| + |d| > 0. \quad (1.14)$$

- (c) If a collection  $(a, b, c, d, f, F)$  of numbers  $a, b, c, d$ , and polynomials  $f, F$  satisfy (1.3), then so is the collection  $(\lambda a, \lambda b, \bar{\lambda}c, \bar{\lambda}d, f/\lambda, F)$  for any complex number  $\lambda \neq 0$ .

- (2) Given a polynomial  $F$  and numbers  $a, b, c$ , and  $d$ , satisfying  $|\alpha| + |\beta| > 0$ , there exists a unique  $f$  satisfying (1.3). Indeed, if  $\alpha \neq 0$ , then the first  $n+1$  equations of system (1.13) completely determine the coefficients  $a_0, a_1, \dots, a_n$  of the polynomial  $f$ . If, however,  $\alpha = 0$  and  $\beta \neq 0$ , then all the coefficients  $a_0, a_1, \dots, a_n$  of the polynomial  $f$  are completely determined by  $n+1$  equations of system (1.13) starting with the second one.
- (3) It follows from (1.5) that the roots of  $g(z)$  turn out to be the roots of  $G(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z)$ . Also, if  $\alpha \neq 0$  and  $\beta^2 \neq 4|\alpha|^2$ , then  $z_0, z_1 = 1/\bar{z}_0$ , where

$$z_0 = \frac{-\beta + \sqrt{\beta^2 - 4|\alpha|^2}}{2\alpha}, \quad (1.15)$$

are the roots of  $g(z)$ . Therefore,  $F$  and the coefficients  $a, b, c, d$  are connected by the following relations:

$$(\bar{a}z_j + \bar{b})F(z_j) - (c + dz_j)F^*(z_j) = 0, \quad j = 0, 1. \quad (1.16)$$

If  $\alpha \neq 0$  and  $\beta^2 = 4|\alpha|^2$ , then  $z_1 = z_0$  is a multiple root of  $g(z)$ , and, therefore,

$$\begin{aligned} (\bar{a}z_0 + \bar{b})F(z_0) - (c + dz_0)F^*(z_0) &= 0, \\ (\bar{a}z_0 + \bar{b})F'(z_0) + \bar{a}F(z_0) - (c + dz_0)F^{*'}(z_0) - dF^*(z_0) &= 0. \end{aligned} \quad (1.17)$$

Finally, if  $\alpha = 0$  and  $\beta \neq 0$ , then the linear function  $g(z) = \beta z$  has the only root  $z_0 = 0$ . Hence it follows from (1.5) (see also (1.10)) that

$$\begin{aligned} \bar{b}F(0) - cF^*(0) &= 0, \\ \bar{d}F(0) - aF^*(0) &= 0. \end{aligned} \quad (1.18)$$

## 2. Functional equations: solubility conditions

Given the polynomial (1.2), consider the solubility problem for the functional equation (1.3) with respect to unknown coefficients  $a, b, c, d$  and a polynomial  $f$ . To treat the above problem, we will use the necessary conditions for the solubility of (1.3) given by (3.1)–(3.8) (depending on whether the root  $z_0$  of  $g(z)$  satisfies  $0 < |z_0| < 1$ ,  $|z_0| = 1$ , or  $z_0 = 0$ ).

Assume  $z_0$  and  $z_1$  to be given and consider system (1.16) with respect to unknown  $a, b, c, d$ . We will try to find a solution to (1.16) in such a way that (1.3) will have a solution with respect to  $f$ . Also, given  $z_0$ , we will follow the same way regarding system (1.17).

To describe the solubility conditions for the functional equation (1.3), it is convenient to introduce the notion of a regular point.

A point  $z$  is called *regular* with respect to the polynomial  $F$  if the following conditions are satisfied:

$$\begin{aligned} F(z) \cdot F^*(z) &\neq 0, \quad |F(z)| \neq |F^*(z)|, \quad \text{for } |z| \neq 1, \\ (n+1)|F(z)|^2 &\neq 2\Re[\overline{F(z)}F'(z)], \quad \text{for } |z| = 1. \end{aligned} \quad (2.1)$$

Observe that the notion of a regular point is introduced with respect to the unit circle. It follows immediately from the definition of a regular point that if  $z_0$  is regular, then so is  $z_1 = 1/\bar{z}_0$  and vice versa.

According to the definition of polynomial  $F^*$ , the rational function

$$A(z) = \frac{F^*(z)}{F(z)} \quad (2.2)$$

satisfies the identity

$$\overline{A(z)} \cdot A\left(\frac{1}{\bar{z}}\right) \equiv 1. \quad (2.3)$$

In addition,  $|A(z)| \neq 1$  for all regular points  $z, |z| \neq 1$ .

