*Research Article*

# An Approach of Statistical Corrections to Interactions in Hadron Resonance Gas

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We propose a new model for hadrons with quantum mechanical attractive and repulsive interactions sensitive to some spatial correlation length parameter inspired by the Beth-Uhlenbeck quantum mechanical nonideal gas model (Uhlenbeck and Beth, 1937). We confront the thermodynamics calculated using our model with a corresponding recent lattice data at four different values of the baryon chemical potential, $\mu_b = 0, 170, 340, 425$ MeV over temperatures ranging from 130 MeV to 200 MeV and for five values for the correlation length ranging from 0 to 0.2 fm. For equilibrium temperatures up to the vicinity of the chiral phase transition temperature $\simeq 160$ MeV, a decent fitting between the model and the lattice data is observed for different values of $r$, especially at $(\mu_b, r) = (170, 0.05), (340, 0.1)$, and $(340, 0.15)$, where $\mu_b$ is in MeV and $r$ is in fm. For the vanishing chemical potential, the uncorrelated model ($r = 0$), which corresponds to the ideal hadron resonance gas model, seems to offer the best fit. The quantum hadron correlations seem to be more probable at nonvanishing chemical potentials, especially within the range $\mu_b \in [170, 340\, \mathrm{MeV}]$.

## 1. Introduction

One of the key goals of the ultrarelativistic nuclear collisions is to gain information on the hadron-parton phase diagram, which is characterized by different phases and different types of the phase transitions [1]. Quantum Chromodynamics (QCD), the gauge field theory that describes the strong interactions of colored quarks and gluons and their colorless bound states, has two important intensive state parameters at equilibrium, namely, temperature $T$ and baryon chemical potential $\mu_b$. A remarkable world-wide theoretical and experimental effort has been dedicated to the study of strongly interacting matter under extreme condition of temperature and baryon chemical potential. The lattice QCD simulations provide an *a priori* nonperturbative regularization of QCD that makes it compliant with analytic and computational methods with no model assumptions other than QCD itself being needed to formulate the theory. The temperature and density (chemical potential) dependence of the bulk thermodynamic quantities, commonly summarized as the equation

of state (EoS), provides the most basic characterization of equilibrium properties of the strongly interacting matter. Its analysis within the framework of lattice QCD has been refined ever since the early calculations performed in pure $SU(N)$ gauge theories [2]. The EoS at vanishing chemical potentials does already provide important input into the modeling of the hydrodynamic evolution of hot and dense matter created in heavy-ion collisions [3, 4]. While this is appropriate for the thermal conditions met in these collisions at the LHC and the highest RHIC beam energies, knowledge of the EoS at nonvanishing baryon, strangeness, and electric charge chemical potentials is indispensable for the hydrodynamic models of the conditions met in the beam energy scan (BES) at RHIC [5] and in future experiments at facilities like FAIR at GSI and NICA at JINR [6, 7].

Bulk thermodynamic observables such as pressure, energy density, and entropy density as well as the second-order quantities such as the specific heat and velocity of sound have now been obtained at vanishing chemical potentials for the three lightest quark flavors [8]. By the analysis of the chiral transition

temperature, $T_c \simeq 154 \pm 9$ MeV [9], it has also been shown that the bulk thermodynamic observables change smoothly in the transition region probed. Due to the well-known sign problem encountered in the lattice QCD formulations at finite chemical potential, a direct calculation of the EoS at nonzero chemical potential is unfortunately not fully reliable. Lots of effort have been made to circumvent the divergences at the nonzero chemical potential, such as the Taylor expansion of the thermodynamic potential on course lattices besides other sophisticated computational techniques [10–15] which made it possible to conduct calculations covering the range $0 \leq \mu_b/T \leq 3$ that is expected to be explored with the BES at RHIC by varying the beam energies in the range $7.7 \leq \sqrt{s_{NN}} \leq 200$ GeV [16]. A promising approach in this quest is the investigation of hadron production. The hadron resonance gas (HRG) is customarily used in the lattice QCD calculations as a reference for the hadronic sector [17, 18]. At low temperatures, they are found to be in quite good agreement with the HRG model calculations [19], although some systematic deviations have been observed, which may be attributed to the existence of additional resonances which are not taken into account in HRG model calculations based on well-established resonances listed by the particle data group [20] and perhaps the need to extend the model to incorporate interactions.

In the HRG model, the thermodynamics of a strongly interacting system is conjectured to be approximated as an ideal gas composed of hadron resonances with masses $\lesssim 2$ GeV [3, 19] that are treated as a free gas, exclusively in the hadronic phase, i.e., below $T_c$. Therefore, the hadronic phase in the confined phase of QCD could be modeled as a noninteracting gas of the hadron resonances. It is reported in recent literature that the standard performance of the HRG model seems to be unable to describe all the available data that is predicted by recent lattice QCD simulations [21, 22]. The conjecture to incorporate various types of interactions has been worked out in various studies [4, 23–25]. When comparing the thermodynamics calculated within the HRG framework with the corresponding data obtained using lattice QCD methods, one has to decide how to incorporate interactions among t/he hadrons.

