

Research Article

Newton-Kantorovich and Smale Uniform Type Convergence Theorem for a Deformed Newton Method in Banach Spaces

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Newton-Kantorovich and Smale uniform type of convergence theorem of a deformed Newton method having the third-order convergence is established in a Banach space for solving nonlinear equations. The error estimate is determined to demonstrate the efficiency of our approach. The obtained results are illustrated with three examples.

1. Introduction

In this paper, we study the problem of approximating a unique solution x^* of a nonlinear operator equation

$$F(x) = 0, \quad (1)$$

where F is a Fréchet-differentiable operator defined on an open convex Ω of a Banach space X with values in a Banach space Y .

There are many iterative methods (see [1–3]), which have been used for finding a solution of (1). For example, the well-known iterative method for solving (1) is Newton's method defined by

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad (n \geq 0) \quad (x_0 \in \Omega). \quad (2)$$

Under the appropriate assumptions, Newton's method is the second-order convergence. Kantorovich (see [4]) presented the famous convergence result regarding a solution of (1). Many Newton-Kantorovich type of convergence theorems were given in papers [5–11]. Frontini and Sormani (see [12]) presented a new deformed Newton method with

$$\int_{x_n}^x f'(t) dt \approx (x - x_n) f'\left(\frac{x_n + x}{2}\right). \quad (3)$$

The deformed Newton method can be written as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/2f'(x_n))}, \quad (4)$$

where f is a real or a complex function. In papers [13–17], the local convergence theorem has been established and the deformed method in a real or a complex space was discussed.

In the paper, we generalize the deformed Newton method [18] in a Banach space. The deformed Newton method [18] is shown as follows:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ x_{n+1} &= x_n - F'\left(\frac{x_n + y_n}{2}\right)^{-1} F(x_n), \end{aligned} \quad (5)$$

where F is defined on an open convex subset Ω of a Banach space X with values in a Banach space Y , $F(x)$ has Fréchet derivatives in Ω , and $F'(x)^{-1}$ exists.

We establish Newton-Kantorovich and Smale uniform type convergence theorem (see [18]) for the deformed Newton method with the third-order in a Banach space with new sufficient conditions for the existence of a well-defined sequence which converges to a unique solution x^* of (1).

2. Main Results

Denote $g(t) = \int_0^t (t-u)L(u)du - t + \eta$, $u \in (0, R)$, $\eta > 0$, and suppose $L(u)$, $L'(u)$ are the positive and nondecreasing continuous functions, $\lim_{t \rightarrow R^+} g(t) = g(R^+) > 0$, $\int_0^R L(u)du > 1$, $\int_0^\alpha L(u)du = 1$ for $\alpha \in (0, R)$, $\beta = \alpha - \int_0^\alpha (\alpha - u)L(u)du = \int_0^\alpha uL(u)du$.

Assume that sequences $\{t_n\}$, $\{s_n\}$ are generated by the following formulae [18]:

$$\begin{aligned} s_n &= t_n - g'(t_n)^{-1} g(t_n), \\ t_{n+1} &= t_n - g'\left(\frac{t_n + s_n}{2}\right)^{-1} g(t_n), \quad t_0 = 0. \end{aligned} \quad (6)$$

Firstly, we give some lemmas.

Lemma 1. *If $\eta \leq \beta$, then the function $g(t)$ has two positive real roots r_1, r_2 ($0 < r_1 \leq \alpha \leq r_2 < R$).*

Proof. Because $g(0) = \eta > 0$, $g(R^+) > 0$, and $g''(t) = L(t) > 0$, we know that $g(t)$ is the convex function for $t \in (0, R)$. Hence, α is a unique positive root of $g'(t) = \int_0^t L(u)du - 1$. So, the necessary and sufficient condition that $g(t)$ has two positive roots for $t \in (0, R)$ is that the minimum of $g(t)$ satisfies the condition $g(\alpha) \leq 0$, which holds for $\eta \leq \beta$. This completes the proof of Lemma 1. \square

Lemma 2. *Suppose the sequences $\{t_n\}$, $\{s_n\}$ are generated by (6). Then, for $\eta \leq \beta$, the sequences $\{t_n\}$, $\{s_n\}$ are increasing and converge to the minimum positive root of $g(t)$, and*

$$0 \leq t_n \leq s_n \leq t_{n+1} < r_1. \quad (7)$$

Proof. Denote

$$U(x) = x - \frac{g(x)}{g'(x)}, \quad V(x) = x - \frac{g(x)}{g'((x+U(x))/2)}. \quad (8)$$

