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Research Article

Bound State Solution Schrödinger Equation for Extended Cornell Potential at Finite Temperature

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In this paper, we study the finite temperature-dependent Schrödinger equation by using the Nikiforov-Uvarov method. We consider the sum of the Cornell, inverse quadratic, and harmonic-type potentials as the potential part of the radial Schrödinger equation. Analytical expressions for the energy eigenvalues and the radial wave function are presented. Application of the results for the heavy quarkonia and B_c meson masses are in good agreement with the current experimental data except for zero angular momentum quantum numbers. Numerical results for the temperature dependence indicates a different behaviour for different quantum numbers. Temperature-dependent results are in agreement with some QCD sum rule results from the ground states.

1. Introduction

It is a well-known fact that the potential models in quantum mechanics are very accurate in reproducing experimental data for meson spectroscopy [1] at zero temperature. However, one needs to consider spin-dependent potentials in the Schrödinger equation in order to describe the relativistic effects [2, 3]. There exist few potentials which are highly important because of their exact solubility within the Schrödinger equation for which all spectra of radial n_r and orbital l quantum states can be obtained analytically [2–5]. Except for the exact solvable potentials, others are solved by either approximation or numerical methods.

Several potentials such as the exponential-type including the Hulthén-, Manning–Rosen-, Woods–Saxon-, and Eckart-type potentials are also currently being investigated by several researchers. Among the particularly interesting potentials which play an important role in the quark-antiquark bound states include the so-called Cornell potential and a mixture of it with the harmonic oscillator potential and Morse potential as discussed in [6–10].

As an analytical method, the Nikiforov-Uvarov (NU) method is one of the widely applicable methods for solving the Schrödinger equation. The quarkonia system in a hot and dense matter media is studied in Ref. [11], where the authors studied the quarkonium dissociation in anisotropic plasma in hot and dense media by analytically solving the multidimensional Schrödinger equation via the NU method for the real part of the potential. The NU method was successfully applied for solving the radial Schrödinger equation in the presence of an external magnetic field and the Aharonov-Bohm flux fields in [12, 13]. The inverse square root potential, which is a long-range potential and a combination of the Coulomb, linear, and harmonic potentials, is often used to describe quarkonium states.

The study of the heavy quark resonances in a nonrelativistic regime and the thermal environment shows the importance of the color screening radius below which binding is impossible [14]. The theoretical investigation of this effect for the charmonium resonance is investigated in [15]. Considering the finite temperature for the Cornell potential within the D-dimensional Schrödinger equation by using the NU method is presented in [16]. However, the behaviour of bound states of heavy quarks in a strongly interacting medium close to the deconfinement temperature T_c is largely uncertain such that various models predict the mass being constant, increasing and decreasing with temperature increments. One of the first applications of a nonrelativistic lattice QCD to the study of heavy quarkonia at finite temperature is presented in [17].

Temperature-dependent Schrödinger equations for different potentials by different methods are studied in [18–21]. In Ref. [22] a modified radial Schrödinger equation for the sum of the Cornell and inverse quadratic potentials at finite temperature is solved. In Refs. [23, 24], the radial and hyperradial Schrödinger equations are analytically solved using the NU and SUSYQM methods, in which the heavy quarkonia potential is introduced at finite temperature with the baryon chemical potential. For numerical solutions, one may, for instance, have a look in [25].

The Cornell potential is extensively used to describe the mass spectrum of the heavy quark and antiquark systems at zero temperature [26, 27]. It is the sum of linear and Coulomb terms which are responsible for the confinement and quark-antiquark interaction at short distances, respectively.

The bound state solutions to the wave equations under the quark-antiquark interaction potential such as the ordinary, extended, and generalized Cornell potentials and combined potentials such as the Cornell with other potentials have attracted much research interest in atomic and high-energy physics within ordinary and supersymmetric quantum mechanics methods as in [28–35].

Recently, Ikot et al. [36] reported the approximate solutions of the Schrödinger equation with the central generalized Hulthén and Yukawa potential within the framework of the functional method. The obtained wave function and the energy levels are used to study the Shannon entropy, the Renyi entropy, the Fisher information, the Shannon-Fisher complexity, the Shannon power, and the Fisher-Shannon product in both position and momentum spaces for the ground and first excited states.

The exact solution of the Schrödinger equation for the new anharmonic oscillator, double ring-shaped oscillator, and quantum system with a nonpolynomial oscillator potential related to the isotonic oscillator was also widely studied in Refs. [37–39]. The relativistic Levinson theorem was also studied in detail in Ref. [40], and the authors obtained the modified relativistic Levinson theorem for noncritical cases.

It is still an open question if the appropriate potential which describes the interaction between a quark and an antiquark can be found more precisely. It would be interesting to test the following potential for the arbitrary orbital quantum number $l \neq 0$ at finite temperature by using the NU method [41]:

$$V(r) \equiv A \cdot r - \frac{B}{r} + \frac{C}{r^2} + D \cdot r^2. \tag{1}$$

Here, A, B, C, and D are constant potential parameters, respectively.

The rest of the paper is organized as follows. The temperature-dependent radial Schrödinger equation for the sum of the Cornell, inverse quadratic, and harmonic oscillator-type potentials is introduced in Section 2 and solved using the NU method in Section 3. In Section 4, we apply the results to the mass spectrum of heavy mesons at zero and nonzero temperatures in Section 5, respectively. Finally, we end up with some concluding remarks in Section 6.