Assume that  $z_0$  is a regular point of  $F$ ,  $A_0 = A(z_0)$ ,  $A_1 = 1/\bar{A}_0$ , and  $\sigma_0, \sigma_1$  are arbitrary complex numbers. Consider the linear system

$$\begin{aligned} \bar{a}z_0 + \bar{b} &= \sigma_0 A_0, & \bar{a} + \bar{b}\bar{z}_0 &= \sigma_1 A_1, \\ c + d z_0 &= \sigma_0, & c\bar{z}_0 + d &= \sigma_1, \end{aligned} \quad (2.4)$$

with unknown  $a, b, c$ , and  $d$ . It should be pointed out that any solution to system (2.4) is also a solution to (1.16) for  $z_0 \neq 0$ , as well as a solution to (1.18) for  $z_0 = 0$ .

For  $|z_0| < 1$  system, (2.4) has the unique solution

$$\begin{aligned} \bar{a}(1 - |z_0|^2) &= \sigma_1 A_1 - \bar{z}_0 \sigma_0 A_0, & \bar{b}(1 - |z_0|^2) &= \sigma_0 A_0 - \bar{z}_0 \sigma_1 A_1, \\ c(1 - |z_0|^2) &= \sigma_0 - z_0 \sigma_1, & d(1 - |z_0|^2) &= \sigma_1 - \bar{z}_0 \sigma_0. \end{aligned} \quad (2.5)$$

With the above  $a, b, c,$  and  $d$  on hands, the coefficients  $\alpha, \beta$  from formula (1.6) satisfy the equalities

$$\alpha(1 - |z_0|^2) = \bar{z}_0(|A_1|^2 - 1)(|\sigma_0 A_0|^2 - |\sigma_1|^2), \quad (2.6)$$

$$\beta(1 - |z_0|^2) = -(|A_1|^2 - 1)(|\sigma_0 A_0|^2 - |\sigma_1|^2). \quad (2.7)$$

**THEOREM 2.1.** *Let  $z_0$  with  $|z_0| < 1$  be a regular point of the polynomial  $F$ , and assume the numbers  $\sigma_0$  and  $\sigma_1$  to satisfy the condition*

$$|\sigma_0 A_0| \neq |\sigma_1|. \quad (2.8)$$

*Let, further,  $a, b, c,$  and  $d$  be defined by (2.5). Then the functional equation (1.3) admits a solution  $f(z)$ .*

*Proof.* To begin with, consider the case  $z_0 \neq 0$ . It follows from (2.8) and (2.6) that  $\alpha \neq 0$ . Formulae (2.4) and (2.3) provide that  $z_0$  and  $z_1 = 1/\bar{z}_0$  are the roots of the polynomial  $g(z)$  (1.6):

$$\begin{aligned} g(z_0) &= (a + bz_0)(\bar{a}z_0 + \bar{b}) - (c + dz_0)(\bar{c}z_0 + \bar{d}) = \bar{\sigma}_1 \bar{A}_1 \sigma_0 A_0 - \sigma_0 \bar{\sigma}_1 = \bar{\sigma}_1 \sigma_0 - \sigma_0 \bar{\sigma}_1 = 0, \\ g(z_1) &= z_1^2 \overline{g(z_0)} = 0. \end{aligned} \quad (2.9)$$

On the other hand, the numbers  $z_0$  and  $z_1$  are the roots of the polynomial

$$G(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z). \quad (2.10)$$

Therefore, by the Bezout theorem, the rational function

$$f(z) = \frac{G(z)}{g(z)} \quad (2.11)$$

is, in fact, a polynomial satisfying the functional equation (1.5). Now the statement of the theorem in the considered case follows from Lemma 1.1.

Assume now  $z_0 = 0$ . The condition (2.8) and equalities (2.6), (2.7) yield  $\alpha = 0$  and  $\beta \neq 0$ , that is,  $g(z) = \beta z$  is a linear function. In addition,  $z = 0$  is a root of the polynomial  $G(z)$ . Therefore,  $f(z) = G(z)/g(z)$  is a polynomial satisfying (1.5), and again the statement of the theorem in the considered case follows from Lemma 1.1.

Theorem 2.1 is completely proved.  $\square$

Assume now that a regular point  $z_0$  of the polynomial  $F$  belongs to the unit circle  $|z| = 1$ . Given numbers  $c$  and  $d$ , consider system (1.18) with unknown  $a, b$ . By solving system (1.18) one obtains

$$\bar{a}F^2 = \Delta \cdot c + (FF^* + z\Delta)d, \quad \bar{b}F^2 = (FF^* - z\Delta)c - z^2\Delta d, \quad (2.12)$$

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where

$$\begin{aligned} \Delta &= z^n \Delta_0, \quad \Delta_0 = (n+1)|F|^2 - 2\Re[\bar{F}F'z], \\ F &= F(z), \quad F' = F'(z), \quad F^* = F^*(z), \quad z = z_0. \end{aligned} \quad (2.13)$$