On $[0, r_1]$, we know $g(t) > 0$, $g'(t) < 0$, $g''(t) > 0$, and $g''(t)$ is increasing. Denoting $y = (x + U(x))/2 = x - g(x)/2g'(x)$, then

$$U'(x) = \frac{g(x)g''(x)}{g'(x)^2} > 0,$$

$$\begin{aligned} [g'(y) - g'(x)] &= g''(\xi)(y - x) = -g''(\xi) \frac{g(x)}{2g'(x)}, \\ \xi &\in (x, y), \end{aligned}$$

$$\begin{aligned} V'(x) &= 1 - \left(g'(x)g'(y) - \frac{1}{2}g(x)g''(y) \right) \\ &\quad \times 1 + \left(\frac{g(x)g''(x)}{g'(x)^2} \right) \end{aligned}$$

$$\begin{aligned} &\times (g'(y)^2)^{-1} \\ &= \frac{1}{g'(y)} [g'(y) - g'(x)] + \frac{g(x)g''(y)}{2g'(y)^2} \\ &\quad + \frac{g(x)^2g''(x)g''(y)}{2g'(x)^2g'(y)^2} \geq -\frac{g''(\xi)}{g'(y)} \cdot \frac{g(x)}{2g'(x)} \\ &\quad + \frac{g(x)g''(y)}{2g'(y)^2} = -\frac{g(x)g''(\xi)}{2g'(y)^2g'(x)} [g'(y) - g'(x)] \\ &\quad + \frac{g(x)g''(y) - g(x)g''(\xi)}{2g'(y)^2} = \frac{g(x)g''(\xi)}{2g'(y)^2g'(x)} \\ &\quad \cdot \frac{g(x)g''(\xi)}{2g'(x)} + \frac{g(x)g''(y) - g(x)g''(\xi)}{2g'(y)^2} > 0. \end{aligned} \quad (9)$$

Therefore, $U(x)$, $V(x)$ are increasing on $[0, r_1]$. Thus, for $x \in [0, r_1]$, $U(x) < U(r_1) = r_1$, $V(x) < V(r_1) = r_1$. Moreover,

$$s_n = U(t_n), \quad t_{n+1} = V(t_n), \quad t_0 = 0 < r_1; \quad (10)$$

hence we can easily prove Lemma 2 by the induction.

Suppose X and Y are the Banach spaces, $\Omega \subset X$ is an open convex subset, $F : \Omega \subset X \rightarrow Y$ has the second-order Fréchet derivative, $F'(x_0)^{-1}$ exists for $x_0 \in \Omega$, and the following conditions hold:

$$\begin{aligned} \|F'(x_0)^{-1}F(x_0)\| &\leq \eta, \quad \|F'(x_0)^{-1}F''(x_0)\| \leq L(0), \\ \|F'(x_0)^{-1}(F''(y) - F''(x))\| &\leq L(y) - L(x), \end{aligned} \quad (11)$$

$$\leq \int_{\rho(x)}^{\rho(\overline{x}, y)} L'(u)du, \quad x, y \in \Omega, \quad \rho(\overline{x}, y) < \alpha,$$

where $\rho(x) = \|x - x_0\|$ and $\rho(\overline{x}, y) = \|y - x\| + \|x - x_0\|$. \square

Lemma 3. *Suppose F satisfies (11) and $\|x - x_0\| < \alpha$. Then $F'(x)^{-1}$ exists, and*

$$\begin{aligned} \|F'(x_0)^{-1}F''(x)\| &\leq g''(\|x - x_0\|), \\ \|F'(x)^{-1}F'(x_0)\| &\leq -\frac{1}{g'(\|x - x_0\|)}. \end{aligned} \quad (12)$$

Proof. Firstly, by the conditions (11), we know that

$$\begin{aligned} \|F'(x_0)^{-1}F''(x)\| &\leq \|F'(x_0)^{-1}F''(x_0)\| \\ &\quad + \|F'(x_0)^{-1}F''(x) - F'(x_0)^{-1}F''(x_0)\| \\ &\leq L(0) + \int_0^{\|x - x_0\|} L'(u)du \\ &= L(\|x - x_0\|) = g''(\|x - x_0\|). \end{aligned} \quad (13)$$

Secondly, we know $g'(t) < 0$ for $t < \alpha$. Hence

$$\begin{aligned}
 \|F'(x_0)^{-1}F'(x) - I\| &= \|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \\
 &= \left\| F'(x_0)^{-1} \int_0^1 F''(x_0 + t(x - x_0)) \right. \\
 &\quad \left. \times (x - x_0) dt \right\| \\
 &\leq \int_0^1 g''(t\|x - x_0\|) \|x - x_0\| dt \\
 &= g'(\|x - x_0\|) - g'(0) \\
 &= g'(\|x - x_0\|) + 1 < 1.
 \end{aligned} \tag{14}$$

