Hindawi
Abstract and Applied Analysis
Volume 2021, Article ID 5568636, 18 pages
https://doi.org/10.1155/2021/5568636

Research Article

Guaranteed Deterministic Approach to Superhedging: Case of Binary European Option

Sergey N. Smirnov and Andrey Yu. Zanochkin

Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, Leninskie Gory 1-52, 119991 Moscow, Russia

Correspondence should be addressed to Sergey N. Smirnov; s.n.smirnov@gmail.com

Received 16 January 2021; Accepted 31 May 2021; Published 18 June 2021

Academic Editor: Lucas Jodar

For the superreplication problem with discrete time, a guaranteed deterministic formulation is considered: the problem is to guarantee coverage of the contingent liability on sold option under all admissible scenarios. These scenarios are defined by means of a priori defined compacts dependent on price prehistory: the price increments at each point in time must lie in the corresponding compacts. In a general case, we consider a market with trading constraints and assume the absence of transaction costs. The formulation of the problem is game theoretic and leads to the Bellman–Isaacs equations. This paper analyses the solution to these equations for a specific pricing problem, i.e., for a binary option of the European type, within a multiplicative market model, with no trading constraints. A number of solution properties and an algorithm for the numerical solution of the Bellman equations are derived. The interest in this problem, from a mathematical prospective, is related to the discontinuity of the option payoff function.

1. Introduction

1.1. Literature Review. One of the first publications to develop a guaranteed deterministic approach is an article by Kolokoltsov [1], published in 1998. To the best of our knowledge, this was the first work to explicitly articulate this approach to pricing and hedging contingent clams. Implicitly, however, some mathematical tools for a guaranteed deterministic approach were already present in 1994 in the first edition of the book by Dana and Jeanblanc-Picqué [2] (Sections 1.1.6 and 1.2.4). The result of the first part of [1] (the case of a single risky asset and a convex payout function on European option) follows from [2]. The guaranteed deterministic approach is closely related to a class of market models called interval models in [3], especially to the ideas and results of Kolokoltsov published in [3] (Chapters 11–14), including the independent discovery of the game-theoretic interpretation of risk-neutral probabilities under the assumption of no trading constraints; we find this interpretation to be quite important from an economic point of view. One can also consider guaranteed deterministic approach to be a Merton-type approach, which goes back to 1973; see [4] (no reference probability measure is used in this seminal work). Note that we share an idea, suggested in an unpublished work of Carassus and Vargiolu about 15 years ago and finally published in [5]: in order to get a meaningful theory, it is reasonable to assume the boundedness of price increments.

Formally, from the contemporary point of view (the guaranteed deterministic approach was developed by us in the late 90s (although at that period we were not aware of Kolokoltsov's paper), but published (primarily in Russian) only in the last three years, together with some recent new results), the guaranteed deterministic approach to the superhedging problem can be classified as a specific pathwise (or pointwise) approach addressing uncertainty in market modelling by defining a set of deterministic market scenarios (described in detail in the next section), a result of an agent's beliefs. Or it can be formally described in terms "quasisure" approach (we refer to [6, 7] for these two robust modelling approaches and for detailed review of large literature focusing on robust approach to mathematical finance), by the choice of a collection of probabilistic models (possible priors) for the market. In our case, all these probabilities initially (but can be enlarged to a family of probabilities which is a mixed extension of pure "market" strategies) are Dirac measures (but certainly not all of them). However, it is to stress that we adopt an alternative interpretation to the common robust approach to pricing of contingent liabilities. Our interpretation, as already mentioned above, is game theoretic: we deal with a deterministic dynamic two-player zero-sum game of "hedger" against "market." A family of probabilities appears as a secondary notion, thanks to the introduction of mixed strategies of the "market."

We deem to be related to our approach a formulation of the upper hedging price based on the game-theoretic probability, presented in [8].

1.2. Problem Statement. The present paper joins a series of publications (in particular, [9] describes the market model in detail and provides a literature review) [9–15] that develop a financial market model consistent with an uncertain deterministic price evolution with discrete time: asset prices evolve deterministically under uncertainty described using a priori information about possible price increments. Namely, they are assumed to lie in the given compacts that depend on the prehistory of the prices (such a model is an alternative to the traditional probabilistic market model (in our proposed deterministic approach, the reference probability measure is not initially set, as it is supposed in the probabilistic approach, see, e.g., [16])).

The proposed approach allows us to simplify the mathematical technique to a certain extent and make the formulation of statements more understandable for economists. The advantages of the approach include game-theoretic interpretation (in the absence of trading constraints, this interpretation provides an economically important explanation for the emergence of risk-neutral probabilities as one of the properties of the most unfavourable mixed market strategies).

The market model described above explores the problem of option pricing, by which we mean nondeliverable (for the risk management purposes, mainly nondeliverable contracts are used) over-the-counter contracts whose payoffs depend on the evolution of underlying asset prices up to the time of expiration. The writer of an option assumes a contingent liability that, unlike contingent liabilities on insurance policies, can be protected from market risk by hedging in markets (by means of transactions in underlying assets and risk-free assets). One of the most important ways to hedge the contingent liability of a sold option is through superreplication (this term originated because conditional liabilities cannot be replicated in incomplete markets (this is only possible in complete markets)) or in other words superhedging (we prefer to use the second of the two equivalent terms). The problem of option pricing in superhedging is to determine the minimum level of funds at the initial moment required by the seller (in other words, it is the premium charged to the buyer of the option if the seller uses superhedged pricing), which guarantee, if an appropriate hedging strategy is chosen, the coverage of the contingent liability under the option sold (remind that the corresponding payments under depend on the prehistory of prices). In general, we consider American-style options (American-style options) in which the seller's counterparty (the option holder) can exercise the option (i.e., demand payment in accordance with the rules set out in that contract) at any time, up to the expiration of the option. Note that European- and Bermuda-type options can be seen as a case of American options, subject to certain regularity conditions, including "no arbitrage" condition, in a certain sense.

Let us now formalize the above construction for the superhedging problem. The main premise of the proposed approach is to specify "uncertain" price dynamics by assuming a priori information about price movements at time t, namely, that the increments (the increments are taken "backward," i.e., $\Delta X_t = X_t - X_{t-1}$, where X_t is the vector of discounted prices at time t; the i-th component of this vector represents the unit price of the i-th asset) ΔX_t of discounted prices (we assume that the risk-free asset has a constant price equal to 1) lie in a priori defined compacts (the dot denotes the variables describing the evolution of prices. More precisely, this is the prehistory $\bar{x}_{t-1} = (x_0, \cdots, x_{t-1}) \in (\mathbb{R}^n)^t$ for K_t, while for the functions v_t^* and g_t introduced below, this is the history $\bar{x}_t = (x_0, \cdots, x_t) \in (\mathbb{R}^n)^{t+1}$) $K_t(\cdot) \subseteq \mathbb{R}^n$, where the point denotes the prehistory of prices up to and including time $t-1$, $t = 1, \cdots, N$. We denote by $v_t^*(\cdot)$ the infimum of the portfolio value at time t, at a known prehistory that guarantees, given some choice of an acceptable hedging strategy, the coverage of the current and future liabilities arising with respect to possible payoffs on the American option.

The corresponding Bellman–Isaacs equations in discounted prices arise directly from an economic sense by choosing, at step t, the "best" admissible hedging strategy (vector h describes the size of positions taken in assets, i.e., the i-th component of this vector represents the number of units of the i-th asset being bought or sold) $h \in D_t(\cdot) \subseteq \mathbb{R}^n$ for the "worst-case" scenario $y \in K_t(\cdot)$ of (discounted) prices increments for given functions $g_t(\cdot)$, describing the potential option payoff. Thus, we obtain the following recurrence relations (the sign denotes the maximum, and $hy = \langle h, y \rangle$ is the scalar product of vector h on vector y):

$$\begin{gathered} v_N^*(\bar{x}_N) = g_N(\bar{x}_N), \\ v_{t-1}^*(\bar{x}_{t-1}) = g_{t-1}(\bar{x}_{t-1}) \vee \inf_{h \in D_t(\bar{x}_{t-1})} \sup_{y \in K_t(\bar{x}_{t-1})} [v_t^*(\bar{x}_{t-1}, x_{t-1} + y) - hy], \\ t = N, \cdots, 1, \end{gathered} \tag{1}$$

where $\bar{x}_{t-1} = (x_0, \cdots, x_{t-1})$ describes the prehistory with respect to the present moment t. The conditions for the validity of (1) are formulated in Theorem 3.1 of [17].

Multivalued mappings $x \mapsto K_t(x)$ and $x \mapsto D_t(x)$, as well as functions $x \mapsto g_t(x)$, are assumed to be given for all $x \in (\mathbb{R}^n)^t$, $t = 1, \cdots, N$. Therefore, the functions $x \mapsto v_t^*(x)$ are given by equation (1) for all $x \in (\mathbb{R}^n)^t$. In equation (1), the functions v_t^*, as well as the corresponding suprema and infima, take values in the extended set of real numbers $\mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$, a two-point compactification (the neighbourhoods of points $-\infty$ and $+\infty$ are $[-\infty, a)$, a

$\in \mathbb{R}$ and $(b, +\infty]$, $b \in \mathbb{R}$, respectively) of $\mathbb{R}$.The derivation of equation (1) is easily obtained by a kind of "engineering reasoning." In informal economic language, this can be explained as follows. Assuming for simplicity that suprema and infima in equation (1) are attained, let $t \le N$; by the current (present) time $t-1$, we know the (discounted) price history $x_1, \cdots, x_{t-1}$. The portfolio value V_{t-1} when hedging the contingent liability of a sold American option should first be no less than the current liability, equal to the potential payout $g_t(x_1, \cdots, x_{t-1})$, to guarantee its coverage. Second, the portfolio value at the next moment $V_t = V_{t-1} + H_t \Delta X_t$ (here, the strategy H_t is formed at moment $t-1$ and can only depend on the prehistory of prices $x_1, \cdots, x_{t-1}$) should provide a guaranteed coverage of the contingent claim under any scenario $\Delta X_t = y \in K_t(x_1, \cdots, x_{t-1})$ of price movements at step t; hence, it should be not less than $v_t^*(x_1, \cdots, x_{t-1}, x_{t-1} + y)$. Thus, to cover future liabilities, the portfolio value V_{t-1} when an admissible hedging strategy $H_t = h \in D_t(x_1, \cdots, x_{t-1})$ is used should be no less than $v_t^*(x_1, \cdots, x_{t-1}, x_{t-1} + y) - hy$ under the worst-case scenario $y \in K_t(x_1, \cdots, x_{t-1})$ of price movements at step t, i.e., for $y \in K_t(x_1, \cdots, x_{t-1})$ that maximizes the expression $v_t^*(x_1, \cdots, x_{t-1}, x_{t-1} + y) - hy$. The resulting value is minimized by choosing a strategy $h \in D_t(x_1, \cdots, x_{t-1})$) to evaluate the required reserves to cover future potential payoffs. It remains to put $v_t^*(x_1, \cdots, x_{t-1})$ equal to the maximum amount of current liabilities and the amount of reserves for future potential payments.

We deem a trajectory on the time interval $[0, t] = \{0, \cdots, t\}$ of asset prices $(x_0, \cdots, x_t) = \bar{x}_t$ to be possible if $x_0 \in K_0$, $\Delta x_1 \in K_1(x_0), \cdots, \Delta x_t \in K_t(x_0, \cdots, x_{t-1})$; $t = 0, 1, \cdots, N$. Let us denote by B_t the set of possible trajectories of asset prices on the time interval $[0, t]$; thus,

$$B_t = \{(x_0, \cdots, x_t) \colon x_0 \in K_0, \Delta x_1 \in K_1(x_0), \cdots, \Delta x_t \in K_t(x_0, \cdots, x_{t-1})\}. \tag{2}$$

One of the conditions for the validity of (1) is the assumption of boundedness of payoff functions g_t formulated in Theorem 3.1 from [9], due to which the functions v_t^* are bounded from above. The assumption is as follows.

$$\begin{gathered}\text{There exist constants } C_t \ge 0 \text{ such that for each } t = 1, \cdots, N\\ \text{and all possible trajectories } \bar{x}_t = (x_0, \cdots, x_t) \in B_t\\ g_t(x_0, \cdots, x_t) \le C_t.\end{gathered} \tag{3}$$

Throughout the following, we will assume that the assumptions listed in Theorem 3.1 of [9] as well as those listed in (2) of Remark 3.1 of [9] are met.