Take the coefficients  $a, b, c,$  and  $d$  satisfying (2.12) and define the polynomial  $g(z)$  by means of formula (1.6). Formula (2.12) provides the following relations for  $\alpha$  and  $\beta$ :

$$z\alpha|F|^4 = \Delta_0[|F|^2(|c|^2 - |d|^2) - \Delta_0|c + zd|^2], \quad (2.14)$$

$$\beta|F|^4 = 2\Delta_0[\Delta_0|c + zd|^2 - |F|^2(|c|^2 - |d|^2)]. \quad (2.15)$$

**THEOREM 2.2.** *Let  $z_0, |z_0| = 1$  be a regular point of the polynomial  $F$  and let the numbers  $c, d$  satisfy the condition:*

$$\Delta_0|c + z_0d|^2 \neq |F(z_0)|^2(|c|^2 - |d|^2). \quad (2.16)$$

*If the coefficients  $a, b, c,$  and  $d$  satisfy relation (2.12), then the functional equation (1.3) has the unique solution  $(a, b, c, d, f)$ .*

*Proof.* It follows from (1.6) and (2.15)–(3.1) that  $\beta = -2\alpha z_0 \neq 0$ . Since the coefficient  $\beta$  is real and  $z_0\bar{z}_0 = 1$ , one obtains the equality  $\bar{\alpha} = \alpha z_0^2$ , that is,  $g(z) \equiv \alpha(z - z_0)^2$ . At the same time,  $z_0$  is a multiple root to the polynomial  $G(z) = (\bar{\alpha}z + \bar{b})F(z) - (c + dz)F^*(z)$ . Therefore, the rational function  $f(z) = G(z)/g(z)$  is, in fact, a polynomial satisfying the functional equation (1.5). To complete the proof of Theorem 2.1 it remains to apply Lemma 1.1. The theorem follows.  $\square$

Combining Theorems 2.1 and 2.2 with property (2) of solutions to (1.3) one can effectively compute the numbers  $a, b, c, d$  and the coefficients  $a_0, a_1, \dots, a_n$  of the polynomial  $f$ . Indeed, assume, for instance, that  $z = 0$  is a regular point of the polynomial  $F$ . Then the regularity condition for the point  $z = 0$  along with condition (2.8) take the form

$$b_0 \cdot b_{n+1} \neq 0, \quad |b_0| \neq |b_{n+1}|, \quad |cb_{n+1}| \neq |db_0|; \quad (2.17)$$

also, the equalities (2.6) and (2.7) take the form

$$\bar{a}b_{n+1} = d\bar{b}_0, \quad \bar{b}b_0 = c\bar{b}_{n+1}. \quad (2.18)$$

In this case the leading coefficient  $\alpha$  of the polynomial  $g(z)$ , that is determined by  $a, b, c, d$ , is equal to zero. Hence, in order to determine unknown coefficients  $a_0, a_1, \dots, a_n$  from system (1.13), one has

$$\begin{aligned} 0 &= \bar{b}b_0 - c\bar{b}_{n+1}, \\ \beta a_0 &= \bar{a}b_0 - d\bar{b}_{n+1} + \bar{b}b_1 - c\bar{b}_n, \\ \beta a_{k-1} &= \bar{a}b_{k-1} - d\bar{b}_{n-k+2} + \bar{b}b_k - c\bar{b}_{n-k+1}, \quad k = 2, \dots, n, \\ \beta a_n &= \bar{a}b_n - d\bar{b}_1 + \bar{b}b_{n+1} - c\bar{b}_0, \\ 0 &= \bar{a}b_{n+1} - d\bar{b}_0. \end{aligned} \quad (2.19)$$

Observe that the first and the last equations in (2.19) coincide with (2.18). According to (2.17) and (2.18), one obtains the following relation for the coefficient  $\beta$  of the polynomial  $g(z)$ :

$$\beta = \left( |b_{n+1}|^2 - |b_0|^2 \right) \left( \frac{|c|^2}{|b_0|^2} - \frac{|d|^2}{|b_{n+1}|^2} \right) \neq 0. \quad (2.20)$$

Thus the coefficients  $a_0, a_1, \dots, a_n$  of the polynomial  $f$  can be uniquely determined from system (2.19).

It follows from Theorems 2.1 and 2.2 that the existence of a regular point for the polynomial  $F$  is a sufficient condition for the existence of a solution to the functional equation (1.3).

It turns out that the existence of a regular point for the polynomial  $F$  is intimately connected to the linear (in)dependence of the polynomials  $F$  and  $F^*$  in the complex linear space of (complex) polynomials. To be more precise, *there exists a regular point for  $F$  if and only if  $F$  and  $F^*$  are linearly independent*. This statement is a direct consequence of the following lemma.