By Banach Theorem, we know $F'(x)^{-1}$ exists, and

$$\begin{aligned}
 \|F'(x)^{-1}F'(x_0)\| &\leq \frac{1}{1 - \|F'(x_0)^{-1}F'(x) - I\|} \\
 &= -\frac{1}{g'(\|x - x_0\|)}.
 \end{aligned} \tag{15}$$

This completes the proof of Lemma 3. \square

Lemma 4. Suppose X and Y are Banach spaces, Ω is an open convex of the Banach space X , $F : \Omega \subset X \rightarrow Y$ has the second-order Fréchet derivative, and the sequences $\{x_n\}$, $\{y_n\}$ are generated by (5). Then, for any natural number n , the following formula holds:

$$\begin{aligned}
 &F(x_{n+1}) \\
 &= \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1-t) dt \\
 &\quad \times (x_{n+1} - y_n)^2 \\
 &\quad + \frac{1}{2} \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1-t) dt \\
 &\quad \times (x_{n+1} - y_n)(y_n - x_n) \\
 &\quad + \frac{1}{2} \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1-t) dt \\
 &\quad \times (y_n - x_n)(x_{n+1} - y_n) \\
 &\quad + \frac{1}{4} \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1-t) dt \\
 &\quad \times (y_n - x_n)^2 \\
 &\quad - \frac{1}{4} \int_0^1 F''\left(\frac{x_n + y_n}{2} - t\left(\frac{y_n - x_n}{2}\right)\right)(1-t) dt \\
 &\quad \times (y_n - x_n)^2.
 \end{aligned} \tag{16}$$

Proof. By (5), we have

$$\begin{aligned}
 F(x_{n+1}) &= F(x_{n+1}) - F\left(\frac{x_n + y_n}{2}\right) \\
 &\quad - F'\left(\frac{x_n + y_n}{2}\right)\left(x_{n+1} - \frac{x_n + y_n}{2}\right) \\
 &\quad + F\left(\frac{x_n + y_n}{2}\right) + F'\left(\frac{x_n + y_n}{2}\right) \\
 &\quad \times \left(x_{n+1} - \frac{x_n + y_n}{2}\right) \\
 &= \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right)(1-t) dt \\
 &\quad \times \left(x_{n+1} - \frac{x_n + y_n}{2}\right)^2 + F\left(\frac{x_n + y_n}{2}\right) \\
 &\quad + F'\left(\frac{x_n + y_n}{2}\right)\left(x_{n+1} - \frac{x_n + y_n}{2}\right), \\
 &F\left(\frac{x_n + y_n}{2}\right) + F'\left(\frac{x_n + y_n}{2}\right)\left(x_{n+1} - \frac{x_n + y_n}{2}\right) \\
 &= F\left(\frac{x_n + y_n}{2}\right) + F'\left(\frac{x_n + y_n}{2}\right) \\
 &\quad \times \left(x_{n+1} - x_n - \frac{y_n - x_n}{2}\right) \\
 &= F\left(\frac{x_n + y_n}{2}\right) - F(x_n) - F'\left(\frac{x_n + y_n}{2}\right)\frac{y_n - x_n}{2} \\
 &= -\frac{1}{4} \int_0^1 F''\left(\frac{x_n + y_n}{2} - t\left(\frac{y_n - x_n}{2}\right)\right)(1-t) dt \\
 &\quad \times (y_n - x_n)^2.
 \end{aligned} \tag{17}$$

Hence

$$\begin{aligned}
 F(x_{n+1}) &= \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right) \\
 &\quad \times (1-t) dt (x_{n+1} - y_n)^2 \\
 &\quad + \frac{1}{2} \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right) \\
 &\quad \times (1-t) dt (x_{n+1} - y_n)(y_n - x_n) \\
 &\quad + \frac{1}{2} \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right) \\
 &\quad \times (1-t) dt (y_n - x_n)(x_{n+1} - y_n) \\
 &\quad + \frac{1}{4} \int_0^1 F''\left(\frac{x_n + y_n}{2} + t\left(x_{n+1} - \frac{x_n + y_n}{2}\right)\right) \\
 &\quad \times (1-t) dt (y_n - x_n)^2
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4} \int_0^1 F'' \left(\frac{x_n + y_n}{2} - t \left(\frac{y_n - x_n}{2} \right) \right) (1-t) dt \\
& \times (y_n - x_n)^2.
\end{aligned} \tag{18}$$