This paper considers the problem of superhedging pricing of a binary option (European type) for a multiplicative one-dimensional market model, under the assumption of no trading constraints. A number of solution (1) properties are obtained, in particular, continuity except a single point. In addition, an algorithm for obtaining a "semi-implicit" solution (1), represented in the form of a piecewise rational function, is proposed. The interest to this problem is caused by the fact that the payout function is discontinuous, and therefore, the results concerning the case of continuous payout functions given in [12, 13] are not applicable here.

2. Auxiliary Results

Throughout the discussion below, we refer only to discounted prices. The price of the risk-free asset (after discounting, see [9]) is identically equal to 1. According to the terminology proposed in [9], for risky assets, the price dynamics (trading constraints) belongs to the Markov type if $K_t(\cdot)$ (respectively $D_t(\cdot)$) depend only on the price value at the previous moment, i.e., $K_t(\cdot)$ (respectively $D_t(\cdot)$) can be represented in the following form:

$$K_t(x_0, \cdots, x_{t-1}) = K_t^\star(x_{t-1}), \tag{4}$$

respectively,

$$D_t(x_0, \cdots, x_{t-1}) = D_t^\star(x_{t-1}), \tag{5}$$

for $t = 1, \cdots, N$.Let us formulate some simple but useful statements.

Proposition 1. *If price dynamics and trade constraints are of the Markov type and the payoff functions depend only on the current price, i.e., for $t = 1, \cdots, N$ are represented in the form*

$$g_t(x_0, \cdots, x_t) = g_t^\star(x_t), \tag{6}$$

then the solutions of the Bellman–Isaacs equation (1) also depend only on the current price, i.e., for $t = 1, \cdots, N$, they can be represented in the following form:

$$v_t^*(x_0, \cdots, x_t) = v_t^\star(x_t). \tag{7}$$

Proof. It follows directly from the form of the Bellman–Isaacs equation (1). □

Proposition 2. *Let assumptions (2) and (4) be satisfied, trading constraints be absent (in this case, condition (5) is obviously fulfilled), i.e., $D_t(\cdot) \equiv \mathbb{R}^n$, and the condition NDAO of no arbitrage opportunities be satisfied (in this case, the NDAO condition is equivalent to a geometric one: 0 lies in the relative interior of convex hull of $K_t(\cdot)$), $t = 1, \cdots, N$; see [10]). Then, for European options, the solutions of the Bellman–Isaacs equation (1) are monotonically decreasing in time, i.e.,*

$$v_0^\star(x) \ge v_1^\star(x) \ge \cdots \ge v_N^\star(x). \tag{8}$$

Proof. When there are no trading constraints and the condition NDAO of no arbitrage opportunities is fulfilled, we can assume that this is a special case of American options with payout functions (in principle, a weaker condition NDSA of no guaranteed arbitrage is sufficient for this; see [10])

$$g_t(x) \equiv 0, t = 0, \cdots, N-1, g_N = g. \quad (9)$$

□

Using Proposition 1 and the theorem proved in [11], we obtain for $t = 1, \cdots, N$ that the representation (7) holds and the following equality is valid

$$v_{t-1}^{\star}(x) = \sup \left\{ \int v_t^{\star}(x+y) Q(dy), Q \in \mathscr{M}_t(x) \right\}, \quad (10)$$

where $\mathscr{M}_t(x)$ is the set of probability measures on $K_t^{\star}(x)$ with a finite support (in fact, it is sufficient to consider the set of measures with the number of support points not exceeding $n+1$) satisfying the martingality condition (more precisely, the price increments form a martingale difference sequence): $\int y\, Q(dy) = 0$. In particular, $\delta_0 \in \mathscr{M}_t(x)$, where δ_0 is a probability measure centred at point 0, and thus,

$$v_{t-1}^{\star}(x) \geq v_t^{\star}(x). \quad (11)$$

Proposition 3. *Let for the one-dimensional model (that is, for a model with one risky asset (and one riskless asset)) the assumptions of Proposition 1 be satisfied and the payoff functions $g_t^{\star}$, $t = 1, \cdots, N$ be monotonically nondecreasing (respectively, monotonically nonincreasing). Then, the solutions of the Bellman–Isaacs equations $v_t^{\star}$, $t = 1, \cdots, N$ are also monotonically nondecreasing (respectively, monotonically nonincreasing).*

Proof. This follows directly from the form of the Bellman–Isaacs equation (1). □

Further, we consider a one-dimensional market model, where, in a multiplicative representation, the dynamics of the discounted price of a risky asset are described by the following relations (according to the terminology proposed in [9], in this case, the price dynamics refer to a multiplicative-independent type):

$$X_t = M_t X_{t-1}, t = 1, \cdots, N, \quad (12)$$

where (here, the prices and multipliers are considered as "uncertain" values (a deterministic analogue of random variables)) the multiplier

$$M_t \in [\alpha, \beta], 0 < \alpha < \beta. \quad (13)$$

The trading constraints are absent and the condition NDAO of no arbitrage opportunities is fulfilled, which in our case is equivalent to the following inequalities:

$$\alpha < 1 < \beta. \quad (14)$$

A model of this kind was first proposed by Kolokoltsov [1].

(1) If the function $v_t^{\star}$ satisfies the Lipschitz condition on some interval $[a, b]$, then the function also satisfies the Lipschitz condition on the (narrower) interval $[a/\alpha, b/\beta]$, and on this interval, the Lipschitz constant for $v_{t-1}^{\star}$ does not exceed the Lipschitz constant for $v_t^{\star}$ on the interval $[a, b]$

(2) If there is an upper estimate of the Bellman function $v_t^{\star}(x) \leq cx + d$ for $x \in [a, b]$, then $v_{t-1}^{\star}(x) \leq cx + d$ for $x \in [a/\alpha, b/\beta]$

(3) If the payoff functions $g_s^{\star}$, $s = 1, \cdots, N$ are upper semicontinuous, then the strict inequality $v_t^{\star}(x) < cx + d$ for $x \in [a, b]$ entails a strict inequality $v_{t-1}^{\star}(x) < cx + d$ for $x \in [a/\alpha, b/\beta]$

(4) If $x_1 > 0$, $x_1 < x_2$, and $x_2/x_1 \leq \beta/\alpha$, then for $x \in [x_2/\beta, x_1/\alpha]$ the inequality $v_{t-1}^{\star}(x) \geq cx + d$ holds, where

$$c = \frac{v_t^{\star}(x_2) - v_t^{\star}(x_1)}{x_2 - x_1}, \quad d = v_t^{\star}(x_1). \quad (15)$$

Proposition 4. *Let the model of price dynamics be described by relations (12), (13), and (14); we fix $t \in \{1, \cdots, N\}$. Then, the following statements hold for the European option.*

Proof.

(1) Let us use the multiplicative analogue of formula (10) for the European option:

$$v_{t-1}^{\star}(x) = \sup \left\{ \int v_t^{\star}(mx) Q(dm), Q \in \mathscr{N} \right\}, \quad (16)$$

where $\mathscr{N}$ is the set of probability measures on $[\alpha, \beta]$ with a finite support (it is sufficient to consider the set of measures with the number of support points not exceeding $n+1$) satisfying the multiplicative martingality condition: $\int m\, Q(dm) = 1$. Denote the Lipschitz constant for $v_t^{\star}$ on the interval $[a, b]$ by L. Since for $x \in [a/\alpha, b/\beta]$ the inclusion $[\alpha x, \beta x] \subseteq [a, b]$ holds, for any points x_1 and x_2 such that $a/\alpha \leq x_1 \leq x_2 \leq b/\beta$ we have the following inequalities:

$$|v_{t-1}^{\star}(x_2) - v_{t-1}^{\star}(x_1)| \leq \sup \left\{ \int |v_t^{\star}(mx_2) - v_t^{\star}(mx_1)| Q(dm), Q \in \mathscr{N} \right\} \leq L|x_2 - x_1| \int m Q(dm) = L|x_2 - x_1|. \quad (17)$$

(2) Given the inclusion $[\alpha x, \beta x] \subseteq [a, b]$ for $x \in [a/\alpha, b/\beta]$, for any $Q \in \mathscr{N}$ we have

$$\int v_t^{\star}(mx) Q(dm) \leq cx \int m\, Q(dm) + d = cx + d, \quad (18)$$

whence, according to (16), we obtain $v_{t-1}^{\star}(x) \leq cx + d$.

(3) Under the assumptions made, because the supremum in (16) is attained (see [13]) for some measure $Q_{t,x} \in \mathcal{N}$, then

$$\int v_t^\star(mx) Q_{t,x}(dm) < cx \int m\, Q_{t,x}(dm) + d = cx + d. \quad (19)$$

(4) For $x \in [x_2/\beta, x_1/\alpha]$, choose $m_1 = x_1/x$ and $m_2 = x_2/x$; we have then $\alpha \le m_1 < m_2 \le \beta$. Consider a measure $Q \in \mathcal{N}$ concentrated at points (the probabilities of these points are uniquely determined from the normalization and martingality conditions; therefore, Q depends on t, x, x_1, and x_2) m_1 and m_2. Thanks to the choice of constants c and d in (15), the functions $m \mapsto v_t^\star(mx)$ and $m \mapsto c\,mx + d$ coincide at the points of the support of measure $Q \in \mathcal{N}$, and we obtain the following equality:

$$\int v_t^\star(mx) Q(dm) = cx \int m\, Q(dm) + d = cx + d, \quad (20)$$

whence, using (16), we obtain the required inequality. □

3. Binary Option of European Type

3.1. General Case of the Support of Distribution of Uncertain Multiplier. Within the framework of the price dynamics model described by relations (12), (13), and (14), we are interested in the superhedging problem within the guaranteed deterministic approach for a European-type binary option. Without limiting the generality, we can assume that the strike price is equal to 1. Let us consider a binary call option (the case of a binary put option can be investigated using similar methods) whose payoff function g at the expiration moment is equal to

$$g(x) = \mathbb{I}_{[1,+\infty)}(x), \quad (21)$$

where $\mathbb{I}_A$ is an indicator function of set A. Note that Proposition 2 is applicable in the case of our model, and thus, the solutions of the Bellman–Isaacs equation (1) are monotonically nonincreasing over time. By virtue of the condition of the absence of NDAO arbitrage opportunities, as noted above, the European option superhedging problem is reduced to the American option superhedging problem, with the payoff functions described by (9), i.e., with zero payoff functions except for the expiration moment (21). Thus, Proposition 1 is applicable, and a representation of the form (7) holds for the solution of the corresponding Bellman–Isaacs equations. Hereinafter, we will consider our problem as a superhedging of an American option with zero payoff functions except for the expiration moment. Since the terminal payoff function is monotonically nondecreasing, Proposition 3 is applicable. Thus, the solutions to the corresponding Bellman–Isaacs equations are also monotonically nondecreasing, or equivalently, by notation (26), the functions $v_s^\star, s = 0, \cdots, N$ are monotonically nondecreasing. Therefore, these functions can have discontinuities of the first kind (jumps) only. In addition, as the payoff function g is upper semicontinuous and the multivalued mappings $K_t(\cdot)$ and $D_t(\cdot)$ are continuous, the solutions to the Bellman–Isaacs equations $v_s^\star, s = 0, \cdots, N$ are also upper semicontinuous; see [12]. For monotonically nondecreasing functions, upper semicontinuity is equivalent to their right continuity. Since the solutions of the Bellman–Isaacs equations $v_s^\star, s = 0, \cdots, N$ are upper semicontinuous, a game equilibrium takes place (at each time step); see [14]. In this case, according to the results of [14], for the saddle point, the most unfavourable mixed strategies are achieved in the class of distributions concentrated in no more than two points. To find the solution to the Bellman equations (after separating the pricing problem from the hedging problem), it is sufficient (see [15]) to construct at each step $t = 1, \cdots, N$ on the interval $[\alpha x, \beta x]$ (upper semicontinuous) concave envelope $\tilde{v}_t^\star$ of Bellman function $v_t^\star$ and set $v_{t-1}^\star(x) = \tilde{v}_t^\star(x)$.