LEMMA 2.3. *The following conditions are equivalent:*

- (a)  *$F$  and  $F^*$  are linearly independent in the complex linear space of (complex) polynomials;*
- (b) *the identity  $|F(z)| \equiv |F^*(z)|$  is satisfied;*
- (c) *the identity*

$$2 \operatorname{Re} [\overline{F(w)} F'(w) w] \equiv (n+1) |F(w)|^2 \quad \forall |w| = 1 \quad (2.21)$$

*is satisfied.*

*Proof.* Assume (a) is satisfied:  $F^* = C \cdot F$  for some nonzero complex number  $C$ . Then, according to the definition of the polynomial  $F^*$ , one has the following equality for the coefficients  $b_0, b_{n+1}$ :  $\bar{b}_0 = C b_{n+1}$ ,  $\bar{b}_{n+1} = C b_0$ , from which it follows that  $|C| = 1$ , and therefore,  $|F^*(z)| \equiv |F(z)|$ .

Thus (a) implies (b).

Assume, further,  $|F(z)| \equiv |F^*(z)|$ . Applying to this identity the change of coordinates  $z = rw$ ,  $r \geq 0$ ,  $|w| = 1$  and using the definition of  $F^*$ , one obtains

$$r^{2n+2} F(r^{-1}w) \overline{F(r^{-1}w)} \equiv F(rw) \overline{F(rw)}. \quad (2.22)$$

Differentiating the last identity with respect to the real argument  $r$  at the point  $r = 1$  we obtain (c).

Thus (b) implies (c).

Finally, assume that (c) is satisfied and show that  $F$  and  $F^*$  are linearly dependent. Consider the function  $\overline{F(w)}/F(w)$  on the unit circle  $|w| = 1$ , where  $F(w) \neq 0$ , and show that this function has a continuous extension over the unit circle. Assume that  $F$  vanishes at some point  $w_0$  belonging to the unit circle. Then we have the following representation:  $F(z) = (z - w_0)^k F_1(z)$ , where  $k \geq 1$  is an integer,  $F_1(w_0) \neq 0$ .

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The conditions

$$\lim_{w \rightarrow w_0} \frac{\bar{w} - \bar{w}_0}{w - w_0} = -\frac{1}{w_0^2}, \quad |w| = 1, \quad (2.23)$$

yield

$$\lim_{w \rightarrow w_0} \frac{\overline{F(w)}}{F(w)} = (-1)^k w_0^{-2k} \frac{\overline{F_1(w_0)}}{F_1(w_0)}, \quad |w| = 1, \quad (2.24)$$

providing the continuity of the function  $\overline{F(w)}/F(w)$ .

Differentiating the function  $w^{n+1}\overline{F(w)}/F(w)$ ,  $w = \exp(it)$  with respect to  $t$  at the points where  $F(w) \neq 0$  and using condition (c), one obtains

$$\frac{d}{dt} \frac{w^{n+1}\overline{F(w)}}{F(w)} = iw^{n+1} \left\{ \frac{(n+1)\overline{F(w)} - \overline{F'(w)}w}{F(w)} - \frac{\overline{F(w)}F'(w)w}{F^2(w)} \right\} = 0. \quad (2.25)$$

The above equality along with the continuity of the function  $w^{n+1}\overline{F(w)}/F(w)$  on the unit circle yield that the latter function is, in fact, constant, that is  $w^{n+1}\overline{F(w)} = C \cdot F(w)$  or, equivalently,  $F^*(w) = C \cdot F(w)$ ,  $|w| = 1$ . Therefore, the coefficients of the polynomials  $C \cdot F$  and  $F^*$  coincide. The linear dependence of the polynomials  $F$  and  $F^*$  is established and the proof of Lemma 2.3 is complete.  $\square$

### 3. An algorithm for computing the number of roots in the unit circle

In what follows we will be interested in the case when the coefficients  $a, b, c, d$  satisfy the following additional condition:

$$|a + bw| \geq |c + dw|, \quad |w| = 1. \quad (3.1)$$

LEMMA 3.1. *Linear functions  $a + bz, c + dz$  satisfy condition (3.1) if and only if the numbers  $\alpha = \bar{a}b - \bar{c}d$ ,  $\beta = |a|^2 + |b|^2 - |c|^2 - |d|^2$  satisfy the inequality*

$$2|\alpha| \leq \beta. \quad (3.2)$$

*Proof.* The equality

$$|a + bz|^2 - |c + dz|^2 = |a|^2 + 2\Re(\bar{a}bz) + |b|^2|z|^2 - |c|^2 - 2\Re(\bar{c}dz) - |d|^2|z|^2 \quad (3.3)$$

yields, for  $z = e^{it}$ ,  $t \in [0, 2\pi]$ ,

$$|a + bz|^2 - |c + dz|^2 = \beta + 2\Re[\alpha e^{it}]. \quad (3.4)$$

Combining this with the equality

$$\min_t \Re[\alpha e^{it}] = -|\alpha|, \quad (3.5)$$

one obtains the equivalence of conditions (3.1) and (3.2). The lemma is proved.  $\square$



Observe that Lemma 3.1 allows one to verify effectively the validity of condition (3.1) for the coefficients  $a, b, c, d$  determined by regular points of the polynomial  $F$ .