This completes the proof of Lemma 4. \square

Theorem 5. Suppose X and Y are Banach spaces, $\Omega \subset X$ is an open convex subset, $F : \Omega \subset X \rightarrow Y$ satisfies condition (11), $\eta \leq \beta$, and

$$\overline{S(x_0, r_1)} = \{x \mid \|x - x_0\| \leq r_1, x \in X\} \subset \Omega. \tag{19}$$

Then the sequence $\{x_n\}_{n \geq 0}$ generated by (5) is well defined, $x_n \in \overline{S(x_0, r_1)}$, and converges to the unique solution x^* in $S(x_0, \alpha)$ and

$$\|x_n - x^*\| \leq r_1 - t_n. \tag{20}$$

Proof. By induction, we can prove that the following formulae hold:

$$\begin{aligned}
& \|x_n - x_0\| \leq t_n; \\
& \|F'(x_n)^{-1} F'(x_0)\| \leq -g'(t_n)^{-1}; \\
& \|y_n - x_n\| \leq s_n - t_n; \\
& \|y_n - x_0\| \leq s_n; \\
& \left\| F' \left(\frac{x_n + y_n}{2} \right)^{-1} F'(x_0) \right\| \leq -g' \left(\frac{t_n + s_n}{2} \right)^{-1}; \\
& \|x_{n+1} - y_n\| \leq t_{n+1} - s_n; \\
& \|x_{n+1} - x_n\| \leq t_{n+1} - t_n.
\end{aligned} \tag{21}$$

In fact, by Lemma 2 we know $t_n < r_1$ for any natural number n . It is easy to prove that for $n = 0$ the above formulae hold. Suppose the above formulae also hold for $n > 0$. Then

$$\begin{aligned}
& \|x_{n+1} - x_0\| \leq \|x_{n+1} - x_n\| \\
& + \|x_n - x_0\| \leq t_{n+1} - t_n + t_n = t_{n+1}.
\end{aligned} \tag{22}$$

By Lemma 3, we get

$$\begin{aligned}
& \|F'(x_{n+1})^{-1} F'(x_0)\| \leq -g'(\|x_{n+1} - x_0\|)^{-1} \\
& \leq -g'(t_{n+1})^{-1}.
\end{aligned} \tag{23}$$

By Lemmas 3 and 4 and the fact that $-g'(t)^{-1}$, $g''(t)$ are positive and increasing on $[0, \alpha)$, we have

$$\begin{aligned}
& \left\| F'(x_0)^{-1} \left[F'' \left(\frac{x_n + y_n}{2} + t \left(x_{n+1} - \frac{x_n + y_n}{2} \right) \right) \right. \right. \\
& \quad \left. \left. - F'' \left(\frac{x_n + y_n}{2} - t \left(\frac{y_n - x_n}{2} \right) \right) \right] \right\| \\
& \leq \int_0^{t\|x_{n+1} - x_n\|} L' \left(u + \left\| \frac{x_n + y_n}{2} - t \left(\frac{y_n - x_n}{2} \right) - x_0 \right\| \right) du \\
& \leq \int_0^{t(t_{n+1} - t_n)} L' \left(u + \frac{t_n + s_n}{2} - t \frac{s_n - t_n}{2} \right) du \\
& = L \left(\frac{t_n + s_n}{2} + t \left(t_{n+1} - \frac{t_n + s_n}{2} \right) \right) \\
& \quad - L \left(\frac{t_n + s_n}{2} - t \left(\frac{s_n - t_n}{2} \right) \right) \\
& = g'' \left(\frac{t_n + s_n}{2} + t \left(t_{n+1} - \frac{t_n + s_n}{2} \right) \right) \\
& \quad - g'' \left(\frac{t_n + s_n}{2} - t \left(\frac{s_n - t_n}{2} \right) \right), \\
& \left\| F'(x_0)^{-1} F(x_{n+1}) \right\| \\
& \leq \int_0^1 g'' \left(\frac{t_n + s_n}{2} + t \left(t_{n+1} - \frac{t_n + s_n}{2} \right) \right) \\
& \quad \times (1-t) dt (t_{n+1} - s_n)^2 \\
& \quad + \frac{1}{2} \int_0^1 g'' \left(\frac{t_n + s_n}{2} + t \left(t_{n+1} - \frac{t_n + s_n}{2} \right) \right) (1-t) dt \\
& \quad \times (t_{n+1} - s_n) (s_n - t_n) \\
& \quad + \frac{1}{2} \int_0^1 g'' \left(\frac{t_n + s_n}{2} + t \left(t_{n+1} - \frac{t_n + s_n}{2} \right) \right) (1-t) dt \\
& \quad \times (s_n - t_n) (t_{n+1} - s_n) \\
& \quad + \frac{1}{4} \int_0^1 g'' \left(\frac{t_n + s_n}{2} + t \left(t_{n+1} - \frac{t_n + s_n}{2} \right) \right) (1-t) dt \\
& \quad \times (s_n - t_n)^2 \\
& \quad - \frac{1}{4} \int_0^1 g'' \left(\frac{t_n + s_n}{2} - t \left(\frac{s_n - t_n}{2} \right) \right) (1-t) dt \\
& \quad \times (s_n - t_n)^2 = g(t_{n+1}).
\end{aligned} \tag{24}$$