3.2. Cox–Ross–Rubinstein Assumption about the Endpoints of the Uncertain Multiplier Support. The general case of parameters α and β is quite difficult to analyse owing to the chaotic behaviour (including the mutual position) of the products of the form $\alpha^i \beta^j$, where i and j are nonnegative integers, unless $\ln \alpha$ and $\ln \beta$ are rationally commensurable. We choose the simplest case of rational commensurability of $\ln \alpha$ and $\ln \beta$, proposed in the Cox–Ross–Rubinstein model [18], namely, we apply

$$\beta = \alpha^{-1}. \quad (22)$$

In this case, the condition of no arbitrage opportunities (14) is automatically satisfied for $\alpha < 1$. Note that assumption (22) simplifies significantly the analysis: if, at step $s = 1, \cdots, N$, point x, the price value at the previous time, lies in an interval of the form $[\alpha^k, \alpha^{k-1})$, $k = 0, \cdots, s+1$, then the endpoints of the interval $[\alpha x, \alpha^{-1} x)$ of the possible values of the uncertain value X_s given $X_{s-1} = x$, i.e., points αx and $\alpha^{-1} x$, lie in the adjacent intervals $[\alpha^{k+1}, \alpha^k)$ and $[\alpha^{k-1}, \alpha^{k-2})$, respectively. We will say that the points $\alpha^k, k = 0, \cdots, s$ form a *skeleton* at step $s = 1, \cdots, N$. The most unfavourable mixed market strategies in step t for a given price x in the previous step may be nonunique. For example, if $x \in [1, \infty)$, any distribution with the support contained in $[1, \infty)$ and the barycentre x would be such, and if $x \in (0, \alpha^{t-1})$, any distribution with the support contained in $(0, \alpha^{t-1})$ and the barycentre x would be such. At points x where there is a nonuniqueness of the most unfavourable mixed market strategy, we adopt a convention to choose a distribution with barycentre x that has the minimum number of support points to fix the unique “optimal” mixed market strategy. There will never be more than two such points, and hence, given the martingality condition, the corresponding distribution is defined in the only way possible. Due to this convention, the conditional distribution Q_x^s of price X_s given $X_{s-1} = x$, concentrated in no more than two points, will be chosen as the most unfavourable mixed market strategy at step $s = 1, \cdots, N$ (when the maximum in (16) is attained). We call the support of the distribution Q_x^s a *scenario*. When the scenario is a one-point set, $Q_x^s = \delta_x$, where δ_a denotes the probability measure concentrated at a point a. When the scenario is a set of two points, Q_x^s has the following form:

$$Q_x^s = p_s(x)\delta_{a_s(x)} + q_s(x)\delta_{b_s(x)}, \tag{23}$$

where $a_s(x) < b_s(x)$. Given a scenario, the probabilities $p_s(x)$ and $q_s(x)$ are uniquely defined from the normalization condition

$$p_s(x) + q_s(x) = 1 \tag{24}$$

and price martingality condition, whence

$$\begin{aligned} p_s(x) &= \frac{b_s(x) - x}{b_s(x) - a_s(x)}, \\ q_s(x) &= \frac{x - a_s(x)}{b_s(x) - a_s(x)}, \end{aligned} \tag{25}$$

For convenience, we shall use the following notations:

$$u_s(x) = v^{\star}_{N-s}(x), s = 0, \cdots, N. \tag{26}$$

In particular, $u_0 = g$, where g is given by (21). The recurrence relations for $u_s, s = 1, \cdots, N$ are

$$u_s(x) = p_s(x)u_{s-1}(a_s(x)) + q_s(x)u_{s-1}(b_s(x)). \tag{27}$$

In what follows, using notation (26), we will investigate the properties of the solution $u_s, s = 1, \cdots, N$ of the European binary call option superhedging problem, with the payoff function at the expiration moment given by (21), for the market described using relations (12), (13), (14), and (22).

3.3. Solutions of the Bellman Equations for the First Two Steps. For $x < \alpha$, the function u_1 is identically equal to zero because the interval $[\alpha x, \alpha^{-1}x]$ is contained in $(0, 1)$, where the function $u_0 = g$ is zero. For $x \geq 1$, the function u_1 is identically equal to 1 because the (upper semicontinuous) concave envelope $\tilde{u}_0$ of the function u_0 on $[\alpha x, \alpha^{-1}x]$ at x is equal to 1.

Note that in the first step, for $x \in [\alpha, 1)$, the most unfavourable mixed market strategy can be a conditional distribution Q_x^1 concentrated at two points αx and 1, with probabilities $p_1(x)$ and $q_1(x)$, respectively. Formula (25) in this case takes the form

$$\begin{aligned} p_1(x) &= \frac{1 - x}{1 - \alpha x}, \\ q_1(x) &= \frac{(1-\alpha)x}{1 - \alpha x}, \end{aligned} \tag{28}$$

and by (27), the values of function u_1 on the interval $[\alpha, 1)$ are given by the expression

$$u_1(x) = p_1(x)g(\alpha x) + q_1(x)g(1) = q_1(x). \tag{29}$$

Thus, in the interval $[\alpha, 1)$, the scenario $\{\alpha x, 1\}$ is realized, and function u_1 has a hyperbolic form

$$u_1(x) = \frac{(1-\alpha)x}{1 - \alpha x}, \tag{30}$$

which is strictly monotonically increasing and (strictly) convex. At point α, the function u_1 has a single discontinuity (jump), is right-continuous, and

$$u_1(\alpha) = \frac{\alpha}{1+\alpha}. \tag{31}$$

On the right endpoint of interval $[\alpha, 1)$ by (31), we have

$$u_1(1-0) = 1, \tag{32}$$

so that function u_1 is continuous at point 1.

Note that the line passing through the points in the plane of the hyperbola (31) corresponding to the arguments α and 1, i.e., passing through the points with coordinates $(\alpha, u_1(\alpha))$ and $(\alpha^{-1}x, u_1(\alpha^{-1}x))$, is defined by

$$\begin{aligned} \omega_1(z) &= u_1(\alpha) + \frac{u_1\left(\alpha^{-1}x\right) - u_1(\alpha)}{\alpha^{-1}x - \alpha}(z - \alpha) \\ &= \frac{\alpha}{1+\alpha} + \frac{z-\alpha}{(1+\alpha)(1-x)} \\ &= \frac{z - \alpha x}{(1+\alpha)(1+x)}, \end{aligned} \tag{33}$$

which has a root αx, i.e.,

$$\omega_1(\alpha x) = 0. \tag{34}$$

In particular, for $x = \alpha$, we obtain that the line passing through the points of the hyperbola (31) corresponding to the arguments α and 1 have the root α^2. To complete the geometric image, we also note that the tangent at point α to the restriction of the function u_1 to the interval $[\alpha, 1)$, given by the function

$$\varphi_1(z) = u_1(\alpha) + (z-\alpha)u'_1(\alpha + 0) = \frac{\alpha}{1+\alpha} + \frac{z - \alpha}{(1-\alpha)(1+\alpha)^2}, \tag{35}$$

has a root α^3.

The graph of the function u_1 for $\alpha = 0.5$ is shown in Figure 1.

It follows from (33) and (34) that for $x \in [\alpha^2, \alpha)$, the line segment defined by function (23), connecting points with coordinates $(\alpha x, 0)$ and $(\alpha^{-1}x, u_1(\alpha^{-1}x))$, is a (upper semicontinuous) concave envelope $\tilde{u}_1$ of function u_1 on the interval $[\alpha x, \alpha^{-1}x]$, and thus,

$$u_2(x) = \tilde{u}_1(x) = \frac{x - \alpha x}{(1+\alpha)(1-x)}, \tag{36}$$

for $x \in [\alpha^2, \alpha)$. At the right endpoint of the hyperbola (36), the equality

$$u_2(\alpha - 0) = \frac{\alpha}{1+\alpha} \tag{37}$$

holds.

Note that in the second step, for $x \in [\alpha, 1)$, the most unfavourable mixed market strategy (note that when $x = \alpha$ the most unfavourable mixed market strategy is not unique: any distribution with barycentre $x = \alpha$ concentrated at no more than three points: α^2, α, and 1, i.e., a distribution represented as a mixture $p\delta_\alpha + (1-p)((1/(1+\alpha))\delta_{\alpha^2} + (\alpha/(1+\alpha))\delta_1)$, $p \in [0,1]$, is "optimal") can be represented as a conditional distribution of the form $Q_x^2 = p_1(x)\delta_\alpha + p_2(x)\delta_1$. Formula (25) in this case takes the form

$$\begin{aligned} p_2(x) &= \frac{1-x}{1-\alpha}, \\ q_2(x) &= \frac{x-\alpha}{1-\alpha}, \end{aligned} \tag{38}$$

and by (27), the function u_2 on the interval $[\alpha, 1]$, taking into account (31), is an affine function, namely,

$$u_2(x) = p_2(x)u_1(\alpha) + q_2(x)u_1(1) = \frac{x-\alpha^2}{1-\alpha^2}. \tag{39}$$

Specifically,

$$u_2(\alpha) = \frac{\alpha}{1+\alpha}. \tag{40}$$

Given (37), the function u_2 is therefore continuous at point α. The function u_2 is not only continuous at $[\alpha^2, +\infty)$: it turns out that at this point there exists a derivative equal to $(1-\alpha^2)^{-1}$ so that the function u_2 is differentiable at $(\alpha^2, 1)$. It is easily seen that for $x < \alpha^2$ the function u_2 is identically equal to zero, and for $x \geq 1$, the function u_2 is identically equal to one. Because (39) implies $u_2(1-0) = 1$, the function u_2 is continuous at point 1, and hence, the function u_2 is continuous at $[\alpha^2, +\infty)$.

The graph of the function u_2 for $\alpha = 0.5$ is shown in Figure 2.