Arguments based on the S-matrix approach [26–28] suggest that the HRG model includes attractive interactions between hadrons which lead to the formation of resonances. More realistic hadronic models take into account the contribution of both attractive and repulsive interactions between the component hadrons. Repulsive interactions in the HRG model had previously been considered in the framework of the relativistic cluster and virial expansions [27], via repulsive mean fields [29, 30], and via excluded volume (EV) corrections [31–36]. In particular, the effects of EV interactions between hadrons on HRG thermodynamics [37–44] and on observables in heavy-ion collisions [45–52] have extensively been studied in the literature. Recently, repulsive interactions have received renewed interest in the context of lattice QCD data on fluctuations of conserved charges. It was shown that large deviations of several fluctuation observables from the ideal HRG baseline could well be interpreted in terms of repulsive baryon-baryon interactions [23, 52, 53].

The present script is organized as follows: In Section 2, we review the detailed formalism of the conventional ideal (uncorrelated) HRG model, then we develop a nonideal (correlated) statistical correction to the ideal HRG model inspired by the Beth-Uhlenbeck (BU) quantum theory of nonideal gases. The calculations of the HRG thermodynamics based on the proposed correction are discussed in Section 3. Section 4 is devoted to the conclusions and outlook.

## 2. Model Description

In this study, we use the particle interaction probability term originally implemented in the expression for the second virial coefficient worked out in Ref. [54] in order to suggest a statistical correction to the uncorrelated HRG model. Uhlenbeck and Beth suggested a connection between the virial coefficients and the probabilities of finding pairs, triples, and so on, of particles near each other [55]. In the classical limit, which is usually designated by sufficiently high temperatures and/or low particle densities, it was shown that these probabilities (explicit expressions are to follow in the next section) can be expressed by Boltzmann factors so long as the de Broglie wavelength, which is a common measure of the significance of the quantum nonlocality, is small enough compared with the particle spacial extent measured by the particle "diameter" [55] as it was dubbed by Uhlenbeck and Beth themselves. Such a particle diameter can be considered as a measure of the spatial extent within which a particle can undergo hardcore (classical) interactions. Based on a comparison of their model with experimental results on helium molecules, the authors of [55] concluded that at sufficiently low temperatures for which the thermal de Broglie wavelength is comparable with the particle diameter, deviations from the classical excluded volume model due to quantum effects will be significant [55].

An extension has been made by the same authors to the quantum mechanical model of the particle interactions proposed in Ref. [55]. This extension considered the influence of Bose or Fermi statistics in addition to the effect of the inclusion of discrete quantum states for a general interaction potential that is not necessarily central [54]. The expression for the second virial expansion developed in Ref. [55] and extended in Ref. [54] was later generalized using the cluster integral to describe *general* particle interactions provided that such particles do not form bound states [26, 27, 56].