Hence we get

$$\begin{aligned}
& \|y_{n+1} - x_{n+1}\| = \left\| -F'(x_{n+1})^{-1} F(x_{n+1}) \right\| \\
& \leq \left\| -F'(x_{n+1})^{-1} F'(x_0) \right\| \\
& \quad \times \left\| F'(x_0)^{-1} F(x_{n+1}) \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq -g'(t_{n+1})^{-1} g(t_{n+1}) \\
&= s_{n+1} - t_{n+1}, \\
\|y_{n+1} - x_0\| &\leq \|y_{n+1} - x_{n+1}\| \\
&\quad + \|x_{n+1} - x_0\| \leq s_{n+1}.
\end{aligned} \tag{25}$$

By Lemma 3, we get

$$\left\| F' \left(\frac{x_{n+1} + y_{n+1}}{2} \right)^{-1} F'(x_0) \right\| \leq -g' \left(\frac{t_{n+1} + s_{n+1}}{2} \right)^{-1}. \tag{26}$$

Moreover, we have

$$\begin{aligned}
&\|x_{n+2} - y_{n+1}\| \\
&= \left\| F'(x_{n+1})^{-1} F(x_{n+1}) \right. \\
&\quad \left. - F' \left(\frac{x_{n+1} + y_{n+1}}{2} \right)^{-1} F(x_{n+1}) \right\| \\
&= \left\| F' \left(\frac{x_{n+1} + y_{n+1}}{2} \right)^{-1} \left[F' \left(\frac{x_{n+1} + y_{n+1}}{2} \right) \right. \right. \\
&\quad \left. \left. - F'(x_{n+1}) \right] \right. \\
&\quad \left. \times F'(x_{n+1})^{-1} F(x_{n+1}) \right\| \\
&= \left\| F' \left(\frac{x_{n+1} + y_{n+1}}{2} \right)^{-1} F'(x_0) F'(x_0)^{-1} \right. \\
&\quad \times \int_0^1 F'' \left(x_{n+1} + \frac{t}{2} (y_{n+1} - x_{n+1}) \right) dt \\
&\quad \times \frac{y_{n+1} - x_{n+1}}{2} F'(x_{n+1})^{-1} \\
&\quad \times F'(x_0) F'(x_0)^{-1} F(x_{n+1}) \left. \right\| \\
&\leq g' \left(\frac{t_{n+1} + s_{n+1}}{2} \right)^{-1} \\
&\quad \times \int_0^1 g'' \left(t_{n+1} + \frac{t}{2} (s_{n+1} - t_{n+1}) \right) dt \\
&\quad \times \frac{(s_{n+1} - t_{n+1})}{2} g'(t_{n+1})^{-1} g(t_{n+1}) \\
&\leq g' \left(\frac{t_{n+1} + s_{n+1}}{2} \right)^{-1} \\
&\quad \times \left[g' \left(\frac{t_{n+1} + s_{n+1}}{2} \right) - g'(t_{n+1}) \right] \\
&\quad \times g'(t_{n+1})^{-1} g(t_{n+1})
\end{aligned}$$

$$\begin{aligned}
&= g'(t_{n+1})^{-1} g(t_{n+1}) \\
&\quad - g' \left(\frac{t_{n+1} + s_{n+1}}{2} \right)^{-1} \\
&\quad \times g(t_{n+1}) = t_{n+2} - s_{n+1}, \\
\|x_{n+2} - x_{n+1}\| &\leq \|x_{n+2} - y_{n+1}\| \\
&\quad + \|y_{n+1} - x_{n+1}\| \leq t_{n+2} - t_{n+1}.
\end{aligned} \tag{27}$$

Hence, the sequence $\{x_n\}_{n \geq 0}$ generated by (5) is well defined, $x_n \in \overline{S(x_0, r_1)}$, and $\{x_n\}$ converges to the solution $x^* \in \overline{S(x_0, r_1)}$ of (1).