3.4. Solutions of the Bellman Equations: Recurrence Properties. We now fix $s \in \{1, \cdots, N\}$.

Proposition 5. *Outside the interval $[\alpha^s, 1)$, the function u_s takes the following values:*

$$u_s(x) = 0 \; npu \; x < \alpha^s, \tag{41}$$

$$u_s(x) = 1 \; npu \; x < 1. \tag{42}$$

Proof. The relations in (41) are obtained through induction, given the property noted in the previous section, and the endpoints of the interval $[\alpha x, \alpha^{-1}x]$ for $x \in [\alpha^k, \alpha^{k-1})$, $k = 0, \cdots, s+1$ lie in adjacent intervals, that is, $\alpha x \in [\alpha^{k+1}, \alpha^k)$ and $\alpha^{-1}x \in [\alpha^{k-1}, \alpha^{k-2})$. For $s = 1$, this property is established as described in the previous section. Suppose (28) is valid for $s = t$, let us show its validity for $s = t+1$. The function u_t is identically equal to zero for $x < \alpha^{t+1}$, as the interval $[\alpha x, \alpha^{-1}x]$ is contained in $(0, \alpha^t)$, where the function u_t is equal to zero. For $x \geq 1$, the function u_{t+1} is identically equal to 1 because the (upper semicontinuous) concave envelope u_{t+1} of the function u_t on $[\alpha x, \alpha^{-1}x]$ at x is equal to 1. ☐

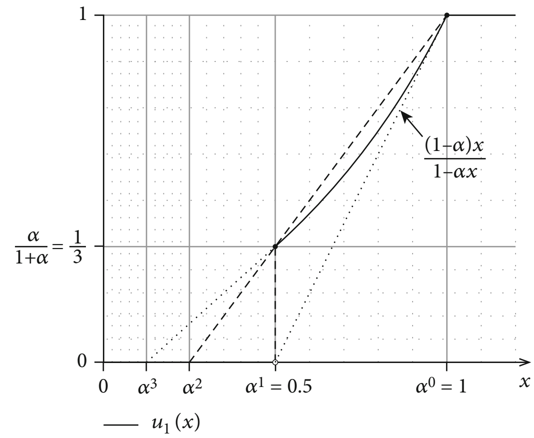

Figure 1: Function $u_1(x)$ for $\alpha = 0.5$.

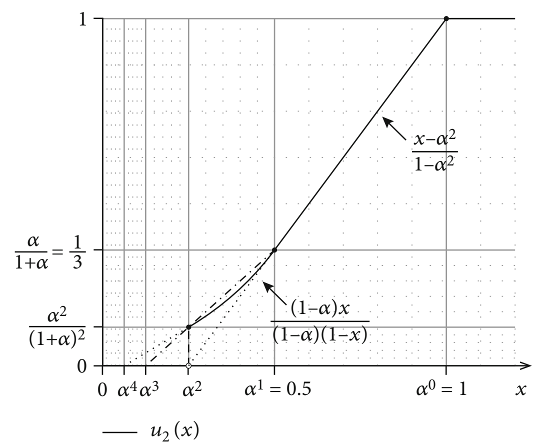

Figure 2: Function $u_2(x)$ for $\alpha = 0.5$.

Proposition 6. *The function u_s, $s = 1, \cdots, N$ has a discontinuity (jump) at point α^s, in which u_s is right continuous, and on the interval $[\alpha^s, \alpha^{s-1})$, the function u_s satisfies the property of self-similarity (owing to the properties of function u_1, on the interval $[\alpha^s, \alpha^{s-1})$, the function u_s is strictly monotonically increasing and strictly convex):*

$$u_s(x) = \left(\frac{\alpha}{1+\alpha}\right)^{s-1} \cdot u_1\left(\alpha^{-(s-1)}x\right). \tag{43}$$

Proof. When $s = 1$, (43) is an identity. Let us make the inductive assumption that (43) holds for $s = t \geq 1$ and check that it holds for $s = t+1$. Substituting $s = t$ in (29) and expression (31) for u_1, we have for $x \in [\alpha^t, \alpha^{t-1})$

$$u_t(x) = \left(\frac{\alpha}{1+\alpha}\right)^{t-1} \cdot u_1\left(\alpha^{t-1}x\right) = \frac{\alpha^{t-1}\left(\alpha^{t-1} - \alpha^t\right)x}{(1+\alpha)^{t-1}\left(1 - \alpha^t x\right)}. \quad (44)$$

□

From geometric similarity considerations, it is clear that for $x \in [\alpha^{t+1}, \alpha^t)$, the concave envelope $\tilde{u}_t$ of the function u_t on the interval $[\alpha x, \alpha^{-1}x]$ is the line segment connecting the points with coordinates $(\alpha x, 0)$ и, $(\alpha^{-1}x, u_t(\alpha^{-1}x))$ given by

$$\omega_t(z) = \frac{u_t\left(\alpha^{-1}x\right) - u_t(\alpha x)}{\alpha^{-1}x - \alpha x}(z - \alpha x), \quad (45)$$

where $u_t(\alpha x) = 0$, and hence, for $x \in [\alpha^{t+1}, \alpha^t)$

$$u_{t+1}(x) = \tilde{u}_t(x) = \omega_t(x) = \frac{\alpha}{1+\alpha} \cdot u_t\left(\alpha^{-1}x\right), \quad (46)$$

which follows from formula (43) for $s = t+1$. Using Proposition 5, we have $u_s(\alpha^s - 0) = 0$, and putting $x = \alpha^s$ in (46), we get

$$u_s(\alpha^s) = \left(\frac{\alpha}{1+\alpha}\right)^s > 0. \quad (47)$$

Thus, u_{s+1} has a jump at point α^{s+1} (where u_{s+1} is right continuous).

Theorem 7.

(1) *For $s = 1, \cdots, N$, the function u_s is convex on each of the intervals $[\alpha^k, \alpha^{k-1})$, $k = 1, \cdots, s$*

(2) *For $x \in [\alpha^k, \alpha^{k-1})$, $k = 1, \cdots, s$, it is sufficient to consider only four scenarios, i.e., the variants of the point locations as $a_s(x)$ and $b_s(x)$ introduced in Section 3.2:*

(I) Scenario $a_s(x) = \alpha^k$ and $b_s(x) = \alpha^{k-1}$

(II) Scenario $a_s(x) = \alpha^k$ and $b_s(x) = \alpha^{-1}x$

(III) Scenario $a_s(x) = \alpha x$ and $b_s(x) = \alpha^{k-1}$

(IV) Scenario $a_s(x) = \alpha x$ and $b_s(x) = \alpha^{-1}x$

Moreover, the number of possible switching scenarios on the intervals $[\alpha^k, \alpha^{k-1})$, $k = 1, \cdots, s$ does not exceed 2.

(3) *For $s = 1, \cdots, N$, the function u_s is piecewise rational on the interval $(0, +\infty)$ or, more precisely, rational on at most $m_s \leq 3s+1$ adjacent intervals, which we shall call rationality intervals (in particular, for $s = 1, \cdots, N$, the function u_s is infinitely differentiable within intervals of rationality interior), with endpoints $d_{s,i}$, $i = 0, \cdots, m_s + 1$; all points of type α^t, $t = 0, \cdots, s$ are endpoints of rationality intervals for the function u_s. The partitioning into rationality intervals for the function u_{s+1} is a refinement of the partitioning into rationality intervals for the function u_s. For the given intervals of rationality of the rational functions represented in the form of an irreducible fraction of polynomials, the degree of polynomials does not exceed s, and this degree on intervals $(0, \alpha^s)$ and $[1, +\infty)$ equals zero; if scenario I is realized, the degree equals 1, whereas is scenario IV is realized, the degree does not exceed $s - 1$*

(4) *For $s = 1, \cdots, N$, the derivative of the function u_s is positive (at points that are endpoints of rationality intervals, a jump in the derivative of the function u_s may occur, but not necessarily so, as seen in the example of the function u_2). In particular, the function u_s is strictly monotone on the interval $[\alpha^s, 1)$*

Proof. For convenience, we write out for scenarios I, II, III, and IV the specific formulas given in the general case by (23), (25), and (27). Note that for those points x for which one of the scenarios I, II, III, and IV holds, the points $a_s(x)$ and $b_s(x)$ belonging to the support of distribution given by (16), and hence, the probabilities $p_s(x)$ and $q_s(x)$ are independent of s, and thus, for these scenarios the carrier points and probabilities will have s omitted. □

For scenario I, when $x \in [\alpha^k, \alpha^{k-1})$, $k = 1, \cdots, s$, $a(x) = \alpha^k$, and $b(x) = \alpha^{k-1}$, the probabilities $p(x)$ and $q(x)$ take the form of affine functions

$$p(x) = \frac{\alpha^{k-1} - x}{a^{k-1} - \alpha^k}, \quad q(x) = \frac{x - \alpha^k}{\alpha^{k-1} - \alpha^k}, \quad (48)$$

and the values of the function u_s are expressed through the values of the function u_{s-1} by the formula

$$u_s(x) = p(x)u_{s-1}\left(\alpha^k\right) + q(x)u_{s-1}\left(\alpha^{k-1}\right). \quad (49)$$

Thus, in the case of scenario I on the interval $[\alpha^k, \alpha^{k-1})$, the function u_s is affine, and in the case of this scenario $x \in (\alpha^k, \alpha^k + \varepsilon)$, for some $\varepsilon > 0$, the function values match:

$$u_s\left(\alpha^k\right) = u_{s-1}\left(\alpha^k\right). \quad (50)$$

In addition, in the case of this scenario, for $x \in (\alpha^{k-1} - \varepsilon, \alpha^{k-1})$, for some $\varepsilon > 0$, the following "matching" relations take place:

$$u_s\left(\alpha^{k-1} - 0\right) = u_{s-1}\left(\alpha^{k-1} - 0\right). \quad (51)$$

For scenario II, when $x \in [\alpha^k, \alpha^{k-1})$, $k = 1, \cdots, s$, $a(x) = \alpha^k$, and $b(x) = \alpha^{-1}x$, the probabilities $p(x)$ and $q(x)$ take the form

$$p(x) = \frac{\alpha^{-1}x - x}{\alpha^{-1}x - \alpha^k}, \quad q(x) = \frac{x - \alpha^k}{\alpha^{-1}x - \alpha^k}, \tag{52}$$

and the values of the function u_s are expressed through the values of the function u_{s-1} by the formula

$$u_s(x) = p(x)u_{s-1}\left(\alpha^k\right) + q(x)u_{s-1}\left(\alpha^{-1}x\right). \tag{53}$$

In this scenario for $x \in (\alpha^k, \alpha^k + \varepsilon)$, for some $\varepsilon > 0$, the "matching" relations take place:

$$u_s\left(\alpha^k\right) = u_{s-1}\left(\alpha^k\right), \tag{54}$$

and in the case of this scenario for $x \in (\alpha^{k-1} - \varepsilon, \alpha^{k-1})$, for some $\varepsilon > 0$, the "matching" relations take place:

$$u_s\left(\alpha^{k-1} - 0\right) = \frac{1}{1+\alpha} u_{s-1}\left(\alpha^k - 0\right) + \frac{\alpha}{1+\alpha} u_{s-1}\left(\alpha^{k-2} - 0\right). \tag{55}$$

For scenario III, when $x \in [\alpha^k, \alpha^{k-1}), k = 1, \cdots, s$, Đ° $a(x) = \alpha x$, and $b(x) = \alpha^{k-1}$, the probabilities $p(x)$ and $q(x)$ are as follows:

$$p(x) = \frac{\alpha^{k-1} - x}{\alpha^{k-1} - \alpha x}, \quad q(x) = \frac{x - \alpha x}{\alpha^{k-1} - \alpha x}, \tag{56}$$

and the values of the function u_s are expressed through the values of the function u_{s-1} by the formula

$$u_s(x) = p(x)u_{s-1}(\alpha x) + q(x)u_{s-1}\left(\alpha^{k-1}\right). \tag{57}$$

In this scenario, for $x \in (\alpha^{k-1} - \varepsilon, \alpha^{k-1})$, for some $\varepsilon > 0$, the "matching" relations take place:

$$u_s\left(\alpha^k\right) = \frac{1}{1+\alpha} u_{s-1}\left(\alpha^{k+1}\right) + \frac{\alpha}{1+\alpha} u_{s-1}\left(\alpha^{k-1}\right), \tag{58}$$

and in this scenario, for $x \in (\alpha^{k-1} - \varepsilon, \alpha^{k-1})$, for some $\varepsilon > 0$, the "matching" relations take place:

$$u_s\left(\alpha^{k-1} - 0\right) = u_{s-1}\left(\alpha^{k-1} - 0\right). \tag{59}$$