2. Temperature-Dependent Radial Schrödinger Equation

The Schrödinger equation in spherical coordinates is given as follows:

$$\nabla^2 \psi + \frac{2\mu}{\hbar^2}[E - V(r)]\psi(r, \theta, \phi) = 0. \tag{2}$$

For the case of the separation of the wave function to the radial and angular parts, we can write the wave function as follows:

$$\psi(r, \theta, \phi) = R(r) Y_{l,m}(\theta, \phi). \tag{3}$$

Consideration of the radial part of the wave function and the Cornell, inverse quadratic, and harmonic-type potentials for the potential part of the Schrödinger equation leads to the following:

$$R''(r) + \frac{2}{r}R'(r) + \frac{2\mu}{\hbar^2}\left[E - \frac{l(l+1)\hbar^2}{2\mu r^2} - V(r)\right]R(r) = 0. \tag{4}$$

This equation gets the following form if we do the replacement for the radial part of the wave function $R(r) \equiv \chi(r)/r$ in (4):

$$\chi''(r) + \frac{2\mu}{\hbar^2}\left[E - \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} - V(r)\right]\chi(r) = 0, \tag{5}$$

where μ is the reduced mass of the quark-antiquark system which is defined in this form: $1/\mu = 1/\mu_q + 1/\mu_{\bar{q}}$.Following the same philosophy for the Cornell potential in [17], one can do the nonzero temperature modification to the constant terms in equation (1) and make the potential term temperature dependent as follows:

$$V(T, r) \equiv A(T, r) \cdot r - \frac{B(T, r)}{r} + \frac{C(T, r)}{r^2} + D(T, r) \cdot r^2, \tag{6}$$

where

$$\begin{aligned} A(T,r) &\equiv \frac{A}{\mu_D(T)\cdot r}\left(1-\exp\{-\mu_D(T)r\}\right),\\ B(T,r) &\equiv B\exp\{-\mu_D(T)r\},\\ C(T,r) &\equiv C\exp\{-\mu_D(T)r\},\\ D(T,r) &\equiv \frac{D}{\mu_D(T)\cdot r}\left(1-\exp\{-\mu_D(T)r\}\right). \end{aligned} \tag{7}$$

Here, $\mu_D(T)$ is the Debye screening mass, which vanishes at $T\longrightarrow 0$. It should be noted that, in this model, the temperature dependence of a potential is contained in a Debye screened mass. Next, using the approximation $\exp(-\mu_D(T)r)=\sum_{n=0}^{\infty}((-\mu_D(T)r)^n/n!)$ up to the second order, which gives a good accuracy when $\mu_D(T)r\ll 1$, we then obtain the following:

$$\begin{aligned} A(T,r) &\equiv \frac{A}{\mu_D(T)\cdot r}\left(1-\exp\{-\mu_D(T)r\}\right)=A-\frac{A}{2}\mu_D(T)r,\\ B(T,r) &\equiv B\exp\{-\mu_D(T)r\}=B\left(1-\mu_D(T)r+\frac{1}{2}\mu_D^2(T)r^2\right),\\ C(T,r) &\equiv C\exp\{-\mu_D(T)r\}=C\left(1-\mu_D(T)r+\frac{1}{2}\mu_D(T)r^2\right),\\ D(T,r) &\equiv \frac{D}{\mu_D(T)\cdot r}\left(1-\exp\{-\mu_D(T)r\}\right)=D-\frac{D}{2}\mu_D(T)r^2. \end{aligned} \tag{8}$$

Then, for $V(T,r)$, we obtain the following:

$$\begin{aligned} V(T,r) &= B\mu_D(T)+\frac{1}{2}C\mu_D^2(T)\\ &\quad+\left(A-\frac{1}{2}B\mu_D^2(T)\right)r-\left(B+C\mu_D(T)\right)\frac{1}{r}\\ &\quad-\left(\frac{1}{2}A\mu_D(T)-D\right)r^2+\frac{C}{r^2}-\frac{D}{2}\mu_D(T)r^3. \end{aligned} \tag{9}$$

As can be seen from the expressions in (8) at $T=0$ zero temperature, $A(T=0)=A$, $B(T=0)=B$, $C(T=0)=C$, and $D(T=0)=D$.

By using the expressions in (8), we may rewrite expression (6) in a more compact way as follows:

$$V(T,r)=F+Gr-\frac{L}{r}-Mr^2+\frac{C}{r^2}-Nr^3, \tag{10}$$

where we used the following substitutions:

$$\begin{aligned} F &\equiv B\mu_D(T)+\frac{1}{2}C\mu_D^2(T),\\ G &\equiv A-\frac{1}{2}B\mu_D^2(T),\\ L &\equiv B+C\mu_D(T),\\ M &\equiv \frac{1}{2}A\mu_D(T)-D,\\ N &\equiv \frac{D}{2}\mu_D(T). \end{aligned} \tag{11}$$

Considering all these in the radial Schrödinger equation (5), we get the following:

$$\chi''(r)+\frac{2\mu}{\hbar^2}\left[E-\frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}-F-Gr+\frac{L}{r}+Mr^2-\frac{C}{r^2}+Nr^3\right]\chi(r)=0. \tag{12}$$

Let us reduce the above equation to the generalized hypergeometric type [41]:

$$\chi''(s)+\frac{\tilde{\tau}}{\sigma}\chi'(s)+\frac{\tilde{\sigma}}{\sigma^2}\chi(s)=0. \tag{13}$$

In order to do that, we do the replacement $r=(1/x)$ in equation (12) which leads to the following:

$$\begin{aligned} &\chi''(x)+\frac{2x}{x^2}\chi'(x)+\frac{2\mu}{\hbar^2}\frac{1}{x^4}\left[E-\frac{\hbar^2}{2\mu}l(l+1)x^2-F\right.\\ &\left.-\frac{G}{x}+Lx+\frac{M}{x^2}-Cx^2+\frac{N}{x^3}\right]\chi(x)=0. \end{aligned} \tag{14}$$

For the solution in equation (14), we introduce the following approximation scheme on the terms G/x, M/x^2, and N/x^3. Let us consider a characteristic radius r_0 of the quark and antiquark system; it is the minimum interval between two quarks at which they cannot collide with each other. This scheme is based on the expansion of G/x, M/x^2, and N/x^3 in a power series around r_0 or $\delta=1/r_0$ in the x-space, up to the second order. One should note that the G-, M-, and N-dependent terms save the original form of equation (14). This approach is similar to the Pekeris approximation [42], which causes a deformation of the centrifugal potential. Hence, after this modified potential, equation (14) can be solved by the NU method. This expansion is done for the new variable $y=x-\delta$, where $\delta=1/r_0$ around $y=0$ as follows:

$$\begin{aligned} \frac{G}{x} &= \frac{G}{y+\delta}=\frac{G}{\delta}\left(1-\frac{y}{\delta}+\frac{y^2}{\delta^2}\right)\\ &= \frac{G}{\delta}\left(1-\frac{x-\delta}{\delta}+\frac{(x-\delta)^2}{\delta^2}\right)\\ &= G\left(\frac{3}{\delta}-\frac{3x}{\delta^2}+\frac{x^2}{\delta^3}\right), \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{M}{x^2} &= \frac{M}{(y+\delta)^2}\\ &= \frac{M}{\delta^2}\left(1+\frac{y}{\delta}\right)^{-2}\\ &= M\left(\frac{6}{\delta^2}-\frac{8x}{\delta^3}+\frac{3x^2}{\delta^4}\right), \end{aligned} \tag{16}$$

$$\begin{aligned}\frac{N}{x^3} &= \frac{N}{(y+\delta)^3}\\ &= \frac{N}{\delta^3}\left(1+\frac{y}{\delta}\right)^{-3}\\ &= N\left(\frac{10}{\delta^3}-\frac{15x}{\delta^4}+\frac{6x^2}{\delta^5}\right).\end{aligned} \tag{17}$$