Now we prove the uniqueness. Suppose y^* is also a solution of (1) on $S(x_0, \alpha)$. We know that $g'(t) < 0$ for $t \in [0, \alpha)$. Then

$$\begin{aligned}
&\left\| F'(x_0)^{-1} \int_0^1 F'(x^* + t(y^* - x^*)) dt - I \right\| \\
&\leq \left\| F'(x_0)^{-1} \int_0^1 \{F'[x^* + t(y^* - x^*)] - F'(x_0)\} dt \right\| \\
&\leq \left\| F'(x_0)^{-1} \int_0^1 \int_0^1 F''(x_0 + s(x^* - x_0 + t(y^* - x^*))) ds dt \right. \\
&\quad \left. \times (x^* - x_0 + t(y^* - x^*)) \right\| \\
&\leq \int_0^1 \int_0^1 g''(s\|x^* - x_0 + t(y^* - x^*)\|) ds dt \\
&\quad \times \|x^* - x_0 + t(y^* - x^*)\| \\
&= \int_0^1 g'(\|x^* - x_0 + t(y^* - x^*)\|) dt - g'(0) \\
&= \int_0^1 g'(\|(1-t)(x^* - x_0) + t(y^* - x_0)\|) dt + 1 < 1.
\end{aligned} \tag{28}$$

By Banach Theorem, we know the inverse of $\int_0^1 F'[x^* + t(y^* - x^*)] dt$ exists and

$$\begin{aligned}
0 &= F(y^*) - F(x^*) \\
&= \int_0^1 F'[x^* + t(y^* - x^*)] dt (y^* - x^*);
\end{aligned} \tag{29}$$

hence we get $y^* = x^*$. This completes the proof of the uniqueness of the solution of (1).

For $m > n$, we know that

$$\begin{aligned}
\|x_m - x_n\| &\leq \|x_m - x_{m-1}\| \\
&\quad + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\| \leq t_m - t_n.
\end{aligned} \tag{30}$$

When $m \rightarrow \infty$, we get

$$\|x_n - x^*\| \leq r_1 - t_n. \tag{31}$$

This completes the proof of Theorem 5. \square

Suppose that $L(u) = \gamma + Ku$, $u \in (0, +\infty)$, $\gamma, K > 0$. Then $\int_{\rho(x)}^{\rho(\bar{x}, y)} L'(u) du = K\|x - y\|$, $g(t) = (1/6)Kt^3 + (1/2)\gamma t^2 - t + \eta\alpha = 2/(\gamma + \sqrt{\gamma^2 + 2K})$, and $\beta = \alpha - (1/6)K\alpha^3 - (1/2)\gamma\alpha^2 = 2(\gamma + 2\sqrt{\gamma^2 + 2K})/3(\gamma + \sqrt{\gamma^2 + 2K})^2$.

Corollary 6. Suppose X and Y are the Banach spaces, Ω is an open convex subset of the Banach space X , $F : \Omega \subset X \rightarrow Y$ has the second-order Fréchet derivative, $F'(x_0)^{-1}$ exists for $x_0 \in \Omega$, and the following conditions hold:

$$\begin{aligned} \|F'(x_0)^{-1}F(x_0)\| &\leq \eta, & \|F'(x_0)^{-1}F''(x_0)\| &\leq \gamma, \\ \|F'(x_0)^{-1}(F''(x) - F''(y))\| &\leq K\|x - y\| \quad x, y \in \Omega, \\ \eta &\leq \frac{2(\gamma + 2\sqrt{\gamma^2 + 2K})}{3(\gamma + \sqrt{\gamma^2 + 2K})^2}, & \overline{S(x_0, r_1)} &\subset \Omega. \end{aligned} \quad (32)$$

Then the sequence $\{x_n\}_{n \geq 0}$ generated by (5) is well defined, $x_n \in \overline{S(x_0, r_1)}$, and $\{x_n\}$ converges to the unique solution x^* on $S(x_0, \alpha)$ of (1), where $r_1 \leq r_2$ are two positive roots of $g(t) = (1/6)Kt^3 + (1/2)\gamma t^2 - t + \eta$.