The extended BU quantum mechanical approach [54] was used quite recently to model the repulsive interactions between baryons in a hadron gas [25]. The second virial coefficient or the excluded volume parameter was calculated in [54] within the extended BU approach [54] and found to be temperature dependent and found also to differ dramatically from the classical excluded volume (EV) model result. Moreover, it was shown in [54] that at temperatures $T = 100 - 200$ MeV, the widely used classical EV model [57–59] underestimates the EV parameter for nucleons at a given value of the nucleon hard-core radius (assumed $\simeq 0.3$ fm) by large factors of 3-4. It was thus concluded in [25] that previous studies,
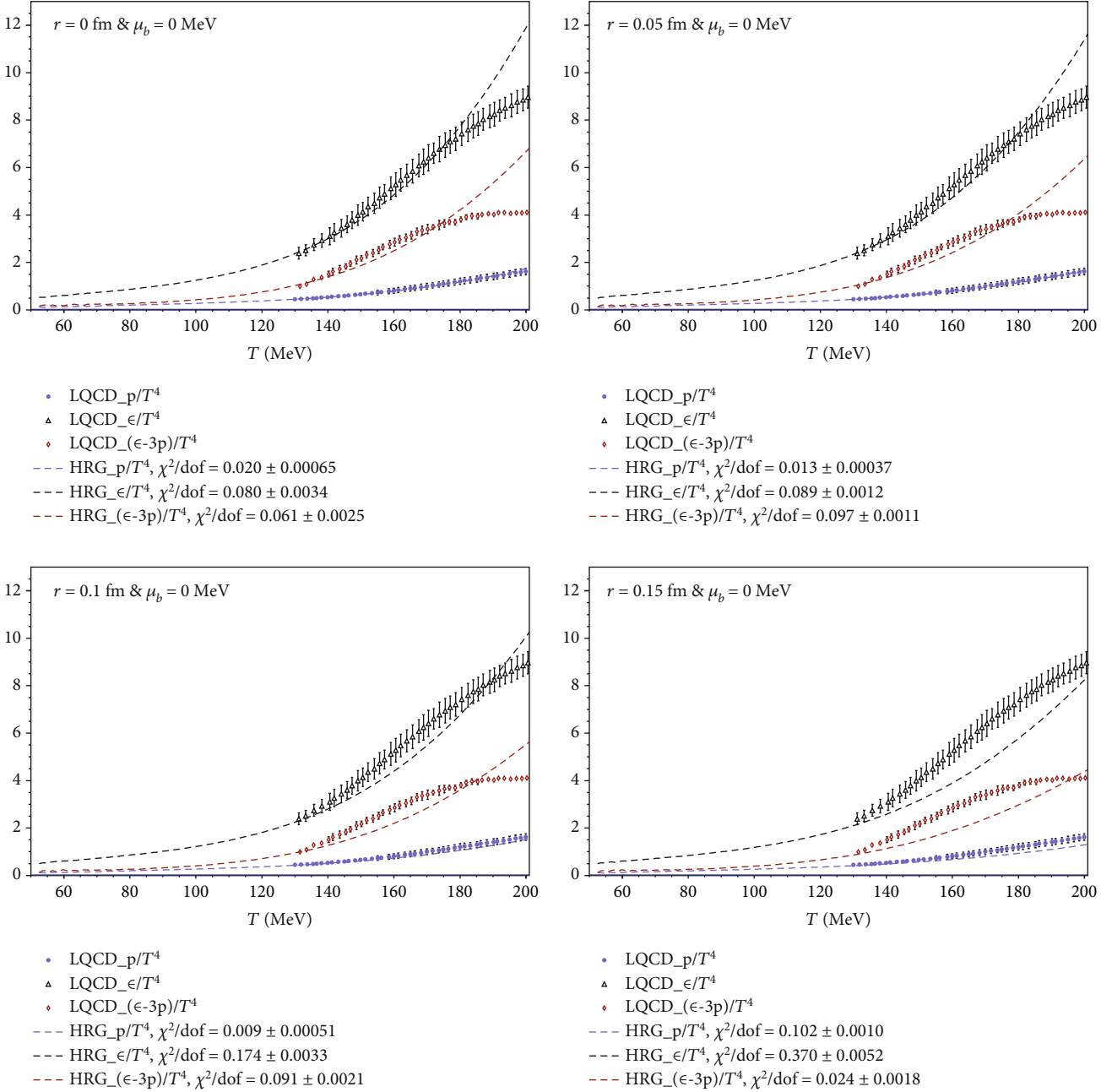
FIGURE 1: Normalized pressure $P/T^4$, normalized energy density $\rho/T^4$, and trace anomaly $(\rho - 3P)/T^4$ (dashed curves) calculated using our statistically corrected HRG model and confronted to the corresponding lattice data taken from Ref. [66] (symbols with error bars), at $\mu_b = 0$ MeV. Comparison is made for four different values of the correlation length $r$.

which employed the hardcore radii of hadrons as an input into the classical EV model, have to be reevaluated using the appropriately rescaled quantum mechanical EV parameters.

In this section, we first introduce the basic formulation of the ideal HRG model. Then, we develop a statistical model inspired by the Beth-Uhlenbeck (BU) quantum theory of nonideal (correlated) gases [54] as a correction to the ideal (uncorrelated) HRG model. We thereby implement in our calculations a modified version of the partition function of a typical ideal hadron gas. In the framework of a bootstrap picture [60, 61], an equilibrium thermal model for an interac-

tion free gas has a partition function $Z(T, \mu, V)$ from which the thermodynamics of such a system can be deduced by taking the proper derivatives.

*2.1. Noncorrelated Ideal HRG.* In a grand canonical ensemble, the partition function reads [3, 4, 19, 62–64].

$$Z(T, V, \mu) = Tr\left[\exp\left(\frac{\mu N - H}{T}\right)\right], \qquad (1)$$

where $H$ is the Hamiltonian combining all relevant degrees of freedom and $N$ is the number of constituents of the statistical

TABLE 1: $\chi^2$/dof statistic for the normalized pressure $P/T^4$, trace anomaly $(\rho - 3P)/T^4$, and normalized energy density $\rho/T^4$ calculated in our statistically corrected hadron resonance gas (HRG) model confronted to the corresponding lattice data taken from Ref. [66, 67], at four values of baryon chemical potential $\mu_b = 0, 170, 340$, and $425$ MeV.

