

Research Article

Amended FRW Metric and Rényi Dark Energy Model

Sarfraz Ali,¹ Sabir Iqbal,² Khuram Ali Khan ,³ and Hamid Reza Moradi ⁴

¹Department of Mathematics, University of Education, Lahore, Pakistan

²Riphah International University Faisalabad Campus, Pakistan

³Department of Mathematics, University of Sargodha, Sargodha, Pakistan

⁴Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran

Correspondence should be addressed to Hamid Reza Moradi; hrmoradi@mshdiau.ac.ir

Received 20 April 2021; Revised 6 May 2021; Accepted 21 May 2021; Published 16 June 2021

Academic Editor: Theophanes Grammenos

Copyright © 2021 Sarfraz Ali et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. The publication of this article was funded by SCOAP³.

This article is devoted to exploring the Rényi holographic dark energy model in the theory of Chern-Simons modified gravity. We studied the deceleration parameter, equation of state, and cosmological plane considering the Amended FRW modal. Modified field equations of -gravity theory gave two independent solutions. In the first case, this model provided the transitional change from deceleration to acceleration compatible with collected observational data. However, it supported a decelerating phase of expansion only in the second case. It was noted that the Equation of State advocated the dominance era under the influence of dark energy in the first case and the second predicted the influence of Λ CDM. In both cases, $\omega < 0, \omega' < 0$ voted that the universe is in a freezing region and its cosmic expansion is more rapidly accelerated in the background of Chern-Simons modified gravity.

1. Introduction

Among the biggest challenges faced by cosmological communities, one is the authentic solution to the current accelerated expansion of the universe. The collected observational data [1–3] prodded the quest for hypothetical models to clarify the expansion issues. These models that drive this acceleration is an intriguing, nonglowing, negative pressure medium, and it contributes roughly two-thirds of the energy contents of the present universe. This is called dark energy (DE) [4, 5], and it is considered one of the existing theoretical possibilities, including modifications of gravity theories [6].

One of the reknown and most fascinating extensions of the Einstein gravity theory is four-dimensional Chern-Simons modified gravity (CSMG) [7] permitting the actualization of the CPT and Lorentz symmetry breaking within the gravity theory. Another fundamental component of this modification comprises the way that it normally includes higher-order derivatives of the metric change. In a dynamical variant, this theory has a (genuine) scalar field, with an axionic-type coupling with the Pontryagin density [8]. The

CS-modified gravity was first studied in the nondynamic composition, which does not have a kinetic expression of the scalar field during the operation and assumes a predetermined space-time function. Chern-Simons, in a couple of decades, much attention has been paid to the dynamical CSMG [9], a more sophisticated form in which the scalar field is assumed as a dynamical.

In recent studies [10–13], the HDE model has been studied broadly and analyzed as $\rho \propto \Lambda^4$ using the connections between IR, UV cut-off, and the entropy such that $\Lambda^3 L^3 \leq S^{3/4}$. Working on the same lines, the relation of IR cut-offs and entropy gives rise to the energy density of the HDE model, which is related to the Bekenstein-Hawking term $S = A/4G$. The vacuum energy density is related to UV cut-off Ricci scalar, Hubble horizon, event horizon, etc., i.e., large-scale structures of the universe are related with the infrared (IR) cut-off. The HDE model perseveres through the choice of IR cut-off model. Enough literature is available on the investigations of a huge number of IR cut-offs in [14–20].

In ongoing investigations to comprehend the nature of the universe, a huge number of DE models have been built

on holographic principle and known as holographic dark energy (HDE) models [18, 21, 22]. Adabi et al. [23] reconstructed the potential and dynamics for the Chaplygin scalar field model according to the evolutionary behavior of ghost DE in the context of Einstein's theory phantom accelerated expansion of the universe. The evolution equation and EoS parameters for the nonflat FRW universe are elaborated using the HDE model with Granda-Oliveros cut-off in [24]. Pasqua et al. [25] investigated HDE and modified the Ricci HDE model in the context of CSMG theory. Ali and Amir [26] discussed the Ricci DE model using the Amended FRW metric in the framework of CSMG theory. Further, [27] also investigated the cosmological analysis of the MHRDE model and reconstructed different models such as dilation, K-essence, quintessence, and tachyon modal in the context of CSMG theory.