Suppose $L(u) = 2\gamma(1 - \gamma u)^{-3}$, $u \in (0, 1/\gamma)$, $g(t) = \eta - t + \gamma t^2/(1 - \gamma t)$, $\alpha = (1 - \sqrt{2}/2)(1/\gamma)$ and $\beta = (3 - 2\sqrt{2})/\gamma$ and for $\|x - x_0\| < \alpha$, $\|F'(x_0)^{-1}F'''(x)\| \leq 6\gamma^2/(1 - \gamma\|x - x_0\|)^4$. Hence, for $\|x - x_0\| + \|y - x\| < \alpha$, we get

$$\begin{aligned} &\|F'(x_0)^{-1}[F''(y) - F''(x)]\| \\ &= \left\| \int_0^1 F'(x_0)^{-1}F'''(x + t(y - x)) dt (y - x) \right\| \\ &\leq \int_0^1 \frac{6\gamma^2}{[1 - \gamma\|x - x_0\| + t\|y - x\|]^4} dt \|y - x\| \\ &\leq \int_0^1 \frac{6\gamma^2}{[1 - \gamma(\|x - x_0\| + t\|y - x\|)]^4} dt \|y - x\| \quad (33) \\ &= \int_{\|x - x_0\|}^{\|x - x_0\| + \|y - x\|} \frac{6\gamma^2}{(1 - \gamma u)^4} u \\ &= \int_{\|x - x_0\|}^{\|x - x_0\| + \|y - x\|} L'(u) du. \end{aligned}$$

Corollary 7 (see [10]). Suppose X and Y are Banach spaces, Ω is an open convex subset of the Banach space X , $F : \Omega \subset$

$X \rightarrow Y$ has the third-order Fréchet derivative, $F'(x_0)^{-1}$ exists for $x_0 \in \Omega$, and the following conditions hold:

$$\begin{aligned} \|F'(x_0)^{-1}F(x_0)\| &\leq \eta, & \|F'(x_0)^{-1}F''(x_0)\| &\leq 2\gamma, \\ \|F'(x_0)^{-1}F'''(x)\| &\leq \frac{6\gamma^2}{(1 - \gamma\|x - x_0\|)^4} \\ &= g'''(\|x - x_0\|), \quad x \in \Omega, \quad (34) \\ \|x - x_0\| &\leq \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{\gamma}, & \eta\gamma &\leq 3 - 2\sqrt{2}, \\ \overline{S(x_0, r_1)} &\subset \Omega. \end{aligned}$$

Then the sequence $\{x_n\}_{n \geq 0}$ generated by (5) is well defined, $x_n \in \overline{S(x_0, r_1)}$, and $\{x_n\}$ converges to the unique solution x^* of (1) on $S(x_0, (1 - 1/\sqrt{2})(1/\gamma))$, where

$$\begin{aligned} r_1 &= \frac{1 + \eta\gamma - \sqrt{(1 + \eta\gamma)^2 - 8\eta\gamma}}{4\gamma}, \\ r_2 &= \frac{1 + \eta\gamma + \sqrt{(1 + \eta\gamma)^2 - 8\eta\gamma}}{4\gamma} \end{aligned} \quad (35)$$

are two positive roots of the equation $g(t) = \eta - t + \gamma t^2/(1 - \gamma t)$.

3. Numerical Examples

In this section, we apply the convergence theorem and show three numerical examples.

Example 1. Consider the equation

$$F(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{3} = 0. \quad (36)$$

We choose the initial point $x_0 = 0$, $\Omega = [-1, 1]$; then

$$\begin{aligned} \eta &= |F'(0)^{-1}F(0)| = \frac{2}{5}, & \gamma &= |F'(0)^{-1}F''(0)| = \frac{2}{5}, \\ K &= \frac{6}{5}, \end{aligned} \quad (37)$$

$$\frac{2(\gamma + 2\sqrt{\gamma^2 + 2K})}{3(\gamma + \sqrt{\gamma^2 + 2K})^2} = \frac{3}{5} > \eta.$$

Hence, by Corollary 6, the sequence $\{x_n\}_{n \geq 0}$ generated by (5) is well defined, and $\{x_n\}$ converges to the solution x^* of (36).

Now, we will analyze errors $\|x_n - x^*\|$ by Corollary 6 (see Table 1). In this case, we take $x_0 = 0$; then $r_1 = 0.462598422 \dots$.

Example 2. Consider the system of equation [18] $F(u, v) = 0$, where

$$F(u, v) = (uv - 1, uv + u - 2v)^T. \quad (38)$$

TABLE 1: Error results for Corollary 6 ($\|x_n - x^*\| \leq r_1 - t_n$).

| Step | $r_1 - t_n$ | Step | $r_1 - t_n$ |
|---------|-----------------------------|---------|-----------------------------|
| $k = 1$ | 1.616985×10^{-2} | $k = 2$ | 2.236349×10^{-6} |
| $k = 3$ | 6.225929×10^{-18} | $k = 4$ | 1.343387×10^{-52} |
| $k = 5$ | $1.349560 \times 10^{-156}$ | $k = 6$ | $1.368249 \times 10^{-468}$ |