| $\mu_b$ (MeV) | $r$ (fm) | $\chi^2$/dof for $P/T^4$ | $\chi^2$/dof for $(\rho - 3P)/T^4$ | $\chi^2$/dof for $\rho/T^4$ |
|---|---|---|---|---|
| 0 | 0 | $0.020 \pm 0.00065$ | $0.061 \pm 0.0025$ | $0.080 \pm 0.0034$ |
| | 0.05 | $0.013 \pm 0.00037$ | $0.097 \pm 0.0011$ | $0.089 \pm 0.0012$ |
| | 0.1 | $0.009 \pm 0.00051$ | $0.091 \pm 0.0021$ | $0.174 \pm 0.0033$ |
| | 0.15 | $0.102 \pm 0.0010$ | $0.024 \pm 0.0018$ | $0.370 \pm 0.0052$ |
| 170 | 0 | $0.002 \pm 0.0002$ | $0.3698 \pm 0.0050$ | $0.036 \pm 0.0010$ |
| | 0.05 | $0.008 \pm 0.00041$ | $0.3190 \pm 0.0045$ | $0.062 \pm 0.002$ |
| | 0.1 | $0.010 \pm 0.00062$ | $0.1452 \pm 0.0011$ | $0.075 \pm 0.0035$ |
| | 0.15 | $0.028 \pm 0.0009$ | $0.069 \pm 0.0017$ | $0.145 \pm 0.0051$ |
| 340 | 0 | $0.020 \pm 0.0014$ | $0.853 \pm 0.0028$ | $0.241 \pm 0.0040$ |
| | 0.05 | $0.010 \pm 0.0016$ | $2.46 \pm 0.0062$ | $1.206 \pm 0.011$ |
| | 0.1 | $0.005 \pm 0.0010$ | $1.31 \pm 0.0075$ | $0.399 \pm 0.0093$ |
| | 0.15 | $0.015 \pm 0.0007$ | $0.335 \pm 0.0029$ | $0.101 \pm 0.005$ |
| 425 | 0 | $0.065 \pm 0.0012$ | $2.12 \pm 0.15$ | $4.28 \pm 0.0007$ |
| | 0.05 | $0.351 \pm 0.0030$ | $6.0 \pm 0.0019$ | $3.3 \pm 0.024$ |
| | 0.1 | $0.035 \pm 0.0091$ | $2.77 \pm 0.0012$ | $1.68 \pm 0.017$ |
| | 0.15 | $0.058 \pm 0.0015$ | $1.44 \pm 0.0038$ | $0.376 \pm 0.0082$ |

ensemble. Equation (1) can be expressed as a sum over all hadron resonances taken from the recent particle data group (PDG) [20] with masses up to 2.5 GeV,

$$\ln Z(T, V, \mu) = \sum_i \ln Z_i(T, V, \mu)$$
$$= V \sum_i \frac{g_i}{2\pi^2} \int_0^\infty \pm p^2 dp \ln \left[ 1 \pm \lambda_i \exp \left( \frac{-\varepsilon_i(p)}{T} \right) \right],$$
(2)

where the pressure can be derived as $T\partial \ln Z(T, V, \mu)/\partial V$ and $\pm$ stands for fermions and bosons, respectively. $\varepsilon_i = (p^2 + m_i^2)^{1/2}$ is the dispersion relation and $\lambda_i$ is the fugacity factor of the $i$-th particle [19],

$$\lambda_i(T, \mu) = \exp \left( \frac{B_i \mu_b + S_i \mu_S + Q_i \mu_Q}{T} \right),$$
(3)

where $B_i(\mu_b)$, $S_i(\mu_S)$, and $Q_i(\mu_Q)$ are baryon, strangeness, and electric charge quantum numbers (their corresponding chemical potentials) of the $i$-th hadron, respectively. From the phenomenological point of view, the baryon chemical potential $\mu_b$—along the chemical freezeout boundary, where the production of particles is conjectured to cease—can be related to the nucleon-nucleon center-of-mass energy $\sqrt{s_{NN}}$ [65].

$$\mu_b = \frac{a}{1 + b\sqrt{s_{NN}}},$$
(4)

where $a = 1.245 \pm 0.049$ GeV and $b = 0.244 \pm 0.028$ GeV$^{-1}$. In addition to pressure, the number and energy density, respectively, and likewise the entropy density and other thermodynamics can straightforwardly be derived from the partition function by taking the proper derivatives

$$n_i(T, \mu) = \sum_i \frac{g_i}{2\pi^2} \int_0^\infty p^2 dp \frac{1}{\exp\left[ (\mu_i - \varepsilon_i(p))/T \right] \pm 1},$$
(5)

$$\rho_i(T, \mu) = \sum_i \frac{g_i}{2\pi^2} \int_0^\infty p^2 dp \frac{-\varepsilon_{i(p)} \pm \mu_i}{\exp\left[ (\mu_i - \varepsilon_i(p))/T \right] \pm 1}.$$
(6)

It should be noticed that both $T$ and $\mu = B_i \mu_b + S_i \mu_S + \cdots$ are related to each other and to $\sqrt{s_{NN}}$ [19]. As an overall thermal equilibrium is assumed, $\mu_S$ is taken as a dependent variable to be estimated due to the strangeness conservation, i.e., at given $T$ and $\mu_b$, the value assigned to $\mu_S$ is the one assuring $\langle n_S \rangle - \langle n_{\bar{S}} \rangle = 0$. Only then, $\mu_S$ is combined with $T$ and $\mu_b$ in determining the thermodynamics, such as the particle number, energy, and entropy. The chemical potentials related to other quantum charges, such as the electric charge and the third-component isospin, can also be determined as functions of $T$, $\mu_b$, and $\mu_S$ and each of them must fulfill the corresponding conservation laws.