For scenario IV, when $x \in [\alpha^k, \alpha^{k-1}), k = 1, \cdots, s$, $a_k(x) = \alpha x$, and $b_k(x) = \alpha^{-1}x$, the probabilities $p(x)$ and $q(x)$ are as follows:

$$p(x) = \frac{1}{1+\alpha}, \quad q(x) = \frac{\alpha}{1+\alpha}, \tag{60}$$

and the values of the function u_s are expressed through the values of the function u_{s-1} by the formula

$$u_s(x) = p(x)u_{s-1}(\alpha x) + q(x)u_{s-1}\left(\alpha^{-1}x\right), \tag{61}$$

and in this scenario, for $x \in (\alpha^k, \alpha^k + \varepsilon)$, for some $\varepsilon > 0$, the "matching" relations take place:

$$u_s\left(\alpha^k\right) = \frac{1}{1+\alpha} u_{s-1}\left(\alpha^{k+1}\right) + \frac{\alpha}{1+\alpha} u_{s-1}\left(\alpha^{k-1}\right), \tag{62}$$

and in this scenario, for $x \in (\alpha^{k-1} - \varepsilon, \alpha^{k-1})$, for some $\varepsilon > 0$, the "matching" relations take place:

$$u_s\left(\alpha^{k-1} - 0\right) = \frac{1}{1+\alpha} u_{s-1}\left(\alpha^k - 0\right) + \frac{\alpha}{1+\alpha} u_{s-1}\left(\alpha^{k-2} - 0\right). \tag{63}$$

Let us show by induction that for $s \geq 2$ the function u_s satisfies the four properties from the formulation of the theorem. For $s = 2$, these properties are satisfied (note that for the function u_2, scenario II takes place on the interval $[\alpha^2, \alpha)$, whereas on the interval $[\alpha, 1)$, scenario I takes place). Suppose that this property is satisfied for $s = t \geq 2$. Let us check its fulfilment for $s = t + 1$. For x from the interval $[\alpha^k, \alpha^{k-1})$, when $k = s + 1$, this follows from formula (43). If $k \leq s$ and point x lies in an interval of the form $[\alpha^k, \alpha^{k-1})$, as mentioned above, the endpoints of the interval $[\alpha x, \alpha^{-1}x]$, that is, points αx and $\alpha^{-1}x$, lie in adjacent intervals $[\alpha^{k+1}, \alpha^k)$ and $[\alpha^{k-1}, \alpha^{k-2})$, respectively. At the points $\alpha^k, k = 1, \cdots, s$, a positive jump is in principle possible (below, we prove that continuity takes place at these points), and continuity to the right takes place. In case when there is a jump at the point α^{k-1}, the function preserves the convexity on the closed interval $[\alpha^k, \alpha^{k-1}]$ if it is convex on the interval $[\alpha^k, \alpha^{k-1})$. Owing to the convexity of the function u_t on the interval $[\alpha^k, \alpha^{k-1}]$, for $x \in [\alpha^k, \alpha^{k-1})$, one may not consider any point of the open interval (α^k, α^{k-1}) as a "candidate" to be a point of the support of the most unfavourable mixed market strategy; it is sufficient to consider only the extreme points α^k Đ, α^{k-1} from the interval $[\alpha^k, \alpha^{k-1}]$.Next, we fix the numbers $a \leq \alpha^k$,$x \in [\alpha^k, \alpha^{k-1})$ and consider a distribution Q concentrated at points a and $y \in [\alpha^{k-1}, \alpha^{k-2})$ with probabilities p and q, respectively, satisfying the condition $pa + qy = x$; subject to normalization, whence

$$p = \frac{y - x}{y - a}, \quad q = \frac{x - a}{y - a}. \tag{64}$$

Let us show that the integral $\int u_t\,dQ$ considered as a function of y, i.e., the function $y \mapsto pu_t(a) + qu_t(y) = V(y)$, is monotonically nondecreasing on $[\alpha^{k-1}, \alpha^{k-2})$, where p and q are given by (64) and are considered as functions of the variable $y \in [\alpha^{k-1}, \alpha^{k-2})$. We shall need the following result from a mathematical analysis. If functions f and g are absolutely continuous on the interval $[a, b]$ and f' and g' are their derivatives (defined almost everywhere with respect to the Lebesgue measure), then functions $f'g$ and f g' are summable (in this case, the product fg is absolutely continuous on the interval $[a, b]$, which can be verified directly by definition, given the boundedness of functions f and g) and

$$f(x)g(x) - f(a)g(a) = \int_a^x f'(z)g(z)\,dz + \int_a^x f(z)g'(z)dz, \tag{65}$$

for $x \in [a, b]$; see Theorem 5 of Section 7 of Chapter IX in [19]. Let us add to this that the convex function is absolutely continuous and one can choose for its derivative an equivalent (it is a function coinciding with the original at almost all points (with respect to the Lebesgue measure)) monotonically nondecreasing at all points; see, e.g., [20], Theorem 24.2, as well as Corollary 24.2.1 and Theorem 24.1.

$$\begin{aligned} V(y) - V\left(\alpha^{k-1}\right) &= \int_{\alpha^{k-1}}^{y} V'(z)\,dz \\ &= \int_{\alpha^{k-1}}^{y} \left[p'u_t(a) + q'u_t(z) + qu'_t(z)\right] dz \\ &= \int_{\alpha^{k-1}}^{y} \left\{q'[u_t(z) - u_t(a)] + qu'_t(z)\right\} dz. \end{aligned} \tag{66}$$

Using the equality

$$q' = -\frac{x-a}{(y-a)^2} = -\frac{1}{y-a}q, \tag{67}$$

we obtain that for $y \in [\alpha^{k-1}, \alpha^{k-2})$.

$$V(y) - V\left(\alpha^{k-1}\right) = \int_{\alpha^{k-1}}^{y} q\left[u'_t(z) - \frac{u_t(z) - u(a)}{z-a}\right] dz. \tag{68}$$

Owing to the convexity (by an inductive assumption) of the function u_t, the expression in square brackets under the integral in (68) is nonnegative almost everywhere, and thus, we obtain that function V as monotonically nondecreasing. Thus, as a "candidate" for the point of the support of distribution $[\alpha^{k-1}, \alpha^{-1}x]$ from the interval $[\alpha^{k-1}, \alpha^{-1}x]$, we can consider only one point, $\alpha^{-1}x$. Similarly, consider a "candidate" point for the support of the distribution $Q_x^{s,k}$ on the left side, i.e., on the interval $[\alpha x, \alpha^k]$. Let us now fix the numbers $b \geq \alpha^{k-1}$,$x \in [\alpha^k, \alpha^{k-1})$, and consider a distribution Q concentrated at points $z \in [\alpha^{k+1}, \alpha^k)$ and b, with probabilities p and q, respectively, satisfying the condition $pz + qb = x$, subject to the normalization, whence

$$\begin{aligned} p &= \frac{b-x}{b-z}, \\ q &= \frac{x-z}{b-z}. \end{aligned} \tag{69}$$

Let us show that the integral $\int u_t\,dQ$ considered as a function of z, i.e., the function $z \mapsto pu_t(z) + qu_t(b) = W(z)$, is monotonically nonincreasing on $[\alpha^{k+1}, \alpha^k)$, where p and q are given by (69) and are considered as functions of the variable z. Using equality

$$p' = \frac{b-x}{(b-z)^2} = \frac{p}{b-z}, \tag{70}$$

it is easy to see that for $z \in [\alpha^{k+1}, \alpha^k)$

$$W(z) - W\left(\alpha^{k+1}\right) = \int_{\alpha^{k+1}}^{z} p\left[u_t{}'(y) - \frac{u_t(b) - u_t(y)}{b-y}\right] dy. \tag{71}$$

Thanks to the convexity of the function u_t, the expression in square brackets under the integral in (71) is almost everywhere nonpositive, and thus, we obtain that function V is monotonically nonincreasing. Therefore, as a "candidate" for the point of the support of distribution $Q_x^{s,k}$ from the interval $[\alpha x, \alpha^k]$, we can consider only the point αx.

Thus, it is sufficient to consider only scenarios I, II, III, and IV to study the variants of the location of points belonging to the support of distribution $Q_x^{s,k}$. Let us now consider different variants leading to the occurrence of one or another scenario depending on the mutual arrangement of four points of the plane, which we will call *key points*, namely, $(\alpha^{k+1}, u_t(\alpha^{k+1}))$, $(\alpha^{k-2}, u_t(\alpha^{k-2} - 0))$, and the line connecting points $(\alpha^k, u_t(\alpha^k))$ and $(\alpha^{k-1}, u_t(\alpha^{k-1}))$, i.e., $\{(\xi, \varphi_{t,k}(\xi)) \colon \xi \in \mathbb{R}\}$, where

$$\varphi_{t,k}(\xi) = u_t\left(\alpha^k\right) + \frac{u_t(\alpha^{k-1}) - u_t(\alpha^k)}{\alpha^{k-1} - \alpha^k}\left(\xi - \alpha^k\right). \tag{72}$$

(1) If the points of the plane $(\alpha^{k+1}, u_t(\alpha^{k+1}))$ and $(\alpha^{k-2}, u_t(\alpha^{k-2} - 0))$ do not lie above the line joining $(\alpha^k, u_t(\alpha^k))$ and $(\alpha^{k-1}, u_t(\alpha^{k-1}))$, i.e., using notations (72)

$$\begin{aligned} \varphi_{t,k}\left(\alpha^{k+1}\right) &\geq u_t\left(\alpha^{k+1}\right), \\ \varphi_{t,k}\left(\alpha^{k-2} - 0\right) &\geq u_t\left(\alpha^{k-2} - 0\right), \end{aligned} \tag{73}$$

then scenario I is realized, for any $x \in [\alpha^k, \alpha^{k-1})$.

(2) If the point of the plane $(\alpha^{k+1}, u_t(\alpha^{k+1}))$ is not above and the point $(\alpha^{k-2}, u_t(\alpha^{k-2}-0))$ is above the line joining $(\alpha^k, u_t(\alpha^k))$ and $(\alpha^{k-1}, u_t(\alpha^{k-1}))$, i.e.,

$$\begin{aligned} \varphi_{t,k}\left(\alpha^{k+1}\right) &\geq u_t\left(\alpha^{k+1}\right), \\ \varphi_{t,k}\left(\alpha^{k-2}-0\right) &< u_t\left(\alpha^{k-2}-0\right), \end{aligned} \tag{74}$$

then denoting

$$y_k = \inf\left\{x \in \left[\alpha^k, \alpha^{k-1}\right): \varphi_{t,k}(\alpha x) < u_t(\alpha x)\right\}, \tag{75}$$

we obtain that scenario I is realized for $x \in [\alpha^k, y_k]$ and scenario II is realized for $x \in (y_k, \alpha^{k-1})$.

(3) If the point of the plane $(\alpha^{k+1}, u_t(\alpha^{k+1}))$ lies above and the point $(\alpha^{k-2}, u_t(\alpha^{k-2}-0))$ lies not above the line joining $(\alpha^k, u_t(\alpha^k))$ and $(\alpha^{k-1}, u_t(\alpha^{k-1}))$, i.e.,

$$\begin{aligned} \varphi_{t,k}\left(\alpha^{k+1}\right) &< u_t\left(\alpha^{k+1}\right), \\ \varphi_{t,k}\left(\alpha^{k-2}-0\right) &\geq u_t\left(\alpha^{k-2}-0\right), \end{aligned} \tag{76}$$

then denoting

$$z_k = \sup\left\{x \in \left[\alpha^k, \alpha^{k-1}\right): \varphi_{t,k}(\alpha^{-1}x) < u_t\left(\alpha^{-1}x\right)\right\}, \tag{77}$$

we obtain that scenario III is realized for $x \in [\alpha^k, z_k)$ and scenario I is realized for $x \in [z_k, \alpha^{k-1})$.