We get the following equation by substituting equations (15), (16), and (17) into equation (14):

$$\begin{aligned}&\chi''(x)+\frac{2x}{x^2}\chi'(x)+\frac{2\mu}{\hbar^2}\frac{1}{x^4}\left[\left(E-F-\frac{3G}{\delta}+\frac{6M}{\delta^2}+\frac{10N}{\delta^3}\right)\right.\\ &+\left(\frac{3G}{\delta^2}+L-\frac{8M}{\delta^3}-\frac{15N}{\delta^4}\right)x\\ &\left.+\left(-\frac{\hbar^2}{2\mu}l(l+1)-\frac{G}{\delta^3}+\frac{3M}{\delta^4}+\frac{6N}{\delta^5}-C\right)x^2\right]\chi(x)=0.\end{aligned} \tag{18}$$

In equation (18), we introduce new variables for making the differential equation more compact:

$$\begin{aligned}H &\equiv -\frac{2\mu}{\hbar^2}\left(E-F-\frac{3G}{\delta}+\frac{6M}{\delta^2}+\frac{10N}{\delta^3}\right),\\ P &\equiv \frac{2\mu}{\hbar^2}\left(\frac{3G}{\delta^2}+L-\frac{8M}{\delta^3}-\frac{15N}{\delta^4}\right),\end{aligned} \tag{19}$$

$$Q \equiv \frac{2\mu}{\hbar^2}\left(-\frac{\hbar^2}{2\mu}l(l+1)-\frac{G}{\delta^3}+\frac{3M}{\delta^4}+\frac{6N}{\delta^5}-C\right). \tag{20}$$

Finally, equation (18) gets the following more compact form:

$$\chi''(x)+\frac{2x}{x^2}\chi'(x)+\frac{1}{x^4}\left[-H+Px+Qx^2\right]\chi(x)=0. \tag{21}$$

3. NU Method Application

In this section, we will apply the NU method for defining the energy eigenvalues. A comparison of equation (21) and equation (13) leads us to the following redefinitions:

$$\begin{aligned}\tilde{\tau}(x) &= 2x,\\ \sigma(x) &= x^2,\\ \tilde{\sigma}(x) &= \left(-H+Px+Qx^2\right).\end{aligned} \tag{22}$$

Consider the following factorization:

$$\chi(x)=\phi(x)y(x). \tag{23}$$

For the appropriate function $\phi(x)$, (21) takes the form of the well-known hypergeometric-type equation. The appropriate $\phi(x)$ function has to satisfy the following condition:

$$\frac{\phi'(x)}{\phi(x)}=\frac{\pi(x)}{\sigma(x)}, \tag{24}$$

where function $\pi(x)$ is the maximum degree of a polynomial with one variable and is defined as follows:

$$\begin{aligned}\pi(x) &= \frac{\sigma'-\tilde{\tau}}{2}\pm\sqrt{\left(\frac{\sigma'-\tilde{\tau}}{2}\right)^2-\tilde{\sigma}+k\sigma}\\ &= \frac{2x-2x}{2}\pm\sqrt{H-Px-Qx^2+kx^2}\\ &= \pm\sqrt{(k-Q)x^2-Px+H}.\end{aligned} \tag{25}$$

Finally, we get the following hypergeometric-type equation:

$$\sigma(x)y''(x)+\tau(x)y'(x)+\bar{\lambda}y(x)=0, \tag{26}$$

where $\bar{\lambda}$ and $\tau(x)$ read

$$\begin{aligned}\bar{\lambda} &= k+\pi'(x),\\ \tau(x) &= \tilde{\tau}(x)+2\pi(x).\end{aligned} \tag{27}$$

The constant parameter k can be defined by utilizing the condition that the expression under the square root has a double zero, i.e., its discriminant is equal to zero. Hence, we obtain the following:

$$k=\frac{1}{4H}\left(P^2+4HQ\right). \tag{28}$$

Now, substituting equation (28) into equation (25) leads us to the following expression for $\pi(x)$:

$$\pi(x)=-\frac{1}{2\sqrt{H}}(Px-2H). \tag{29}$$

According to the NU method, out of the two possible forms of the polynomial $\pi(x)$, we select the one for which the function $\tau(x)$ has the negative derivative. Another form is not suitable for physical reasons. Therefore, the suitable functions for $\pi(x)$ and $\tau(x)$ have the following forms:

$$\begin{aligned}\pi(x) &= -\frac{1}{2\sqrt{H}}(Px-2H),\\ \tau(x) &= 2x-\frac{Px}{\sqrt{H}}+2\sqrt{H},\end{aligned} \tag{30}$$

and their derivatives are as follows:

$$\begin{aligned}\pi'(x) &= -\frac{P}{2\sqrt{H}},\\ \tau'(x) &= 2-\frac{P}{\sqrt{H}}.\end{aligned} \tag{31}$$

Table 1: Mass spectra of charmonium resonances in GeV.

States	Present paper	Experimental results [44]	States	Present paper	Experimental results [44]
$J/\psi(1s)$	3.098	3.097	$1p$	3.256	3.511
$\psi(2s)$	3.687	3.686	$2p$	3.780	3.922
$3s-\psi(4040)$	4.042	4.039	$3p$	4.100	
$4s-\psi(4260)$	4.272	4.259	$4p$	4.310	
$5s-\psi(4415)$	4.429	4.421	$5p$	4.456	
$6s$	4.540		$1d$	3.505	3.774

Table 2: Mass spectra of bottomonium resonances in GeV.