LEMMA 3.2. *Let  $z_0, |z_0| \leq 1$ , be a regular point of the polynomial  $F$ . Assume that the numbers  $a, b, c, d$  are determined by equalities (2.5) with*

$$(|A_0| - 1)(|\sigma_0 A_0| - |\sigma_1|) > 0, \quad |z_0| < 1, \quad (3.6)$$

or that they satisfy (2.12) with

$$\Delta_0 \left[ \Delta_0 |c + z_0 d|^2 - |F(z_0)|^2 (|c|^2 - |d|^2) \right] > 0, \quad |z_0| = 1. \quad (3.7)$$

Then  $a, b, c, d$  satisfy (3.1).

The statement following below provides an important property of solution  $(a, b, c, d, f)$  to the functional equation (1.3).

THEOREM 3.3. *Assume that the polynomial  $F$  does not contain roots on the unit circle  $|z| = 1$ . Suppose, further, that the coefficients  $a, b, c, d$  satisfy condition (3.1) and*

$$|ad - bc| + \beta > 0, \quad \beta = |a|^2 + |b|^2 - |c|^2 - |d|^2. \quad (3.8)$$

Then the polynomial  $f$  as well as any polynomial of the parameterized family

$$G_\lambda(z) = (a + bz)f(z) + \lambda(c + dz)f^*(z), \quad 0 \leq \lambda \leq 1, \quad (3.9)$$

does not contain roots on the unit circle  $|z| = 1$ .

*Proof.* Arguing indirectly, one obtains the existence of numbers  $w, |w| = 1$ , and  $\lambda \in [0, 1]$  such that

$$G_\lambda(w) = (a + bw)f(w) + \lambda(c + dw)f^*(w) = 0. \quad (3.10)$$

Since  $\bar{w}w = 1$ , one has

$$f^*(w) = \overline{w^n f\left(\frac{1}{\bar{w}}\right)} = w^n \overline{f(w)}, \quad (3.11)$$

from which it follows that

$$(a + bw)f(w) + \lambda(c + dw)w^n \overline{f(w)} = 0. \quad (3.12)$$

By condition,

$$G_1(w) = (a + bw)f(w) + (c + dw)w^n \overline{f(w)} = F(w) \neq 0, \quad (3.13)$$

therefore,  $f(w) \neq 0$  and  $0 \leq \lambda < 1$ . Now, using the equality (3.12), we obtain

$$|a + bw| = \lambda |c + dw|, \quad (3.14)$$

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from which it follows (see Lemma 3.1) that  $|a + bw| = 0$ ,  $|c + dw| = 0$ . The latter equalities yield  $|a| = |b|$ ,  $|c| = |d|$ ,  $ad - bc = 0$  that contradicts condition (3.8) and the result follows.  $\square$

Denote by  $\kappa(F)$  the number of roots of the polynomial  $F$  (counted according to their multiplicity) belonging to the open unit disc  $|z| < 1$ . As noted, the computing  $\kappa(F)$  maybe a reduction to the generalized Rauss-Hurwitz problem of determining the number of roots of a polynomial in a semiplane (see, for instance, [1–3]). The above results allow us to construct an iterative process for computing  $\kappa(F)$ , namely, the following theorem.

**THEOREM 3.4.** *Assume that the polynomial  $F$  does not have roots on the unit circle  $|z| = 1$ . If the polynomials  $F$  and  $F^*$  are linearly dependent, then*

$$\kappa(F) = \frac{n+1}{2}, \quad (3.15)$$

otherwise

$$\kappa(F) = \kappa(G_0), \quad (3.16)$$

where  $G_0(z) = (a + bz)f(z)$  and the collection  $(a, b, c, d, f)$  satisfying (3.1), (3.8) is a solution to the functional equation (1.3).

*Proof.* Assume the polynomials  $F$  and  $F^*$  to be linearly dependent, that is  $F^*(z) \equiv CF(z)$ . Then the following presentation takes place:

$$F(z) = b_{n+1}(z - z_1)^{\alpha_1} \cdots (z - z_m)^{\alpha_m} \left(z - \frac{1}{\bar{z}_1}\right)^{\alpha_1} \cdots \left(z - \frac{1}{\bar{z}_m}\right)^{\alpha_m}, \quad (3.17)$$

where

$$|z_s| < 1, \quad s = 1, \dots, m, \quad 2(\alpha_1 + \cdots + \alpha_m) = n + 1. \quad (3.18)$$

From this it follows that the number  $n + 1$  is even and

$$\kappa(F) = \alpha_1 + \cdots + \alpha_m = \frac{(n+1)}{2}. \quad (3.19)$$

Assume now the polynomials  $F$  and  $F^*$  to be linearly independent. Then, by Lemma 2.3,  $F$  admits a regular point  $z_0$ ,  $|z_0| \leq 1$ . Therefore, by Lemmas 1.1, 2.3, and 3.2, there exists a collection  $(a, b, c, d, f)$  satisfying conditions (3.1), (3.8) and being a solution to the functional equation (1.3). Combining Theorem 3.3 and the Rouché theorem one obtains  $\kappa(F) = \kappa(G_0)$ , where  $G_0(z) = (a + bz)f(z)$ .