2.2. Quantum-Statistically Correlated HRG. In the beginning of Section 2, we qualitatively motivated a statistical correction scenario to the HRG model inspired by the generalized BU quantum mechanical particle interaction model. In the following, we are going to draw a quantitative picture of such a correction and how it can be smoothly blended with the
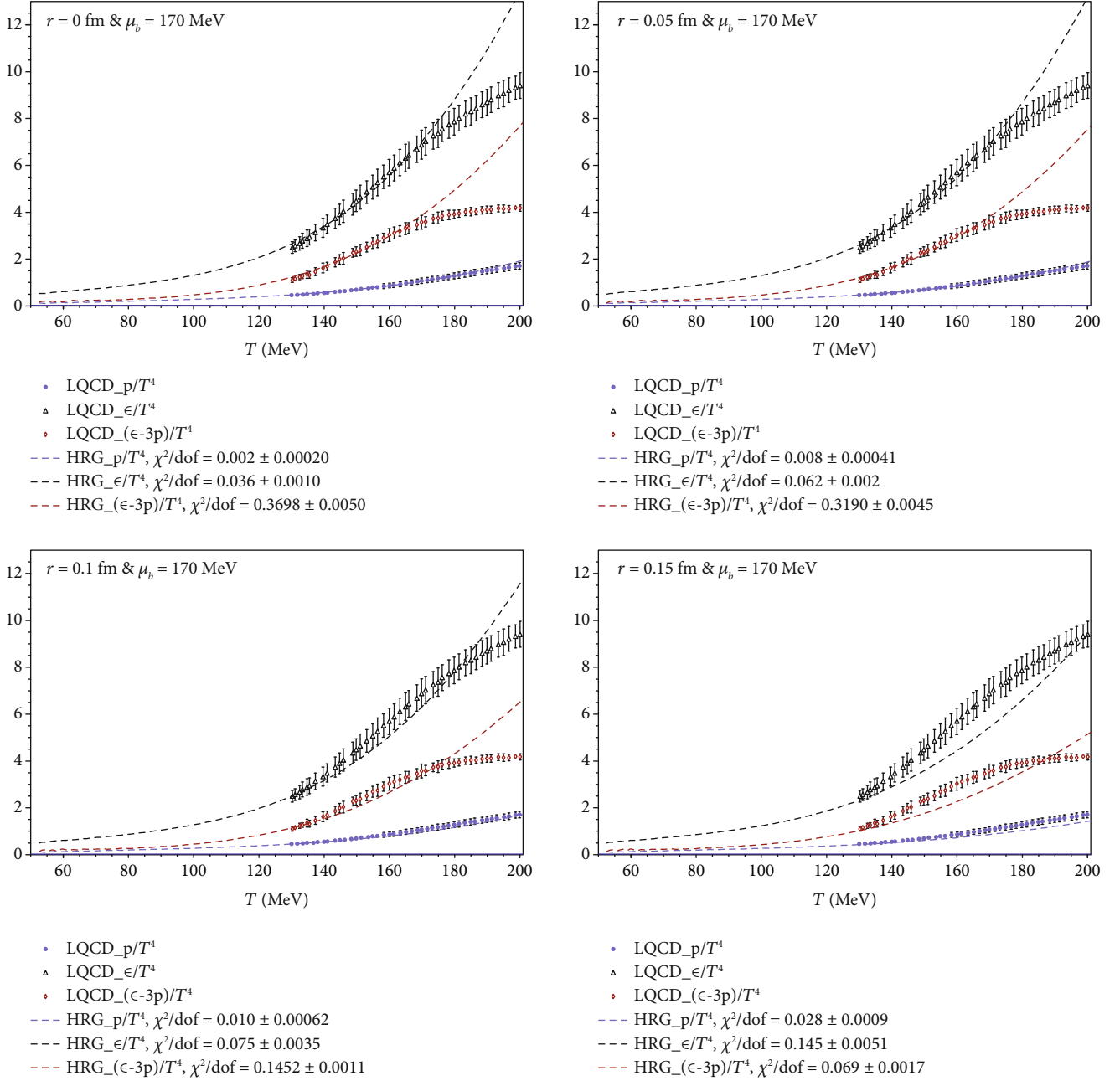
FIGURE 2: The same as in Figure 1 but at $\mu_b = 170$ MeV.