States	Present paper	Experimental results [44]	States	Present paper	Experimental results [44]
$Y(1s)$	9.460	9.460	$1p$	9.619	9.899
$Y(2s)$	10.023	10.023	$2p$	10.114	10.260
$Y(3s)$	10.355	10.355	$3p$	10.411	
$Y(4s)$	10.567	10.579	$4p$	10.604	
$7s-Y(10860)$	10.887	10.885	$5p$	10.736	
$10s-Y(11020)$	11.021	11.020	$1d$	9.863	10.164

We can define the constant $\bar{\lambda}$ from equation (27) which reads as follows:

$$\bar{\lambda} = \frac{P^2}{4H} + Q - \frac{P}{2\sqrt{H}}. \tag{32}$$

Given a nonnegative integer n_r, the hypergeometric-type equation has a unique polynomial solution of degree n if

$$\bar{\lambda} = \bar{\lambda}_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \quad \text{for } n = 0, 1, 2 \cdots. \tag{33}$$

with the condition $\bar{\lambda}_m \neq \bar{\lambda}_n$ for $m = 0, 1, 2, \cdots, n-1$. Furthermore, it follows that

$$\begin{aligned} \bar{\lambda}_{n_r} &= -n_r\left(2 - \frac{P}{\sqrt{H}}\right) - n_r(n_r - 1) \\ &= -2n_r + \frac{P}{\sqrt{H}}n_r - n_r^2 + n_r \\ &= \frac{\sqrt{P}}{\sqrt{H}}n_r - n_r(n_r + 1), \end{aligned} \tag{34}$$

$$\frac{P^2}{4H} + Q - \frac{P}{2\sqrt{H}} = \frac{P}{\sqrt{H}}n_r - n_r(n_r + 1). \tag{35}$$

We can solve equation (35) explicitly for H and get the following:

$$\sqrt{H} = \frac{P}{(1+2n) \pm \sqrt{1-4Q}}. \tag{36}$$

Substituting equation (36) into equation (19) we obtain the following:

$$\sqrt{-\frac{2\mu}{\hbar^2}\left(E - F - \frac{3G}{\delta} + \frac{6M}{\delta^2} + \frac{10N}{\delta^3}\right)} = \frac{P}{(1+2n) \pm \sqrt{1-4Q}}. \tag{37}$$

From this equation, we can get the energy spectrum as follows:

$$E = F + \frac{3G}{\delta} - \frac{6M}{\delta^2} - \frac{10N}{\delta^3} - \frac{\hbar^2}{2\mu}\left[\frac{(2\mu/\hbar^2)\left((3G/\delta^2) + L - (8M/\delta^3) - (15N/\delta^4)\right)}{(1+2n_r) + \sqrt{1 + 4l(l+1) + (8\mu/\hbar^2)(G/\delta^3) - (24\mu/\hbar^2)(M/\delta^4) - (48\mu/\hbar^2)(N/\delta^5) + (8\mu/\hbar^2)C}}\right]^2. \tag{38}$$

We would like to note that the N-dimensional radial Schrödinger equation for the same potential is solved in [32], which should be the same as our result for $N = 3$ at zero temperature. Similar work with the Wentzel-Kramers-Brillouin approximation method has been studied at $T = 0$ temperature in [43]. At $T = 0$, the zero temperature limit of equation (38) reads as follows:

$$E = \frac{3A}{\delta} + \frac{6D}{\delta^2} - \frac{\hbar^2}{2\mu} \cdot \left[\frac{(2\mu/\hbar^2)\left((3A/\delta^2) + B + (8D/\delta^3)\right)}{(1+2n_r) + \sqrt{1 + 4l(l+1) + (8\mu/\hbar^2)(A/\delta^3) + (24\mu/\hbar^2)(D/\delta^4) + (8\mu/\hbar^2)C}}\right]^2. \quad (39)$$

If we take $C = 0$ in equation (39), we then obtain the following:

$$E = \frac{3A}{\delta} + \frac{6D}{\delta^2} - \frac{\hbar^2}{2\mu} \cdot \left[\frac{(2\mu/\hbar^2)\left((3A/\delta^2) + B + (8D/\delta^3)\right)}{(1+2n_r) + \sqrt{1 + 4l(l+1) + (8\mu/\hbar^2)(A/\delta^3) + (24\mu/\hbar^2)(D/\delta^4)}}\right]^2. \quad (40)$$

If we take $D = 0$ in equation (39), we get [22]

$$E = \frac{3A}{\delta} - \frac{\hbar^2}{2\mu} \cdot \left[\frac{(2\mu/\hbar^2)\left((3A/\delta^2) + B\right)}{(1+2n_r) + \sqrt{1 + 4l(l+1) + (8\mu/\hbar^2)(A/\delta^3) + (8\mu/\hbar^2)C}}\right]^2. \quad (41)$$

If we take $C = 0$ and $D = 0$ in equation (39) we obtain the same result as follows [6]:

$$E = \frac{3A}{\delta} - \frac{\hbar^2}{2\mu} \left[\frac{(2\mu/\hbar^2)\left((3A/\delta^2) + B\right)}{(1+2n_r) + \sqrt{1 + 4l(l+1) + (8\mu/\hbar^2)(A/\delta^3)}}\right]^2. \quad (42)$$

We can also find the radial eigenfunctions by applying the NU method. The relevant $\pi(s)$ function must satisfy the following condition:

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)} = \frac{\pm\left(\left(Px/2\sqrt{H}\right) - \sqrt{H}\right)}{x^2} = \pm\left(\frac{P}{2\sqrt{H}x} - \frac{\sqrt{H}}{x^2}\right). \quad (43)$$

It is not a complicated task to find the following result, after substituting $\pi(x)$ and $\sigma(x)$ into equation (43) and solving a first-order differential equation:

$$\phi(x) = x^{(P/2\sqrt{H})} e^{-(\sqrt{H}/x)}. \quad (44)$$

Table 3: Mass spectra of $b\bar{c}$ resonances in GeV.

States	Present paper	Experimental results [44]	States	Present paper
$B_C^+(1s)$	6.277	6.275	$1p$	6.593
$B_C(2s)^\pm$	6.763	6.872	$2p$	6.875
$3s$	6.945		$3p$	6.700
$4s$	7.033		$4p$	7.061
$5s$	7.081		$5p$	7.098
$6s$	7.111		$1d$	6.831

Experimental Results [44].

Table 4: Mass spectra of charmonium in GeV.