(4) If the points of the plane $(\alpha^{k+1}, u_t(\alpha^{k+1}))$ and $(\alpha^{k-2}, u_t(\alpha^{k-2}-0))$ both lie above the line joining the points $(\alpha^k, u_t(\alpha^k))$ и, $(\alpha^{k-1}, u_t(\alpha^{k-1}))$, i.e.,

$$\begin{aligned} \varphi_{t,k}\left(\alpha^{k+1}\right) &< u_t\left(\alpha^{k+1}\right), \\ \varphi_{t,k}\left(\alpha^{k-2}-0\right) &< u_t\left(\alpha^{k-2}-0\right), \end{aligned} \tag{78}$$

then three possible cases could arise.

(4a) If $y_k < z_k$, where y_k and z_k are given by (75) and (77), respectively, scenario IV is realized for $x \in (y_k, z_k)$, scenario III is realized for $x \in [\alpha^k, y_k)$, and scenario II is realized for x $x \in [z_k, \alpha^{k-1})$

(4b) If $y_k = z_k$, then scenario III is realized for $x \in [\alpha^k, y_k)$, and for $x \in [z_k, \alpha^{k-1})$, scenario II is realized

(4c) If $y_k > z_k$, then scenario III is realized for $x \in [\alpha^k, z_k)$, scenario II is realized for $x \in (y_k, \alpha^{k-1})$, and scenario I is realized for $x \in [z_k, y_k]$

We call the points y_k and z_k given by (75) and (77), respectively, the *switching points of scenarios* (at step t). Note that the switching points of scenarios, as well as some of the points $\alpha^k, k \in \{0, \cdots, t\}$, can be assigned to two scenarios simultaneously. The above analysis of the variants of the location of the four key points of the plane allows us to conclude that the interval $[0, +\infty)$ can be divided into nonintersecting adjacent intervals in which one of the four scenarios is realized; these intervals will be called scenario intervals at step t; such an interval can be subdivided into several rationality intervals.

The endpoints of rationality intervals at step t are points $\alpha^k, k \in \{0, \cdots, t\}$ and possibly switching points of scenarios at all steps up to and including t, if any. Adding point α^{t+1} and possibly scenario switching points at step t (if any, no more than $2t$) to the set $\{d_{t,i}, i = 0, \cdots, m_t + 1\}$ of endpoints of rationality intervals for function u_t, we obtain the set $\{d_{t+1,i}, i = 0, \cdots, m_{t+1} + 1\}$ of endpoints of rationality intervals for function u_{t+1}. It can be easily verified by induction that the function u_{t+1} is piecewise rational; more precisely, it is rational on the rationality intervals that form the subdivision of a scenario interval, given that this claim holds for u_0, using the recurrence relations (49), (53), (57), and (61) for four scenarios and the corresponding probability expressions given by formulae (48), (52), (56), and (60). In particular, the function u_{t+1} is infinitely differentiable on the interior of rationality intervals.

Since the expressions for probabilities are rational functions, representable in the form of irreducible fractions of polynomials of degree unity, the corresponding rational functions represented in the form of irreducible fractions of polynomials have a degree not greater than s on the intervals of rationality (it is easy to see that this degree is equal to 0 on the intervals $(0, \alpha^s)$ and $[1, +\infty)$, is equal to 1 where scenario I is realized, and does not exceed $s-1$ where scenario IV is realized).

In the case of scenario I, formulas (48) and (49) imply that the function u_{t+1} is affine on $[\alpha^k, \alpha^{k-1})$, hence convex, and by the inductive assumption of u_t strict monotonicity, they entail the strict monotonicity (recall that, according to Proposition 3, the solutions of the Bellman equations are nondecreasing (for a nondecreasing payment function)) of u_{t+1} .On the rationality interval contained in $[\alpha^k, \alpha^{k-1})$, on which scenario IV is realized, formulas (60) and (61) directly entail strict monotonicity and convexity of u_{t+1}, owing to the strict monotonicity and convexity u_t (by inductive assumption).

Inside the rationality interval contained in $[\alpha^k, \alpha^{k-1})$, where scenario II is realized and the function u_t is therefore infinitely differentiable, we have, using (53)

$$\begin{aligned} u_{t+1}{}'(x) &= p'(x)u_t\left(\alpha^k\right) + q'(x)u_t(\alpha^{-1}x) + q(x)\alpha^{-1}u_t{}'(\alpha^{-1}x) \\ &= q'(x)\left[u_t(\alpha^{-1}x) - u_t\left(\alpha^k\right)\right] + q(x)\alpha^{-1}u_t{}'(\alpha^{-1}x) > 0, \end{aligned} \tag{79}$$

thanks to the positivity of the derivative function u_t (by inductive assumption) and since

$$q_k{}'(x) = \frac{\alpha^{k-1} - \alpha^k}{\left(\alpha^{-1}x - \alpha^k\right)^2} > 0. \tag{80}$$

Next,

$$\begin{aligned}
q_k{}''(x) &= -\frac{2\alpha^{-1}\left(\alpha^{k-1} - \alpha^k\right)}{\left(\alpha^{-1}x - \alpha^k\right)^3} = -\frac{2\alpha^{-1}q_k{}'(x)}{\alpha^{-1}x - \alpha^k}, \\
u'_{t+1}(x) &= q'_k(x)\left[u_t\left(\alpha^{-1}x\right) - u_t\left(\alpha^k\right)\right] + 2\alpha^{-1}q_k{}'(x)u_t{}'\left(\alpha^{-1}x\right) \\
&\quad + q_k(x)\alpha^2 u'_t\left(\alpha^{-1}x\right) \\
&= 2\alpha^{-1}q_k{}'(x)\left[u_t{}'\left(\alpha^{-1}x\right) - \frac{u_t\left(\alpha^{-1}x\right) - u_t\left(\alpha^k\right)}{\alpha^{-1}x - \alpha^k}\right] \\
&\quad + q_k(x)\alpha^2 u'_t\left(\alpha^{-1}x\right) \geq 0,
\end{aligned} \tag{81}$$

because, owing to the convexity assumption of the function u_t, its second derivative and the expression in square brackets are nonnegative.

Inside the rationality interval contained in $[\alpha^k, \alpha^{k-1})$, where scenario III is realized and therefore the function u_t is infinitely differentiable, using (57), we have

$$\begin{aligned}
u_{t+1}{}'(x) &= p_k{}'(x)u_t(\alpha x) + p_k(x)\alpha u_t{}'(\alpha x) + q_k{}'(x)u_t\left(\alpha^{k-1}\right) \\
&= -p'(x)\left[u_t\left(\alpha^{k-1}\right) - u_t(\alpha x)\right] + p(x)\alpha u_t{}'(\alpha x) > 0,
\end{aligned} \tag{82}$$

using the positivity of the derivative function u_t (by inductive assumption) and inequality

$$p'(x) = -\frac{\alpha^{k-1} - \alpha^k}{\left(\alpha^{k-1} - \alpha x\right)^2} < 0. \tag{83}$$

Next,

$$\begin{aligned}
p''(x) &= -\frac{2\alpha\left(\alpha^{k-1} - \alpha^k\right)}{\left(\alpha^{k-1} - \alpha x\right)^3} = \frac{2\alpha p_k{}'(x)}{\alpha^{k-1} - \alpha x}, \\
u_{t+1}{}''(x) &= -p''(x)\left[u_t\left(\alpha^{k-1}\right) - u_t(\alpha x)\right] + 2\alpha p'(x)u_t{}'(\alpha x) \\
&\quad + p(x)\alpha^2 u'_t(\alpha x) \\
&= 2\alpha p'(x)\left[u_t^{\prime(\alpha x)} - \frac{u_t\left(\alpha^{k-1}\right) - u_t(\alpha x)}{\alpha^{k-1}x - \alpha x}\right] \\
&\quad + q(x)\alpha^2 u_t{}''\left(\alpha^{-1}x\right) \geq 0,
\end{aligned} \tag{84}$$

because, owing to the convexity of the function u_t, its second derivative is nonnegative and the expression in square brackets is nonpositive.

It is still necessary to investigate the behaviour of the function u_{t+1} at the switching points of scenarios y_k and z_k for options (2), (3), and (4), leading to the occurrence of one or the other scenario and possibly to their switching. If variant (2) of the arrangement of the key points takes place and the right derivative (at the left endpoint of the interval, where the function u_t is right continuous, the right derivative coincides with the limit on the right side of the derivative, owing to the continuity of the derivatives (from the inductive assumption)) of the function u_t at point α^{k-1} is not less than the slope of the line joining the points $((\alpha^k, u_t(\alpha^k))$ and $(\alpha^{k-1}, u_t(\alpha^{k-1}))$, i.e.,

$$u_t{}'\left(a^{k-1} + 0\right) \geq \frac{u_t\left(\alpha^{k-1}\right) - u_t\left(\alpha^k\right)}{\alpha^{k-1} - \alpha^k}, \tag{85}$$

then scenario II is realized for any $x \in [\alpha^k, \alpha^{k-1})$.

If, on the other hand,

$$u_t{}'\left(a^{k-1} + 0\right) < \frac{u_t\left(\alpha^{k-1}\right) - u_t\left(\alpha^k\right)}{\alpha^{k-1} - \alpha^k}, \tag{86}$$

then at the point $y_k \in (\alpha^k, \alpha^{k-1})$ given by (75), there is a transversal intersection between the graph of the convex function u_t with the line, which is the graph of the function $\varphi_{t,k}$, given by (77), satisfying $u_t(\alpha^{-1}y_k) = \varphi_{t,k}(\alpha^{-1}y_k)$ and such that for its derivative we have

$$u_t{}'\left(\alpha^{-1}y_k + 0\right) > \varphi_{t,k}{}'\left(\alpha^{-1}y_k\right) = \frac{u_t\left(\alpha^{k-1}\right) - u_t\left(\alpha^k\right)}{\alpha^{k-1} - \alpha^k}, \tag{87}$$

and thus,

$$\begin{aligned}
u_{t+1}{}'\left(y_k + 0\right) &= q_t{}'\left(y_k\right)\left[u_t\left(\alpha^{-1}y_k\right) - u_t\left(\alpha^k\right)\right] \\
&\quad + q_t\left(y_k\right)\alpha^{-1}u_k{}'\left(\alpha^{-1}y_k + 0\right) \\
&>> \frac{\alpha^{k-1} - \alpha^k}{\left(\alpha^{-1}y_k - \alpha^k\right)^2}\left[\varphi_{t,k}\left(\alpha^{-1}y_k\right) - \varphi_{t,k}\left(\alpha^k\right)\right] \\
&\quad + \frac{y_k - \alpha^k}{\alpha^{-1}y_k - \alpha^k}\alpha^{-1}\varphi_{t,k}{}'\left(\alpha^{-1}y_k\right) \\
&== \frac{u_t\left(\alpha^{k-1}\right) - u_t\left(\alpha^k\right)}{\alpha^{k-1} - \alpha^k} \\
&= u_{t+1}{}'\left(y_k - 0\right).
\end{aligned} \tag{88}$$

Therefore, at point $y_k \in (\alpha^k, \alpha^{k-1})$, the convexity of the function u_{t+1} is not violated, but the function u_{t+1} is not differentiable at this point, i.e., there is a "jump" in its derivative.

Similarly, option (3) of key point location can be investigated; in this case, if

$$u_t{}'\left(a^k - 0\right) \leq \frac{u_t\left(\alpha^{k-1}\right) - u_t\left(\alpha^k\right)}{\alpha^{k-1} - \alpha^k}, \tag{89}$$

then scenario III is realized for any $x \in [\alpha^k, \alpha^{k-1})$, and if (89) is not satisfied, then there is a transversal intersection at the point $z_k \in (\alpha^k, \alpha^{k-1})$ of the graph of the convex function u_t and a line, which is the graph of the function $\varphi_{t,k}$ given by

(77); in addition, the convexity of the function u_{t+1} is not violated at point $y_k \in (\alpha^k, \alpha^{k-1})$, but the function u_{t+1} is not differentiable at this point (there is a derivative jump).