The proof of Theorem 3.4 is complete.  $\square$

It is easy to see that the number  $\kappa(G_0)$  satisfies the equality

$$\kappa(G_0) = \kappa(f) + \varepsilon, \quad (3.20)$$

where  $\varepsilon = 1$  for  $|a| < |b|$  and  $\varepsilon = 0$ , otherwise. A simple argument shows that if the numbers  $a, b, c, d$  are determined by a regular point  $z_0$ , then  $\varepsilon$  can be evaluated according to

the formulae

$$\varepsilon = \frac{1 + \operatorname{sign}(|F^*(z_0)| - |F(z_0)|)}{2}, \quad |z_0| < 1, \quad \varepsilon = \frac{1 - \operatorname{sign}(\Delta_0)}{2}, \quad |z_0| = 1. \quad (3.21)$$

Thus, under the assumptions of Theorems 2.1 and 2.2, one has

$$\kappa(F) = \frac{1 + \operatorname{sign}(|F^*(z_0)| - |F(z_0)|)}{2} + \kappa(f), \quad |z_0| < 1, \quad (3.22)$$

$$\kappa(F) = \frac{1 - \operatorname{sign}(\Delta_0)}{2} + \kappa(f), \quad |z_0| = 1. \quad (3.23)$$

Formulae (3.22) and (3.23) give rise to a recurrent procedure for the computation of  $\kappa$ . Indeed, they allow one to compute  $\kappa(F)$ , where  $F$  is a polynomial of degree  $n + 1$ , based on  $\kappa(f)$ , where  $f$  is a polynomial of degree  $n$  and its coefficients are completely determined by coefficients of  $F$ .

Observe that if  $z_0 = 0$ , then formula (3.22) takes the form

$$\kappa(F) = \frac{1 + \operatorname{sign}(|b_{n+1}| - |b_0|)}{2} + \kappa(f). \quad (3.24)$$

Here the coefficients of  $f$  can be determined from system (2.19).

#### 4. Criterion for the absence of roots on the unit circle

Given the coefficients of the polynomials  $F$  and  $F^*$ , one can construct the following  $(2n + 2) \times (2n + 2)$  matrix:

$$M_F = \begin{pmatrix} b_0 & b_1 & \cdot & b_{n+1} & \cdot & 0 \\ \bar{b}_{n+1} & \bar{b}_n & \cdot & \bar{b}_0 & \cdot & 0 \\ 0 & b_0 & \cdot & b_n & \cdot & 0 \\ 0 & \bar{b}_{n+1} & \cdot & \bar{b}_0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & b_1 & \cdot & b_{n+1} \\ 0 & 0 & \cdot & \bar{b}_{n+1} & \cdot & \bar{b}_0 \end{pmatrix}. \quad (4.1)$$

The matrix  $M_F$  coincides with the Sylvester matrix of  $F$  and  $F^*$  (up to a permutation of its lines). Therefore,  $\det(M_F)$  coincides (up to a sign) with the resultant  $R(F, F^*)$  of the polynomials  $F$  and  $F^*$ . Let  $z_s, s = 1, \dots, n + 1$  be all the roots of  $F$  (counted according to their multiplicities). By condition,  $F(0) = b_0 \neq 0$ , therefore, all the roots are different from zero. By definition of the polynomial  $F^*$ , the numbers  $\bar{z}_s^{-1}, s = 1, \dots, n + 1$ , are roots of  $F^*$ . Hence (cf. [5]),  $R(F, F^*)$  can be represented as follows:

$$R(F, F^*) = \bar{b}_0^{n+1} b_{n+1}^{n+1} \prod_{s,t} \left( z_s - \frac{1}{\bar{z}_t} \right). \quad (4.2)$$

Formula (4.2) gives rise to the following criteria for the polynomial  $F$  to have no roots on the unit circle.

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**THEOREM 4.1.** *Let  $\det(M_F)$  be different from zero. Then  $F$  does not have roots on the unit circle  $|z| = 1$ .*

Assume that a collection  $(a, b, 1, 1, f)$  of the numbers  $a, b, c = d = 1$  and a polynomial  $f$  satisfy the functional equation (1.3) and the condition  $\alpha \equiv \bar{a}b - 1 = 0$ . Using  $a, b$  define the following square matrix of order  $2n + 2$ :

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ -\bar{b} & 1 & -\bar{a} & 1 & \cdot & 0 & 0 & 0 & 0 \\ 1 & -a & 1 & -b & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{b} & 1 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & -\bar{b} & 1 & -\bar{a} & 1 \\ 0 & 0 & 0 & 0 & \cdot & 1 & -a & 1 & -b \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.3)$$

It is easy to see that  $\det(P)$  satisfies the following condition:

$$\det P = (-1)^{n-1} \beta^{n-1} (1 - |a|^2), \quad (4.4)$$

where  $\beta = |a|^2 + |b|^2 - 2$ .