State	Present paper	$D = 0$ [22]	$C = 0$ [6]	[45]	[46]	[7]	Exp. [44]
$1s$	3.098	3.097	3.096	3.068	3.078	3.096	3.097
$2s$	3.687	3.687	3.686	3.697	3.581	3.686	3.686
$3s$	4.042	4.041	4.040	4.144	4.085	3.984	4.039
$4s$	4.271	4.271	4.269		4.589	4.150	4.421
$5s$	4.428	4.428	4.425				
$1p$	3.256	3.256	3.255	3.526	3.415	3.433	3.511
$2p$	3.780	3.780	3.779	3.993	3.917	3.910	3.922
$3p$	4.100	4.100					
$4p$	4.310	4.310					
$5p$	4.456	4.456					
$1d$	3.505	3.505	3.504	3.829	3.749	3.767	3.774

Table 5: Mass spectra of bottomonium in GeV.

State	Present paper	$D = 0$ [22]	$C = 0$ [6]	[45]	[46]	[7]	Exp. [44]
$1s$	9.460	9.459	9.460	9.447	9.510	9.460	9.460
$2s$	10.023	10.022	10.023	10.012	10.038	10.023	10.023
$3s$	10.355	10.354	10.355	10.353	10.566	10.280	10.355
$4s$	10.567	10.566	10.567	10.629	11.094	10.420	10.579
$5s$	10.710	10.710				6578	
$1p$	9.619	9.618	9.619	9.900	9.862	9.840	9.899
$2p$	10.114	10.113	10.114	10.260	10.390	10.160	10.260
$3p$	10.411	10.411					
$4p$	10.604	10.604					
$5p$	10.736	10.736					
$1d$	9.863	10.257	9.864	10.155	10.214	10.140	10.164

Furthermore, the other part of the wave function $y_n(x)$ is the hypergeometric-type function whose polynomial solutions are given by the Rodrigues relation:

$$y_n(x) = \frac{C_n}{\rho(x)} \frac{d^n}{dx^n} \left[\sigma^n(x)\rho(x)\right], \quad (45)$$

where C_n is a normalizing constant and $\rho(x)$ is the weight function which is the solution of the Pearson differential equation. The Pearson differential equation and $\rho(x)$ for our problem is given as follows:

$$(\sigma\rho)' = \tau\rho. \tag{46}$$

Therefore, we use equation ((46)) to find the second part of the wave function from the definition of weight function:

$$\rho(x) = x^{(-P/\sqrt{H})} e^{(-2\sqrt{H}/x)}. \tag{47}$$

Considering both parts of the wave function $\phi(x)$ and $y_{n_r}(x)$ within equation (23), we obtain the following:

$$\chi_{n_r}(x) = C_{n_r l} \cdot x^{(P/2\sqrt{H})} \cdot e^{(\sqrt{H}/x)} \cdot \frac{d^n}{dx^n}\left[x^{2n-(P/\sqrt{H})} e^{(-2\sqrt{H}/x)}\right]. \tag{48}$$

As the last step, we do the replacement $x = 1/r$, and using $\chi(r) = rR(r)$ in equation (48) we get the following:

$$\chi_{n_r}(r) = C_{n_r l} \cdot r^{-(P/2\sqrt{H})} e^{\sqrt{H}\cdot r} \left(-r^2 \frac{d}{dr}\right)^n \left[r^{-2n+(P/\sqrt{H})} e^{-2\sqrt{H}\cdot r}\right]. \tag{49}$$

The final form of the radial wave function $R(r)$ reads:

$$R(r) = C_{n_r l} \cdot r^{-1-(P/2\sqrt{H})} e^{\sqrt{H}\cdot r} \left(-r^2 \frac{d}{dr}\right)^n \left[r^{-2n+(P/\sqrt{H})} e^{-2\sqrt{H}\cdot r}\right]. \tag{50}$$

4. Mass Spectrum of the Heavy Quarkonium

We calculate the mass spectra of the heavy quarkonium system, for example, charmonium and bottomonium mesons that are the bound state of quarks and antiquarks. For this we apply the following relation:

$$M = m_q + m_{\bar{q}} + E, \tag{51}$$

where m is the bare mass of a heavy quark. Using expression (39) for the energy spectrum in (51) we get the following equation for heavy quarkonia mass at finite temperature:

$$M = m_q + m_{\bar{q}} + F + \frac{3G}{\delta} - \frac{6M}{\delta^2} - \frac{10N}{\delta^3} - \frac{\hbar^2}{2\mu}\left[\frac{(2\mu/\hbar^2)\left((3G/\delta^2) + L - (8M/\delta^3) - (15N/\delta^4)\right)}{(1+2n) + \sqrt{1 + 4l(l+1) + (8\mu/\hbar^2)(G/\delta^3) - (24\mu/\hbar^2)(M/\delta^4) - (48\mu/\hbar^2)(N/\delta^5) + (8\mu/\hbar^2)C}}\right]^2. \tag{52}$$

Depending on the system which we want to study, we may consider that m_q and $m_{\bar{q}}$ are the bare masses of quarks and antiquarks correspondingly and E is the energy of the system.

By replacing $T = 0$, we obtain the meson mass at zero temperature:

$$M = m_q + m_{\bar{q}} + \frac{3A}{\delta} + \frac{6D}{\delta^2} - \frac{\hbar^2}{2\mu}\left[\frac{(2\mu/\hbar^2)\left((3A/\delta^2) + B + (8D/\delta^3)\right)}{(1+2n) + \sqrt{1 + 4l(l+1) + (8\mu/\hbar^2)(A/\delta^3) + (24\mu/\hbar^2)(D/\delta^4) + (8\mu/\hbar^2)C}}\right]^2. \tag{53}$$

Numerical values for the charmonium mass spectra are presented in Table 1

In this table, we considered the bare mass of the charm quark as $m_c = 1.209$ GeV, and the constant terms fitted with the experimental data via equation (53) as $A = 0.2\,\text{GeV}^2$, $B = 1.244$, $C = 2.9 \times 10^{-3}$, $D = 1.4 \times 10^{-5}$, and $\delta = 0.231$ GeV. If we apply formula (53) to the bottomonium case with the bare mass $m_b = 4.823$ GeV, and the experiment fitted constants use equation (53) as $A = 0.2\,\text{GeV}^2$, $B = 1.569$, $C = 2.0 \times 10^{-3}$, $D = 1.4 \times 10^{-5}$, and $\delta = 0.378$ GeV, we obtain the following results shown in Table 2.