For variant (4) of the four key point location, both conditions (85) and (89) have to be checked. When both of these inequalities are satisfied, scenario IV is realized for any $x \in [\alpha^k, \alpha^{k-1})$, whereas otherwise one or two switching points of the scenarios arise, and for the behaviour of function u_{t+1}, which is similar to the cases considered above, their mutual locations (cases (4a), (4b), and (4c)) must be considered.

Theorem 8. *The function u_s is continuous (thus, subject to Propositions 5 and 6, the function u_s is continuous over the interval $(0, +\infty)$, except for a single point α^s, where it experiences a jump and is right continuous) on the interval $[\alpha^s, +\infty)$ for $s = 0, \cdots, N$.*

Proof. Let us check this property by induction. For $s \in \{0, 1, 2\}$, this property is satisfied. Let it be satisfied for $s = t \geq 2$, and let us show that it is satisfied for $s = t + 1$. On open intervals (α^k, α^{k-1}), $k = 0, \cdots, t+1$, the function u_{t+1} is continuous owing to the convexity. It suffices to check its continuity at points $\alpha^k, k = 0, \cdots, t$. We fix $k \in \{0, \cdots, t-1\}$ and consider an interval of the form $[\alpha^k, \alpha^{k-1})$, which we call left (with respect to point α^{k-1}); the corresponding four key points for this interval (with abscissa $\alpha^{k+1}, \alpha^k, \alpha^{k-1}, \alpha^{k-2}$ and the ordinates being the values of function u_t at these points); the adjacent interval $[\alpha^{k-1}, \alpha^{k-2})$, which are called the right one; and the corresponding four key points (with abscissa $\alpha^k, \alpha^{k-1}, \alpha^{k-2}, \alpha^{k-3}$ and ordinates are the values of function u_t at these points). □

If for the left interval there is variant (1) of the arrangement of key points, then for the right interval the possible variants are (1) or (2); the "matching" relations at the right end of the left interval are given by (51), for scenario I, i.e.,

$$u_{t+1}\left(\alpha^{k-1} - 0\right) = u_t\left(\alpha^{k-1} - 0\right), \tag{90}$$

and at the left end of the right interval by relations (50) and (54) for scenarios I and II, i.e.,

$$u_{t+1}\left(\alpha^{k-1}\right) = u_t\left(\alpha^{k-1}\right), \tag{91}$$

and hence, using the inductive assumption of continuity of u_t, we obtain the continuity of u_{t+1} at the point α^{k-1}.If for the left interval, variant (2) of the arrangement of key points takes place, then for the right interval the possible variants are (3) or (4); the matching conditions at the right endpoint of left interval are given by relation (55) for scenario II, i.e.,

$$u_{t+1}\left(\alpha^{k-1} - 0\right) = \frac{1}{1+\alpha} u_t\left(\alpha^k - 0\right) + \frac{\alpha}{1+\alpha} u_t\left(\alpha^{k-2} - 0\right), \tag{92}$$

and at the left endpoint of right interval we have matching condition (58) for scenario III, i.e.,

$$u_{t+1}\left(\alpha^k\right) = \frac{1}{1+\alpha} u_t\left(\alpha^{k+1}\right) + \frac{\alpha}{1+\alpha} u_t\left(\alpha^{k-1}\right), \tag{93}$$

whence u_{t+1} is continuous at the point α^{k-1}.

If for the left interval there is variant (3) of the arrangement of key points, then for the right interval the possible variants are (1) or (2); the conjugation conditions at the right end of the left interval are set by relation (51) for scenario I, i.e.,

$$u_{t+1}\left(\alpha^{k-1} - 0\right) = u_t\left(\alpha^{k-1} - 0\right), \tag{94}$$

and at the left end of the right interval by relations (50) and (54) for scenarios I and II, i.e.,

$$u_{t+1}\left(\alpha^{k-1}\right) = u_t\left(\alpha^{k-1}\right), \tag{95}$$

whence it follows that u_{t+1} is continuous at the point α^{k-1}.

If for the left interval there is variant (4) of the arrangement of key points, then for the right interval the possible variants are (3) or (4); in addition, the matching relations at the right end of the left interval are set by (55), for scenario II, i.e.,

$$u_{t+1}\left(\alpha^{k-1} - 0\right) = \frac{1}{1+\alpha} u_t\left(\alpha^k - 0\right) + \frac{\alpha}{1+\alpha} u_t\left(\alpha^{k-2} - 0\right), \tag{96}$$

and at the left endpoint of the right interval by relation (58) for scenario III, i.e.,

$$u_{t+1}\left(\alpha^k\right) = \frac{1}{1+\alpha} u_t\left(\alpha^{k+1}\right) + \frac{\alpha}{1+\alpha} u_t\left(\alpha^{k-1}\right), \tag{97}$$

whence it follows that u_{t+1} is continuous at the point α^{k-1}.

Thus, the continuity of u_{t+1} at the points (note that for the interval $[\alpha, 1)$, the possible locations of the key points can only be (1) or (2)) $\alpha^{t-1}, \alpha^{t-2}, \cdots, 1$ is established. Consider now the interval $[\alpha^t, \alpha^{t-1})$ and notice that owing to the properties of the function u_1 and the self-similarity property, established in Proposition 5, the points

$$\left(\alpha^{t+1}, u_t\left(\alpha^{t+1}\right)\right), \left(\alpha^t, u_t\left(\alpha^t\right)\right), \left(\alpha^{t-1}, u_t\left(\alpha^{t-1}\right)\right) \tag{98}$$

lie on the same line. Therefore, depending on the position of the point $(\alpha^{t-2}, u_t(\alpha^{t-2}))$, the possible options for the interval $[\alpha^t, \alpha^{t-1})$ are (1) or (2); at the left endpoint of the interval $[\alpha^t, \alpha^{t-1})$, the matching relations are given by (50) and (54), for scenarios I and II, i.e.,

$$u_{t+1}\left(\alpha^t\right) = u_t\left(\alpha^t\right). \tag{99}$$

From (43) and (31), we have

$$u_t(\alpha^t) = \left(\frac{\alpha}{1+\alpha}\right)^t, \tag{100}$$

and from (43) and (32), we have

$$u_{t+1}(\alpha^t - 0) = \left(\frac{\alpha}{1+\alpha}\right)^t, \tag{101}$$

whence it follows that u_{t+1} is continuous at the point α^t.

The statement of Theorem 8 can be strengthened: the function u_s is even Lipschitz on the interval $[\alpha^s, +\infty)$, for $s = 0, \cdots, N$ (see Theorem 9 below). However, in our opinion, the proof of Theorem 8 is of independent interest because it clarifies well the essence of the problem. For the function f on $[a, b]$, we denote

$$L(f, [a, b]) = \sup\left\{\frac{|f(x_2) - f(x_1)|}{x_2 - x_1} : x_1, x_2 \in [a, b], x_1 < x_2\right\}. \tag{102}$$

If $L(f, [a, b])$ in (102) is finite, then it is the Lipschitz constant of the function f on $[a, b]$.

Theorem 9. *The function u_s satisfies the Lipschitz condition on the interval $[\alpha^s, +\infty)$, for $s = 1, \cdots, N$, with the Lipschitz constants being nonincreasing with respect to s, and $L(u_1, [\alpha, +\infty)) = (1-\alpha)^{-1}$.*

Proof. Note first that if $a = c_0 < c_1 < \cdots < c_{r-1} < c_r = b$, where for $r \geq 2$, then, using notation (102), the following equality holds:

$$L(f, [a, b]) = \vee_{j=1}^{r} L(f, [c_{j-1}, c_j]). \tag{103}$$

Let us check the validity of point 2 of the theorem by induction. For $s = 1$, this statement holds, and in this case, the Lipschitz constant is

$$L(u_1, [\alpha, +\infty)) = u_1{}'(1 - 0) = (1 - \alpha)^{-1}. \tag{104}$$

Suppose that for $s = t$ the Lipschitz constant $L(u_t, [\alpha^t, +\infty)) < \infty$. Applying Proposition 4 with parameters $a = a = \alpha^t$, $b = \alpha^{-1}$, and $\beta = \alpha^{-1}$, we obtain, given (103), that

$$\begin{aligned} L(u_{t+1}, [\alpha^{t-1}, +\infty)) &= L(u_{t+1}, [\alpha^{t-1}, 1]) \leq L(u_t, [\alpha^t, \alpha^{-1}]) \\ &= L(u_t, [\alpha^t, +\infty)) < \infty. \end{aligned} \tag{105}$$

By virtue of the self-similarity (43), established in Proposition 6, as well as the continuity $x \in [\alpha^{t+1}, \alpha^t]$ proved in Theorem 8, we have the relation

$$u_{t+1}(x) = \frac{\alpha}{1+\alpha} u_t(\alpha^{-1}x), \tag{106}$$

whence

$$L(u_{t+1}, [\alpha^{t+1}, \alpha^t]) \leq \frac{1}{1+\alpha} L(u_t, [\alpha^t, \alpha^{t-1}]). \tag{107}$$

As it has been noted above at the proof of continuity, the points

$$(\alpha^{t+1}, u_t(\alpha^{t+1})), (\alpha^t, u_t(\alpha^t)), (\alpha^{t-1}, u_t(\alpha^{t-1})) \tag{108}$$

are on the same straight line, and thus depending on the position of the point $(\alpha^{t-2}, u_t(\alpha^{t-2}))$, the possible locations of the key points for the interval $[\alpha^t, \alpha^{t-1})$ are (1) or (2).

In the case of variant (1), scenario I is realized on the interval $[\alpha^t, \alpha^{t-1})$, and thus, given continuity, the function u_{t+1} is affine on $[\alpha^t, \alpha^{t-1}]$; therefore,

$$L(u_{t+1}, [\alpha^t, \alpha^{t-1}]) = \frac{u_t(\alpha^{t-1}) - u_t(\alpha^t)}{\alpha^{t-1} - \alpha^t}, \tag{109}$$

owing to convexity of the function u_t on the interval $[\alpha^t, \alpha^{t-1})$ and its continuity

$$\frac{u_t(\alpha^{t-1}) - u_t(\alpha^t)}{\alpha^{t-1} - \alpha^t} \leq u_t'(\alpha^{t-1} - 0) = L(u_t, [\alpha^t, \alpha^{t-1}]). \tag{110}$$

Thus,

$$L(u_{t+1}, [\alpha^t, \alpha^{t-1}]) \leq L(u_t, [\alpha^t, \alpha^{t-1}]). \tag{111}$$

In the case of variant (2), scenario I is realized on the interval $[\alpha^t, \alpha^{t-1})$; therefore, the derivative of the function u_{t+1} is given by (79) for $k = t$. Note that

$$q_t{}'(x) = \frac{\alpha^{t-1} - \alpha^t}{(x - \alpha^t)(\alpha^{-1}x - \alpha^t)} q_t(x), \tag{112}$$

whence

$$u_{t+1}{}'(x) = q_t(x)\left[\frac{\alpha^{t-1} - \alpha^t}{x - \alpha^t} \cdot \frac{u_t(\alpha^{-1}x) - u_t(\alpha^t)}{\alpha^{-1}x - \alpha^t} + \alpha^{-1} u_t'(\alpha^{-1}x)\right]. \tag{113}$$

Taking into account (110) and the convexity of the function u_{t+1} on the interval $[\alpha^t, \alpha^{t-1})$ and its continuity, we have

$$\begin{aligned} L(u_{t+1}, [\alpha^t, \alpha^{t-1}]) &= u_{t+1}{}'(\alpha^{t-1} - 0) \\ &\leq \frac{\alpha}{1+\alpha}\left[L(u_t, [\alpha^t, \alpha^{t-1}]) + \alpha^{-1} u'_t(\alpha^{t-2} - 0)\right] \\ &= \frac{\alpha}{1+\alpha} L(u_t, [\alpha^t, \alpha^{t-1}]) + \frac{1}{1+\alpha} L(u_t, [\alpha^{t-1}, \alpha^{t-2}]) \\ &\leq L(u_t, [\alpha^t, \alpha^{t-1}]) \vee L(u_t, [\alpha^{t-1}, \alpha^{t-2}]) \\ &= L(u_t, [\alpha^t, \alpha^{t-2}]). \end{aligned} \tag{114}$$

Thus, the required statement follows from (107), (111), and (114). □

4. Numerical Solution Algorithm

To obtain a "semi-implicit" solution of the Bellman equation, summarizing the results obtained above, the following recurrence algorithm can be proposed. Suppose that at step $s \geq 1$ we obtain a partition of the segment $[\alpha^s, 1)$ into intervals of rationality $[d_{s,i}, d_{s,i+1})$, $i = 0, \cdots, m_s$, and that the set of endpoints of these intervals $d_{s+1,i}$, $i = 0, \cdots, m_{s+1} + 1$ contains the points α^k, $k = 0, \cdots, s$. In addition, suppose that on the intervals $[d_{s,i}, d_{s,i+1})$, $i = 0, \cdots, m_s$ are found explicit (analytic) expressions of the functions u_s and their derivatives (this, in particular, can be found using a symbolic computation) in the form of rational functions. The following steps are performed to find the u_{s+1} function.