ideal HRG model. For a quantum gas of fermions and bosons with mass $m_i$ and correlation (interaction) distance $r$, at temperature $T$ and vanishing $\mu_b$, a two-particle interaction probability of the form

$$1 \pm \exp\left(-4\pi^2 m_i T r^2\right), \tag{7}$$

was first introduced by Uhlenbeck and Beth [54] in an attempt to model the interactions of a quantum gas of particles assuming a general potential and neglecting the possibility for bound state formation. The Boltzmann-like term $\exp\left(-4\pi^2 m_i T r^2\right)$ remains in effect even for an ideal gas, which is a typical approximation at sufficiently high temper-

atures. The $\pm$ sign expresses the apparent attraction (repulsion) between bosons (fermions) due to the change of statistics [54]. Inspired by such a correction, we introduce a correction for the probability term in the expression for the ideal hadron gas partition function given in Equation (1). We propose a new probability term of the form

$$1 \pm \lambda_i \exp\left(\frac{-\varepsilon_i(p)}{T}\right)\left[1 \pm \exp\left(-4\pi^2 m_i T r^2\right)\right]. \tag{8}$$

This corrected probability function obviously incorporates interactions in the hadron resonance gas in the sense of Uhlenbeck and Beth quantum correlations [54] with $r$ being the
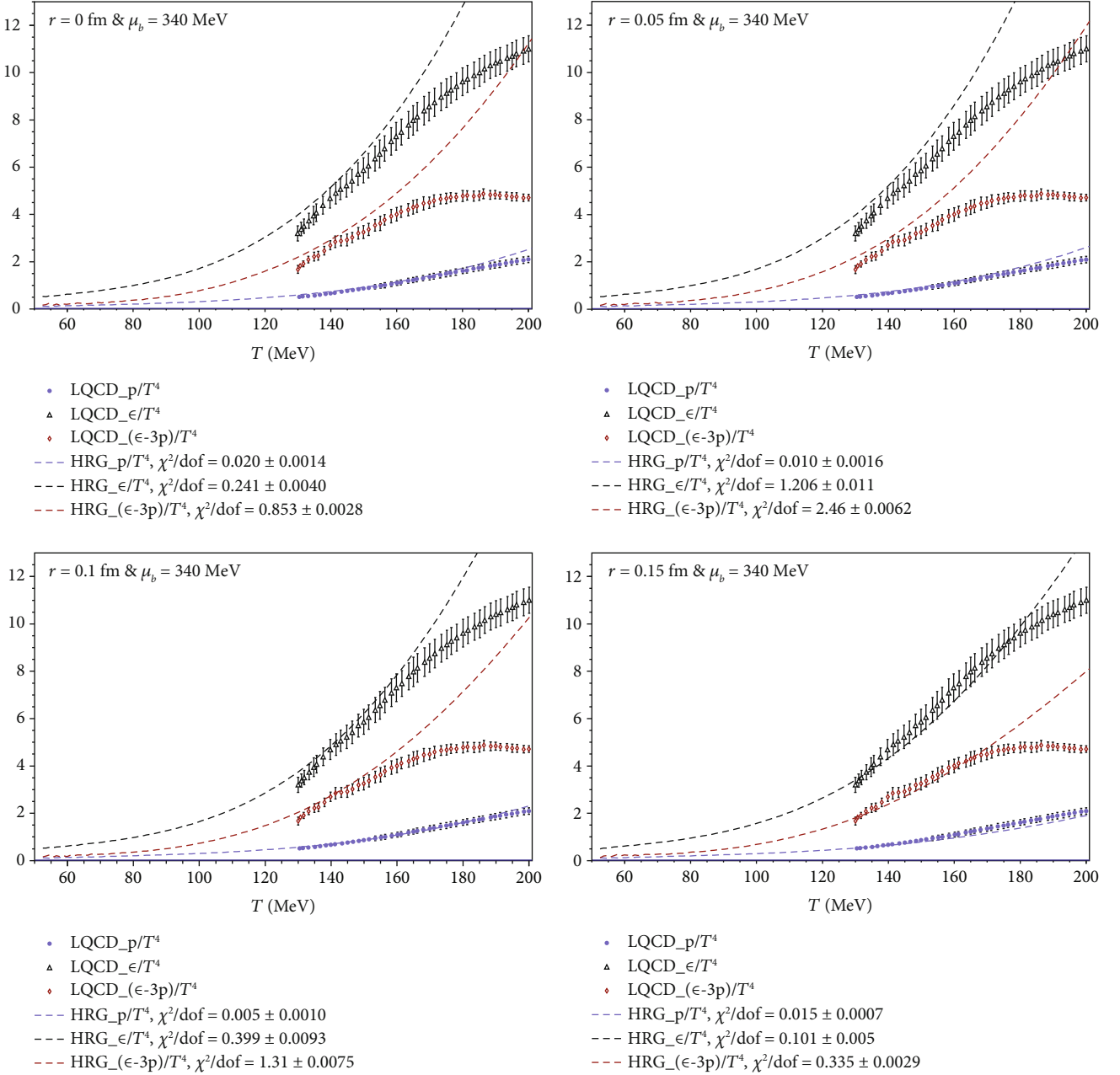
Figure 3: The same as in Figure 1 but at $\mu_b = 340$ MeV.