Let  $M_f$  be a square matrix of order  $2n$  determined by the polynomial  $f$ . Using the matrices  $M_F, M_f$ , and  $P$  one can express a connection between the coefficients of polynomials  $F$  and  $f$ , given in system (2.19), by the following matrix equality:

$$PM_F = \begin{pmatrix} b_0 & b_1 & b_2 & \cdot & & 0 \\ 0 & & & & & 0 \\ 0 & & \beta M_f & & & 0 \\ \cdot & & & & & \cdot \\ 0 & & \cdot & \bar{b}_2 & \bar{b}_1 & \bar{b}_0 \end{pmatrix}. \quad (4.5)$$

Consider the matrix written in the right-hand side of (4.5) more intently: (i) its first and last lines coincide with the corresponding lines of the matrix  $M_F$ ; (ii) the remaining entries (except for zeros related to the first and the last column) are filled out by the elements of the matrix  $\beta \cdot M_f$ . Therefore, equality (4.5) connects  $\det(M_F)$  and  $\det(M_f)$  as follows.

**THEOREM 4.2.** *Let  $c = d = 1$  and assume that the numbers  $a, b$  along with coefficients  $a_0, \dots, a_n$  of the polynomial  $f$  satisfy (2.19),  $\beta \neq 0$ . Then one has*

$$\det M_F = (-1)^{n+1} (|b_{n+1}|^2 - |b_0|^2) \beta^n \det M_f. \quad (4.6)$$

*Proof.* Combining equality (4.5), the above formula for  $\det(P)$  with the standard determinant properties yields

$$\det M_F = \frac{|b_0|^2 \det(\beta M_f)}{\det P} = (-1)^{n+1} (|b_{n+1}|^2 - |b_0|^2) \beta^n \det M_f. \quad (4.7)$$

Assume a collection  $(a, b, 1, 1, f, F)$  of the numbers  $a, b, c = d = 1$  and polynomials  $f, F$  to satisfy the functional equation (1.3) and the condition  $\alpha \equiv \bar{a}b - 1 = 0$ . Consider a sequence of collections  $(a^k, b^k, 1, 1, F^k)$  of numbers  $a^k, b^k, c^k = d^k = 1$  and polynomials

$$F^k(z) = b_{0k} + b_{1k}z + \cdots + b_{kk}z^k, \quad b_{0k} \neq 0, \quad b_{kk} \neq 0, \quad (4.8)$$

of degree  $k$ , satisfying the functional equation

$$(a^k + b^k z)F^k(z) + (1+z)(F^k)^*(z) = F^{k+1}(z), \quad k = n, n-1, \dots, 1, \quad (4.9)$$

and the condition

$$\alpha^k \equiv \overline{a^k} b^k - 1 = 0, \quad (4.10)$$

where  $a^n = a, b^n = b, F^n = f, F^{n+1} = F$ . By Theorem 2.1, if  $z = 0$  is a regular point of the polynomial

$$F^{k+1}(z) = b_{0k+1} + b_{1k+1}z + \cdots + b_{k+1,k+1}z^{k+1}, \quad (4.11)$$

that is,

$$b_{0k+1} \cdot b_{k+1,k+1} \neq 0, \quad |b_{0k+1}| \neq |b_{k+1,k+1}|, \quad (4.12)$$

then the system of (4.9), (4.10) has the unique solution  $(a^k, b^k, F^k)$ . Moreover,

$$\overline{a^k} = \frac{\overline{b_{0k+1}}}{b_{k+1,k+1}}, \quad \overline{b^k} = \frac{1}{a^k}. \quad (4.13)$$

This along with formula (3.24) justify the following relation:

$$\kappa(F) = \sum_{k=1}^{n+1} \frac{1 + \text{sign}(|b_{kk}| - |b_{0k}|)}{2}. \quad (4.14)$$

□

## 5. Application for computing the rotation of a plane vector field

(1) Consider a vector field  $\Phi(x, y) = \{p(x, y), q(x, y)\}$ , where

$$p(x, y) = \sum_{k,j} a_{kj} x^k y^j, \quad q(x, y) = \sum_{k,j} b_{kj} x^k y^j \quad (5.1)$$

are polynomials in real variables  $x$  and  $y$  with real coefficients. Assume that  $\Phi(x, y) \neq 0, (x, y) \in S = \{(x, y) : x^2 + y^2 = 1\}$ . We are interested in computing the rotation  $\gamma(\Phi, S)$ .