(1) The values of the function u_s at the points α^k, $k = 0, \cdots, s$ are calculated, the variants of the key point locations for intervals of the form $[\alpha^k, \alpha^{k-1})$, $k = 1, \cdots, s$, and possible scenarios for this variant are determined

(2) The presence of scenario switching points for intervals of the form $[\alpha^k, \alpha^{k-1})$, $k = 1, \cdots, s$ is determined, and in case of their presence scenario, switching points are found numerically (this, in fact, is equivalent to finding the root of a polynomial of degree not greater than $s + 1$)

For variant (1) of the key point arrangement, scenario I is realized, for any $x \in [\alpha^k, \alpha^{k-1})$.

For variant (2) of the arrangement of key points, the derivative $u_s'(a^{k-1} + 0)$ is calculated and

(i) if inequality (85) holds, then scenario I is realized for any $x \in [\alpha^k, \alpha^{k-1})$

(ii) if inequality (85) is not fulfilled, then the point y_k is found numerically as the only root of the equation on the interval (α^k, α^{k-1}), i.e.,

$$\varphi_{s,k}(\alpha x) = u_s(\alpha x), \tag{115}$$

where the function $\varphi_{t,k}$ is defined by (72); for $x \in (y_k, \alpha^{k-1})$, scenario II is realized, and for $x \in (\alpha^k, y_k]$, scenario I is realized.

For variant (3) of the location of key points, the derivative $u_s'(a^k - 0)$ is calculated and

(i) if inequality (89) holds, then scenario III is realized for any $x \in [\alpha^k, \alpha^{k-1})$

(ii) if inequality (89) is not fulfilled, then the point z_k is found numerically as the only root of the equation on the interval (α^k, α^{k-1}), i.e.,

$$\varphi_{s,k}(\alpha^{-1} x) = u_s(\alpha^{-1} x), \tag{116}$$

for $x \in [\alpha^k, z_k)$, scenario III is realized, and for $x \in [z_k, \alpha^{k-1})$, scenario I is realized.

For variant (4) of the location of key points, both derivatives $u_s'(a^{k-1} + 0)$ and $u_s'(a^k - 0)$ are calculated; the two inequalities (85) and (89) are checked:

(i) if both inequalities (85) and (89) are satisfied, then scenario IV is realized for any $x \in [\alpha^k, \alpha^{k-1})$

(ii) if inequality (85) holds and inequality (89) does not hold, then the switching point of scenarios is found numerically, being the only root z_k of equation (116) on the interval (α^k, α^{k-1}); in this case, scenario III is realized for $x \in [\alpha^k, z_k)$ and scenario IV is realized for $x \in [z_k, \alpha^{k-1})$

(iii) if inequality (89) holds and inequality (85) does not hold, then the switching point of scenarios is found numerically, being the only root y_k of equation (115) on the interval (α^k, α^{k-1}); in this case, scenario II is realized for $x \in [y_k, \alpha^{k-1})$ and scenario I is realized for $x \in [\alpha^k, y_k)$

(iv) if both inequalities (85) and (89) are not satisfied, then two switching points of scenarios are found numerically, being the only root z_k of equation (116) on the interval (α^k, α^{k-1}) and the only root y_k of equation (115) on the interval (α^k, α^{k-1}); three possible cases can arise depending on the mutual location of y_k and z_k

(4a) If $y_k < z_k$, where y_k and z_k are given by (75) and (77), respectively, scenario IV is realized for $x \in (y_k, z_k)$, scenario III is realized for $x \in [\alpha^k, y_k)$, and scenario II for $x \in [z_k, \alpha^{k-1})$

(4b) If $y_k = z_k$, then scenario III is realized for $x \in [\alpha^k, y_k)$ and scenario II is realized for $x \in [z_k, \alpha^{k-1})$

(4c) If $y_k > z_k$, then scenario III is realized for $x \in [\alpha^k, z_k)$, scenario II is realized for $x \in (y_k, \alpha^{k-1})$, and scenario I is realized for $x \in [z_k, y_k]$

Thus, at step t, we obtain a partition of $[0, +\infty)$ into adjacent intervals on which one of the four scenarios is realized; the endpoints of these intervals are points $\alpha^k, k \in \{0, \cdots, s\}$ and possibly switching points of the scenarios at step t (if any).

(3) The partitioning into rationality intervals is constructed: in order to obtain the set $\{d_{s+1,i}, i = 0, \cdots, m_{s+1} + 1\}$ of endpoints for rationality intervals of the function u_{s+1}, the point α^{s+1} is added to the set $\{d_{s,i}, i = 0, \cdots, m_s + 1\}$ and, possibly, points of switching scenarios at step s (if any, note that their number cannot exceed $2t$). Let the sequence $d_{s,i}$, $i = 0, \cdots, m_s + 1$ be increasing (with respect to i). On each interval of the resulting partition $[d_{s+1,i}, d_{s+1,i+1})$, $i = 0, \cdots, m_{s+1}$, one scenario at step s is realized and explicit expressions for the u_s function are

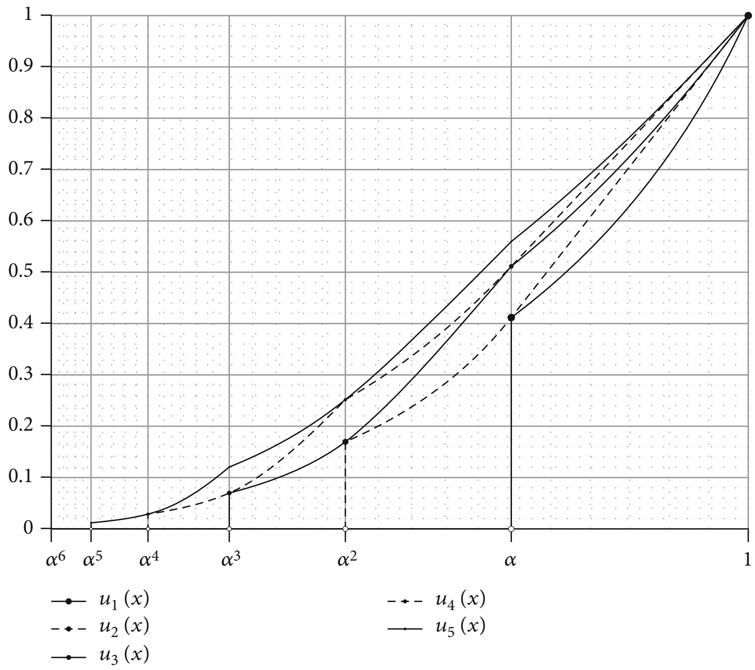

Figure 3: First five iterations, $\alpha = 0.7$.

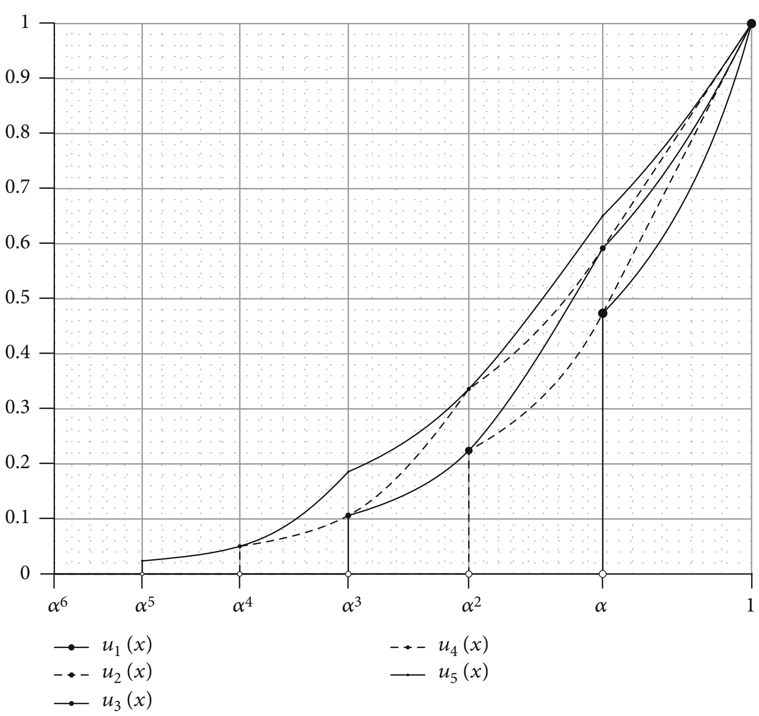

Figure 4: First five iterations, $\alpha = 0.9$.

given, and there are explicit recurrence formulas for four possible scenarios: (49), (53), (57), and (61), that express the u_{s+1} function via the u_s function and preserve rationality. Using the explicit expression for the u_{s+1} function as a rational function, we compute the derivative of ${u_{s+1}}'$ on each partition interval $[d_{s+1,i}, d_{s+1,i+1})$, $i = 0, \cdots, m_{s+1}$

5. Numerical Results

Based on the described algorithm, we have performed the calculations for different values of the parameter α. The results are shown in Figures 3 and 4.

We observed no scenario switching and a smooth conjugation of piecewise convex rational functions on pairs of intervals $[\alpha^{k+1}, \alpha^k]$.

6. Conclusion

This paper considered the problem of pricing a binary call option of the European type in the framework of guaranteed deterministic superhedging approach, for a multiplicative model of price dynamics, with one risky asset and no trading constraints. The main results are obtained for the case when intervals defining possible values of the uncertain price multiplier have endpoints satisfying relation, similar to the assumption of the classical paper of Cox, Ross, and Rubinstein [18]. A number of properties of solutions of Bellman–Isaacs equations (or Bellman equations, arising due to game equilibrium at each time step) are obtained. It is shown that the solutions are numerical functions and are monotonically nondecreasing, continuously to right and piecewise convex, continuous and even Lipschitz, except for one point (in which there is a jump); on the interval from this point to the strike price, the solutions are strictly monotonically increasing, with the Lipschitz constants of the solution not increasing with increasing time to expiration. In addition, the solutions are piecewise rational; this gave us the opportunity to propose an algorithm for constructing a "semi-explicit" solution, i.e., a recurrence construction of solutions in the form of formulas on some intervals; in particular, symbolic calculations can be used. The results of the numerical analysis suggest certain hypotheses about the behaviour of the solutions of Bellman equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

This paper is the English version of the paper [17] initially published in Russian.

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