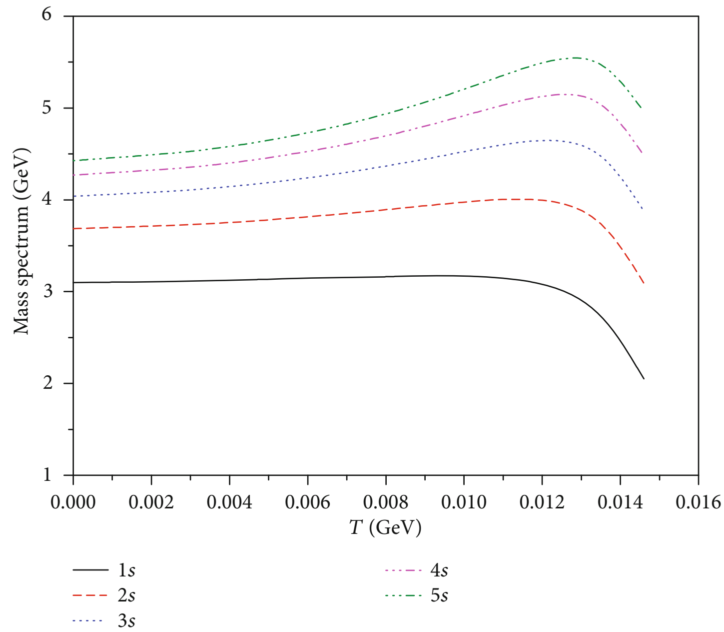

Figure 1: The mass spectrum of charmonium in the 1s, 2s, 3s, 4s, and 5s states as a function of the temperature T with a mass of $m_c = 1.209$ GeV and parameters of $A = 0.2\,\text{GeV}^2$, $B = 1.244$, and $\delta = 0.231$ GeV.

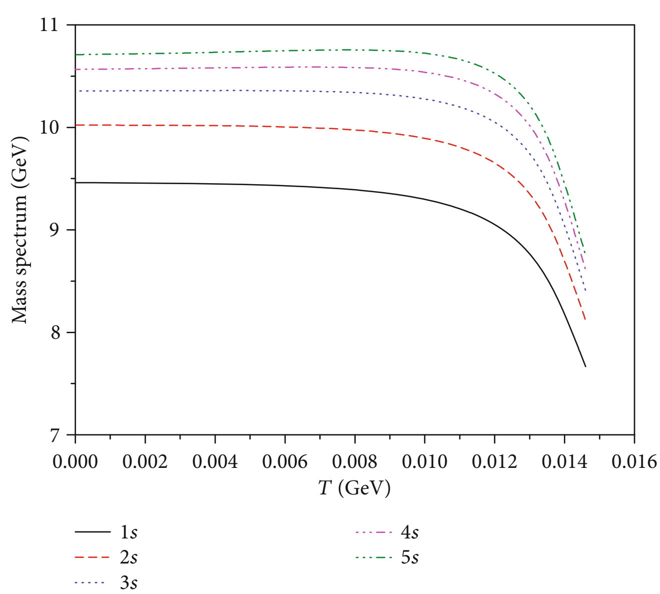

Figure 2: The mass spectrum of bottomonium in the 1s, 2s, 3s, 4s, and 5s states as a function of the temperature T with a mass of $m_b = 4.823$ GeV and parameters of $A = 0.2\,\text{GeV}^2$, $B = 1.569$, and $\delta = 0.378$ GeV.

Considering the $T = 0$ and $D = 0$ limit in equation (52) we get the following expression which is obtained in [22]

$$M = m_q + m_{\bar{q}} + \frac{3A}{\delta} - \frac{\hbar^2}{2\mu} \cdot \left[\frac{(2\mu/\hbar^2)\left((3A/\delta^2) + B\right)}{(1+2n) + \sqrt{1 + 4l(l+1) + (8\mu/\hbar^2)(A/\delta^3) + (8\mu/\hbar^2)C}}\right]^2. \quad (54)$$

The $T = 0$, $C = 0$, and $D = 0$ limits of equation (52) lead to the following formula which fully coincides with the quarkonium mass formula at zero temperature in [6]

$$M = m_q + m_{\bar{q}} + \frac{3A}{\delta} - \frac{\hbar^2}{2\mu} \cdot \left[\frac{(2\mu/\hbar^2)(3A/\delta^2)}{(1+2n) + \sqrt{1 + 4l(l+1) + (8\mu/\hbar^2)(A/\delta^3)}}\right]^2. \quad (55)$$

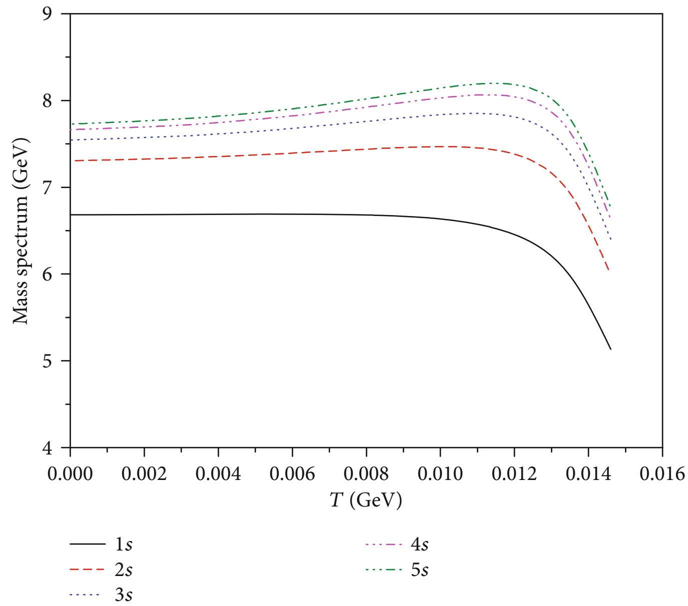

FIGURE 3: The mass spectrum of $b\bar{c}$ for $1s$, $2s$, $3s$, $4s$, and $5s$ states as a function of the temperature T with masses of $m_c = 1.209$ GeV and $m_b = 4.823$ GeV and parameters of $A = 0.2\ \text{GeV}^2$, $B = 1.407$, and $\delta = 0.324$ GeV.