Recall the definition of rotation. Consider the complex presentation of the field  $\Phi$ :

$$p + iq = \exp(i\theta(t)) |p + iq|, \quad p + iq = p(\cot t, \sin t) + iq(\cos t, \sin t), \quad t \in [0, 2\pi], \quad (5.2)$$

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where  $\theta(t)$  is a continuous function. Then (cf. [4])

$$\gamma(\Phi, S) := \frac{1}{2\pi} (\theta(2\pi) - \theta(0)). \quad (5.3)$$

The following statement reduces the computation of rotation of a plane vector field to the computation of the number of roots in the unit disc of some polynomial.

LEMMA 5.1. *Given a (polynomial) plane vector field  $\Phi$ , there exists a unique pair  $(m, F)$ , where  $m$  is an integer and  $F$  is a polynomial in complex variable with complex coefficients,  $F(0) \neq 0$ , satisfying the following condition:*

$$F(z) = z^m (p(x, y) + iq(x, y)), \quad z = x + iy, |z| = 1. \quad (5.4)$$

*Proof.* Take a polynomial  $P(x, y) = p(x, y) + iq(x, y)$  in real variables  $x, y$ . The change of variables

$$(x, y) \rightarrow \left( \frac{1+z^2}{2z}, \frac{i(1-z^2)}{2z} \right) \quad (5.5)$$

determines the rational function in complex variable  $z$ :

$$R(z) = P\left( \frac{1+z^2}{2z}, \frac{i(1-z^2)}{2z} \right). \quad (5.6)$$

The function  $R(z)$  can be represented in the form

$$R(z) = \frac{F(z)}{z^m}, \quad (5.7)$$

where  $F(z)$  is a polynomial satisfying the condition  $F(0) \neq 0$  and  $m$  is an integer. Since

$$x = \frac{1+z^2}{2z}, \quad y = \frac{i(1-z^2)}{2z}, \quad z = x + iy, |z| = 1, \quad (5.8)$$

the pair  $(m, F)$  satisfies (5.4).

To complete the proof of Lemma 5.1, it remains to establish the uniqueness of the pair satisfying (5.4). Suppose that  $(m_1, F_1)$  is another pair satisfying

$$F_1(z) = z^{m_1} (p(x, y) + iq(x, y)), \quad z = x + iy, |z| = 1, \quad (5.9)$$

and  $F_1(0) \neq 0$ . Assuming, without loss of generality, that  $m_1 \geq m$ , one obtains the following equalities for  $F_1$  and  $z^{m_1-m}F$ :

$$F_1(z) = z^{m_1} P(x, y) = z^{m_1-m} z^m P(x, y) = z^{m_1-m} F(z) \quad (5.10)$$

for  $z = x + iy, |z| = 1$ . From this it follows that  $F_1$  and  $z^{m_1-m}F$  coincide. Further, by assumption,  $F(0) \neq 0, F_1(0) \neq 0$ , hence  $m_1 = m$ . Thus  $F_1 = F$  and Lemma 5.1 is completely proved.  $\square$

Let  $\Phi$  and  $F$  be as in Lemma 5.1. Then  $\gamma(\Phi, S)$  and  $\kappa(F)$  satisfy the following relation:

$$\kappa(F) = m + \gamma(\Phi, S). \quad (5.11)$$

(2) Assume a field  $\Phi$  to be given in a parametric form:  $\Phi(t) = \{p(t), q(t)\}$ , where  $p(t), q(t)$  are real continuous  $2\pi$ -periodic functions. Suppose  $\Phi(t) \neq 0, t \in [0, 2\pi]$ . The field  $\Phi$  may be considered as the one defined on the unit circle  $S$ , by assigning to each point  $x = \cos t, y = \sin t$  the vector  $\{p(t), q(t)\}$ . Therefore, the rotation  $\gamma(\Phi, S)$  is correctly defined on  $S$ . Assuming the functions  $p(t), q(t)$  to be smooth enough, one can assign to the field  $\Phi$  the Fourier series of the complex function  $P(t) = p(t) + iq(t)$ :

$$P(t) = \sum_{k=-\infty}^{\infty} c_k \exp(ikt). \quad (5.12)$$

Since the series (5.12) converges uniformly and  $\Phi$  does not vanish, there exists an integer  $N$  such that for all  $t \in [0, 2\pi]$  the following estimate is true:

$$\left| P(t) - \sum_{k=-N}^N c_k \exp(ikt) \right| < |P(t)|. \quad (5.13)$$

Set  $m := \max\{-k : |c_k| > 0, |k| \leq N\}$  and consider the polynomial

$$F(z) = \sum_{k=0}^{N+m} c_{k-m} z^k. \quad (5.14)$$

Using the same arguments as in Section 1 it is easy to check that  $\gamma(\Phi, S)$  and  $\kappa(F)$ , where  $F$  is defined by (5.14), satisfy (5.11).

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