correlation (interaction) length between any two hadrons at equilibrium temperature $T$. Based on our proposed corrected probability function, we modify the noncorrelated HRG partition function $Z(T, \mu, V)$ to have the following form:

$$
\ln Z'(T, V, \mu) = \sum_i V \frac{g_i}{2\pi^2} \int_0^\infty \pm p^2 dp \ln \left[ 1 \pm \lambda_i \exp \left( \frac{-\varepsilon_{i(p)}}{T} \right) \right.
$$
$$
\left. \cdot \left[ 1 \pm \exp \left( 4\pi^2 m_i T r^2 \right) \right] \right], \tag{9}
$$

which apparently sums over all hadron resonances following the same recipe described in motivating Equation (2) for the case of noncorrelated HRG. The thermodynamics of the corre-

lated HRG can thus be calculated by taking the proper derivatives of $\ln Z$ as explicitly stated in the corresponding noncorrelated HRG case discussed above.

## 3. Calculation Results

We confront the data of the thermodynamics calculated using our statistically corrected HRG model based on Equation (9) with the corresponding lattice thermodynamics data from [66, 67] in the temperature range $T \in [130, 200 \text{ MeV}]$. These temperatures are rather typical for the phenomenological applications in the context of heavy-ion collisions and lattice QCD equation-of-state. In Refs. [66, 67], the authors calculated the QCD equation of state using Taylor
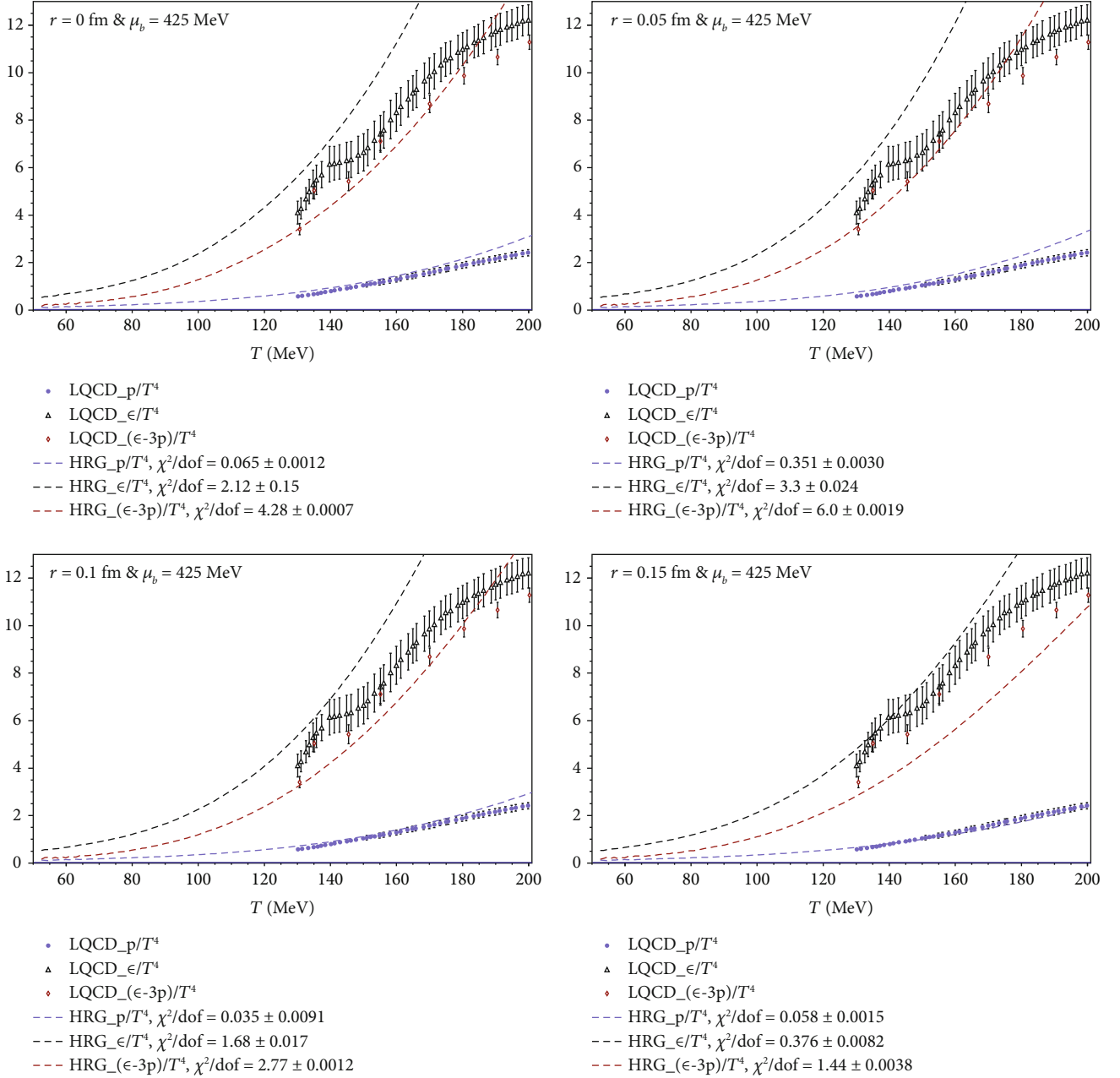
FIGURE 4: The same as in Figure 1 but at $\mu_b = 425$ MeV.