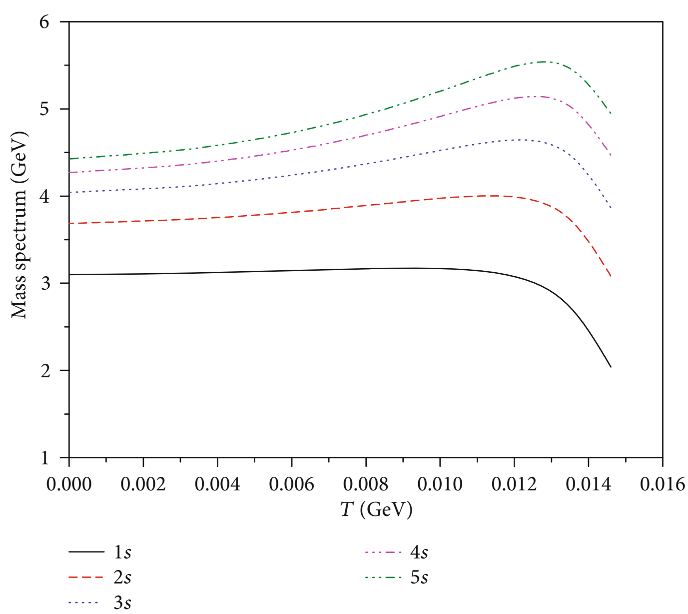

FIGURE 4: The mass spectrum of charmonium for $1s$, $2s$, $3s$, $4s$, and $5s$ states as a function of the temperature T with a mass of $m_c = 1.209$ GeV and parameters of $A = 0.147\ \text{GeV}^2$, $B = 1.204$, $C = 2.8 \times 10^{-3}$, $D = 2.4 \times 10^{-5}$, and $\delta = 0.379$ GeV.

We may conclude that our current results as presented in Table 1 and Table 2 are in good agreement with current available experimental data for all states of charmonium and bottomonium resonances. The main reason why the results for the p and d states are not in good agreement with the experimental data comes from the nonrelativistic calculation which we use throughout our calculations. One needs to consider the spin-spin and spin-orbital interactions terms within the potential. Thus, the reason is not related to the correct choice of the parameters or making a better fit. It is impossible to consider the spin terms within the Schrödinger equation because of its nonrelativistic nature. These terms should be considered within the relativistic equations such as within the Klein-Fock-Gordon and Dirac equations. We are planning to study it in the future since it is out of the scope of the current paper.

In Table 3, we present the mass spectrum results for the B_c mesons with masses $m_c = 1.209$ GeV and $m_b = 4.823$ GeV, and parameters $A = 0.147\ \text{GeV}^2$, $B = 1.204$, $C = 2.8 \times 10^{-3}$, $D = 2.4 \times 10^{-5}$, and $\delta = 0.379$ GeV.

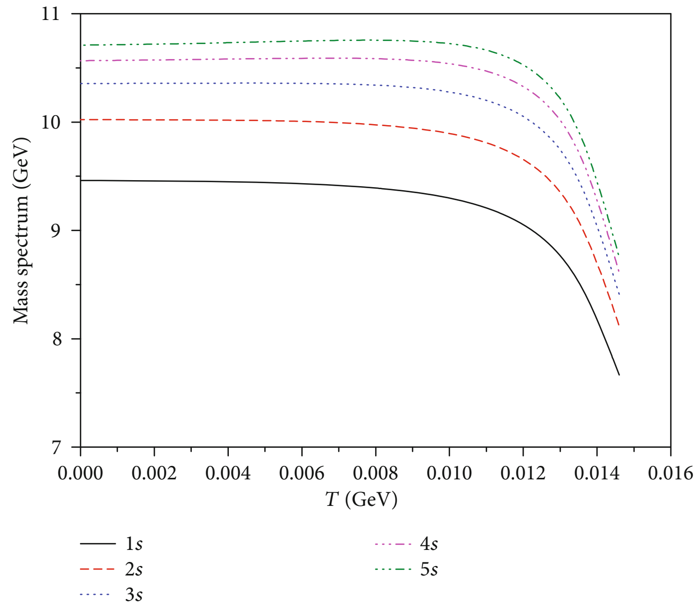

FIGURE 5: The mass spectrum of bc for $1s$, $2s$, $3s$, $4s$, and $5s$ states as a function of the temperature T with a mass of $m_b = 4.823$ GeV and parameters of $A = 0.147\ \mathrm{GeV}^2$, $B = 1.204$, $C = 2.8 \times 10^{-3}$, $D = 2.4 \times 10^{-5}$, and $\delta = 0.379$ GeV.

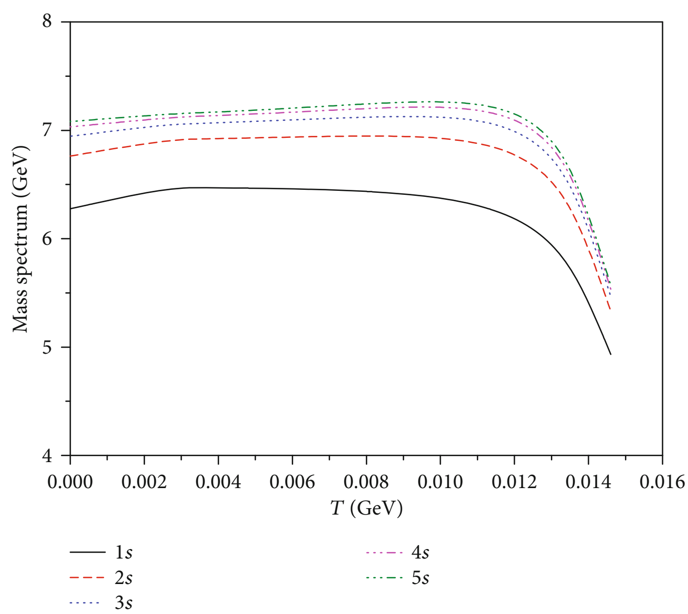

FIGURE 6: The mass spectrum of $b\bar{c}$ for $1s$, $2s$, $3s$, $4s$, and $5s$ states as a function of the temperature T with masses of $m_c = 1.209$ GeV and $m_b = 4.823$ GeV and parameters of $A = 0.147\ \mathrm{GeV}^2$, $B = 1.204$, $C = 2.8 \times 10^{-3}$, $D = 2.4 \times 10^{-5}$, and $\delta = 0.379$ GeV.