Then, we have

$$F'(u, v) = \begin{pmatrix} v & u \\ v+1 & u-2 \end{pmatrix},$$

$$F'(u, v)^{-1} = -\frac{1}{u+2v} \begin{pmatrix} u-2 & -u \\ -v-1 & v \end{pmatrix}, \quad (39)$$

$$F''(u, v) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We choose $x_0 = (u_0, v_0) = (1.75, 1.75)$ and $\Omega = \{x \mid \|x - x_0\| \leq 1.75\}$. We take the max-norm in R^2 and the norm $\|A\| = \max\{|a_{11}| + |a_{12}|, |a_{21}| + |a_{22}|\}$ for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Define the norm of a bilinear operator B on R^2 by

$$\|B\| = \sup_{\|u\|=1} \max_i \sum_{j=1}^2 \left| \sum_{k=1}^2 b_i^{jk} u_k \right|, \quad (40)$$

where $u = (u_1, u_2)^T$ and

$$B = \begin{pmatrix} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \\ b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \end{pmatrix}. \quad (41)$$

Then we get the following results:

$$\eta = \|F'(x_0)^{-1} F(x_0)\| = \frac{9}{14},$$

$$\gamma = \|F'(x_0)^{-1} F''(x_0)\| = \frac{16}{21}, \quad (42)$$

$$K = 0, \quad \frac{2\left(\gamma + 2\sqrt{\gamma^2 + 2K}\right)}{3\left(\gamma + \sqrt{\gamma^2 + 2K}\right)^2} > \eta.$$

This means that the hypotheses of Corollary 6 are satisfied.

Now, we will analyze errors $\|x_n - x^*\|$ by Corollary 6 (see Table 2). In this case, we take $x_0 = (u_0, v_0) = (1.75, 1.75)$; then $r_1 = 1.125$.

Example 3. Consider the following integral equations:

$$x(s) = 1 + \frac{1}{4}x(s) \int_0^1 \frac{s}{s+t} x(t) dt \quad (43)$$

TABLE 2: Error results for Corollary 6 ($\|x_n - x^*\| \leq r_1 - t_n$).

| Step | $r_1 - t_n$ | Step | $r_1 - t_n$ |
|---------|----------------------------|---------|----------------------------|
| $k = 1$ | 2.736486×10^{-1} | $k = 2$ | 3.044252×10^{-2} |
| $k = 3$ | 1.588069×10^{-4} | $k = 4$ | 2.844419×10^{-11} |
| $k = 5$ | 1.636509×10^{-30} | $k = 6$ | 3.116680×10^{-92} |

TABLE 3: Error results for Corollary 7 ($\|x_n - x^*\| \leq r_1 - t_n$).

| Step | $r_1 - t_n$ | Step | $r_1 - t_n$ |
|---------|-----------------------------|---------|----------------------------|
| $k = 1$ | 2.764303×10^{-3} | $k = 2$ | 4.099223×10^{-9} |
| $k = 3$ | 1.344301×10^{-26} | $k = 4$ | 4.741124×10^{-79} |
| $k = 5$ | $2.079868 \times 10^{-236}$ | $k = 6$ | $< 1.0 \times 10^{-500}$ |

and the space $X = C[0, 1]$ with the norm

$$\|x\| = \max_{0 \leq s \leq 1} |x(s)|. \quad (44)$$

This equation arises in the theory of radiative transfer and neutron transport and in the kinetic theory of gases. Define the operator F on X by

$$F(x) = \frac{1}{4}x(s) \int_0^1 \frac{s}{s+t} x(t) dt - x(s) + 1. \quad (45)$$

Then, for $x_0 = 1$, we obtain

$$\eta = \|F'(x_0)^{-1} F(x_0)\| = 0.2652,$$

$$2\gamma = \|F'(x_0)^{-1} F''(x_0)\| = 1.5304 \times 2$$

$$\cdot \frac{1}{4} \max_{0 \leq s \leq 1} \left| \int_0^1 \frac{s}{s+t} dt \right| = 1.5304 \times \frac{\ln 2}{2} = 0.5303, \quad (46)$$

$$\eta\gamma = 0.07032 < 3 - 2\sqrt{2},$$

$$\|F'(x_0)^{-1} F'''(x)\| = 0 < \frac{6\gamma^2}{(1 - \gamma\|x - x_0\|)^4}.$$

This means that the hypotheses of Corollary 7 are satisfied.

Now, we will analyze errors $\|x_n - x^*\|$ by Corollary 7 (see Table 3). In this case, we take $x_0 = 1$; then $r_1 = 0.289222 \dots$.

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