expansions that include contributions from up to the sixth order in the baryon, strangeness, and electric charge chemical potentials. Calculations have been performed with a highly improved staggered quark action in the temperature range $T \in [130, 330 \text{ MeV}]$ using up to four different sets of lattice cut-offs. The lattice data we are confronting with our model are taken from Ref. [66], the total pressure in the $(2 + 1)$-flavor QCD (left) and the total energy density in the $(2 + 1)$-flavor QCD (right) for several values of $\mu_b/T$.

Figure 1 of this letter depicts the temperature dependence of the normalized pressure $P/T^4$, normalized energy density $\rho/T^4$, and trace anomaly $(\rho - 3P)/T^4$ (dashed curves) calculated using our statistically corrected HRG model based on

Equation (9). Moreover, in Figure 1, our model data are confronted with the corresponding lattice data taken from Ref. [66] (symbols with error bars) at the vanishing baryon chemical potential $\mu_b = 0$ MeV. Comparison is made for four different values of the correlation length $r$. Table 1 lists the $\chi^2$/dof statistic for the normalized pressure $P/T^4$, trace anomaly $(\rho - 3P)/T^4$, and normalized energy density $\rho/T^4$ calculated in our statistically corrected hadron resonance gas (HRG) model and confronted with the corresponding lattice data from [66] for four values of the baryon chemical potential $\mu_b = 0$, 170, 340, and 425 MeV.

At the vanishing chemical potential, the best fit to the lattice data occurs for the case of zero correlation length which,

in this case, corresponds to the ideal HRG model. However, a slight exaggeration ($\chi^2$/dof $\simeq 0.06$ in the trace anomaly data) of our model's thermodynamics is observed in the vicinity of the critical phase transition temperature ($T_c \simeq 160$ MeV). In the range $T \in [130, 200$ MeV$]$, the discrepancy between our model thermodynamics and the corresponding lattice data is amplified. Generally, it is obvious that increasing the correlation length emphasizes the mismatch between our model and lattice data.

For the case $\mu_b = 170$ MeV and as it appears in Figure 2, the best fit generally occurs for $r = 0$ and for $r = 0.05$ fm. This good-fit temperature range extends from temperatures well below $T_c$ till $T \gtrsim T_c \simeq 160$ MeV. It is quite obvious here that the model fits the lattice data better compared to the corresponding vanishing chemical potential case(s). However, in the temperature range $T \in [170, 200$ MeV$]$, the mismatch of our model with the lattice data becomes more pronounced compared to the corresponding range of the vanishing chemical potential case(s).

For the case of $\mu_b = 340$ MeV, see Figure 3, the only interesting observation is that for $r = 0.15$ fm, the model data well below and in the vicinity of $T_c$ and up to $T \simeq 170$ MeV significantly approaches the corresponding lattice data. The data mismatch then diverges for higher temperatures.

For the case of $\mu_b = 425$ MeV, Figure 4, the data mismatch is generally too large to suggest any plausible correlation at any of values of $r$ and for all temperatures of interest.

## 4. Conclusions

We confronted a novel statistical correction of the HRG model with recent lattice data [66, 67]. All our model calculations considered in this study do not seem to satisfactorily mimic the corresponding lattice data in the full temperature range under investigation, $T \in [130, 200$ MeV$]$. However, the best matching occurs locally in the vicinity of $T_c$ in the range $T \in [140, 170$ MeV$]$ for the case of $\mu_b = 170$ MeV at zero and $0.05$ fm correlation radii, $r$, respectively. Another remarkable matching between our model data with the corresponding lattice data occurs for the the case of $\mu_b = 340$ MeV, at $0.1$ and $0.15$ fm correlation radii for temperatures $T \lesssim T_c$ and up to $T \simeq 170$ MeV. In the lower temperature range, $T \in [130, 160$ MeV$]$, most of the cases investigated in this research show reasonable match with the corresponding lattice data for different correlation lengths except for the case in which $\mu_b = 425$ MeV where no good fitting is observed for any correlation length.

## Data Availability

The [data type] data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors of this manuscript certify that they have no affiliations with or involvement in any organization or entity with any financial interest or nonfinancial interest in the subject matter or materials discussed in this manuscript.

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