In order to compare our results with the different theoretical works, we present Tables 4 and 5 for the mass spectra of charmonium and bottomonium accordingly.

5. Temperature Dependence of Quarkonia Mass

In this section, we present the numerical results for the temperature dependence for some heavy meson mass spectra.

In Section 4, we have seen that the present potential model for describing the heavy quarkonia and B_c meson mass at zero temperature is quite a good candidate. For studying the temperature dependence, we will follow [15].

For calculating the mass spectra at finite temperature, we use the explicit form of the Debye screening mass $\mu_D(T)$ according to [47]:

$$\mu_D(T) = \gamma \alpha_s(T) T, \tag{56}$$

where $\gamma = 14.652 \pm 0.337$. In the numerical calculations for the running coupling constant $\alpha_s(T)$, we will adopt the following form at finite temperature:

$$\alpha_s(T) = \frac{2\pi}{\left(11 - (2/3)N_f\right) \log\left(T/\Lambda\right)}. \quad (57)$$

Here, we take the critical temperature $T_c = (169 \pm 16)$ MeV from lattice QCD [48] which leads to $\Lambda = \beta T_c = (17.6 \pm 3.2)$ MeV.In the numerical calculations, we will apply as $N_f = 3$ with two light quarks of the same mass u and d and one heavier s. Firstly, we present the graphs for the change of the meson masses within the temperature-dependent Cornell potential in Figures 1–3. Afterwards, we present the same calculation for the Cornell plus the inverse quadratic and harmonic potentials in Figures 4–6. In all these graphs, we see that there exists a substantial decrement in the meson masses around $T = 120$ MeV which corresponds to $T = 0.71T_c$. When we increase the principal quantum number n, the temperature leads an increment in masses of the charmonium and B_c mesons up to some point and then there is a sharp decrement. However, this phenomena is a bit different for the case of bottomonium, such that these states firstly do not change and then start decreasing after some point.

Notice that although the number of the parameters for the Cornell potential is less than the current potential, the results are not so different. Thus, we may conclude that our observation for the temperature-dependent masses does not depend much on the number of parameters. Besides that, our results for the temperature-dependent masses are in agreement with QCD sum rule results for the ground states [49]. One needs to perform more analyses for other resonances in order to check the validity of this agreement between pure nonrelativistic effects and QFT.

These results may open new possibilities for determining the properties of the interactions in a hadronic system. As a conclusion of the results presented in these tables and figures which are based on analytical results, we may draw the following key points: Firstly, temperature-dependent masses either for the Cornell or the Cornell plus inverse quadratic and harmonic-type potentials are very sensitive to the choice of n_r radial and l orbital quantum numbers. Secondly, we may consider the temperature-dependent results valid because of the similar shapes between the Cornell and current potentials. These results are sufficiently accurate for practical purposes.

6. Conclusion

The temperature-dependent Schrödinger equation is investigated by applying the NU method. As a potential part of the Schrödinger equation, the Cornell plus inverse quadratic and harmonic oscillator-type potential is used. Analytical expression for the energy eigenvalues and the radial wave functions is presented. Results are used for describing nonzero and zero temperature mass spectra of heavy quarkonia and B_c. Numerical results are compared with the experimentally well-established resonances, and some predictions are presented for the states which have not been confirmed yet. For instance, we have the predictions for bottomonium, namely, besides the $Y(1s)$, $Y(2s)$, $Y(3s)$, and $Y(4s)$ resonances, we predict the $Y(10860)$ and $Y(11020)$ resonances to be $7s$ and $10s$ states, respectively. For the charmonium states, besides the $J/\psi(1s)$ and $\psi(2s)$ resonances, we predict $3s \longrightarrow \psi(4040)$, $4s \longrightarrow \psi(4260)$, and $5s \longrightarrow \psi(4415)$. We have seen that a zero temperature mass spectrum for $l = 0$ is quite in agreement with current experimental data, while for $l \neq 0$, there is some disagreement with experimental data because of the missing spin-spin and spin-orbital momentum interaction terms within the potential. For the temperature-dependent case, we see the strong dependence of the quarkonia mass spectrum on the quantum numbers. Temperature-dependent results for ground states are in good agreement with the quantum field theoretical approach such as the QCD sum rule results. We have seen that the extension of the Cornell potential leads to slight changes on the masses for nonzero temperature as well as zero temperature.

It seems that this simple potential model, like other nonrelativistic models, is not enough to describe all the features of hadrons within a thermal effect such that having all the hadrons melted at the same temperature contradicts with the lattice data. The most convenient way to compare the prediction of potential models with a direct calculation of quarkonium spectral functions is to calculate the Euclidean meson correlator at finite temperature to compare the lattice data as it was done for the Cornell potential in [50, 51]. It has been shown that even though potential models with certain screened potentials can reproduce qualitative features of the lattice spectral function, such as the survival of the $1s$ state and the melting of the $1p$ state, the temperature dependence of the meson correlators is not reproduced. According to the lattice results [52, 53] the $1s$ charmonium survives up to $1.5T_c$ and the $1p$ charmonium dissolves by $1.16T_c$ and higher excited states disappear near the transition temperature. It is possible that the effects of the medium on quarkonia binding cannot be understood in a simple potential model. However, one can still do this comparison for other potential models such as the one we have used in this paper. One of the main steps for correlator calculation is about the exact solution of the radial Schrödinger equation which is already done in this paper.

The method used in this paper are the systematic ones, and in many cases, it is one of the most concrete works in this area. In particular, the extended Cornell potential can be one of the important potentials, and it deserves special concern in many branches of physics, especially in hadronic, nuclear, and atomic physics.

Consequently, studying for an analytical solution of the modified radial Schrödinger equation for the sum of the Cornell, inverse quadratic-, and harmonic-type potentials within the framework of ordinary quantum mechanics could provide valuable information on elementary particle physics and quantum chromodynamics and open new windows for further investigation.

We can conclude that the theoretical results of this study are expected to enable new possibilities for pure theoretical and experimental physicists, because of the exact and more general nature of the results.

Data Availability

The information given in our tables is available for readers in the original references listed in our work.

Disclosure

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Conflicts of Interest

The authors declare that they have no conflicts of interest.

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