The study of entropies like Tsallis [28], Rényi [29], and Sharma-Mittal [30] HDE models have been carried out for the cosmological and gravitational incidences. The holographic entanglement entropy has been developed by Chen [16] and varying from regular HDE models with Bekenstein entropy, such models have evolved late-time acceleration of the universe. It is tracked down that the Rényi model displayed stable behavior if there is an occurrence of noninteracting universe [29]. Some models like Tsallis, Rényi, and Sharma-Mittal entropies have been investigated by Younus et al. [31], and they concluded the quintessence-like nature of the universe. On these inspirations, we worked on the Rényi HDE utilizing the Amended FRW metric in the context of CSMG theory.

This article is coordinated as follows: in Section 2, the formalism of CSMG theory and its modified field equation for FRW metric is introduced. In Section 3, we examined Rényi HDE model considering the red-shift parameter. Universe evolution is examined in Section 4. Results and conclusions are discussed at the end.

2. Formalism of Chern-Simons Modified Gravity

A very promising modification of General Relativity is CSMG theory which is developed based on leading-order gravitational parity violation. The terminologies of this theory are very standardized to those of peculiarity cancellation widely used in particle physics and string theory. The Einstein Hilbert action is modified as

$$S = S_{\text{EH}} + S_{\text{CS}} + S_{\Theta} + S_{\text{mat}}, \quad (1)$$

The Einstein Hilbert term S_{EH} , CS term S_{CS} , the scalar field S_{Θ} , and an additional undefined matter contributions S_{mat} are Mathematically represented as $\kappa \int_{\nu} d^4x \sqrt{-g} R$, $\alpha/4 \int_{\nu} d^4x \sqrt{-g} \Theta^* RR$, $\beta/2 \int_{\nu} d^4x \sqrt{-g} [g^{ab} (\nabla_{\chi} \Theta) (\nabla_{\psi} \Theta) + 2V(\Theta)]$, and $\int_{\nu} d^4x \sqrt{-g} \mathcal{L}_{\text{mat}}$, respectively. It is mentioned here that the Pontryagin density $*RR$ is expressed as $*RR = \tilde{R}\tilde{R} = *R_b^{\text{acd}} R_{\text{acd}}^b$, and other parameters $\kappa = 1/16\pi G$, the determinant of metric is g , ∇_{χ} represent covariant derivative of Θ , R is a Ricci

scalar, and integrals denoted the volume executed anywhere on the manifold ν and \mathcal{L}_{mat} stands for some matter Lagrangian density executed on ν .

Taking variation of action of Eq. (1) w.r.t to $g_{\chi\psi}$ along with scalar field Θ , a system of field equations for CSMG theory arose in the following form

$$G_{\chi\psi} + \alpha C_{\chi\psi} = -\frac{1}{2\kappa} \left(T_{\chi\psi}^m + T_{\chi\psi}^{\Theta} \right), \quad (2)$$

$$g^{\chi\psi} \nabla_{\chi} \nabla_{\psi} \Theta = -\frac{\kappa\alpha}{4} *RR, \quad (3)$$

where $G_{\chi\psi}$ is Einstein tensor, α coupling constant, and $C_{\chi\psi}$ is Cotton tensor defined as

$$C_{\chi\psi} = -\frac{1}{2\sqrt{-g}} \left[v_{\zeta} \varepsilon^{\zeta\mu\nu\eta} \nabla_{\nu} R_{\eta}^{\nu} + \frac{1}{2} v_{\zeta\lambda} \varepsilon^{\zeta\nu\eta\lambda} R_{\nu\eta}^{\lambda\mu} \right], \quad (4)$$

where $v_{\zeta} = \nabla_{\zeta} \Theta$ and $v_{\zeta\lambda} = \nabla_{\zeta} \nabla_{\lambda} \Theta$. The tensor $T_{\chi\psi}$ consists on matter and scalar field, mathematically described as

$$T_{\chi\psi}^m = (\rho + p) U_{\chi} U_{\psi} - p g_{\chi\psi}, \quad (5)$$

$$T_{\chi\psi}^{\Theta} = \eta \left(\partial_{\chi} \Theta \right) \left(\partial_{\psi} \Theta \right) - \frac{\eta}{2} g_{\chi\psi} \left(\partial^{\alpha} \Theta \right) \left(\partial_{\alpha} \Theta \right). \quad (6)$$

3. Amended FRW Model in CS Modified Gravity

FRW model is used to calculate the homogeneous, isotropic, and expanding universe. Cosmologists are agreed that the FRW model is the best choice for the approximation of homogeneous, isotropic, and expanding universe. There are some equivalent formalism of FRW metric also found in literature to refer the spacetimes that are useful in the following manner:

$$\begin{aligned} ds^2 &= dt^2 - a^2(t) f^2(r) [dr^2 + r^2 d\Phi^2] \\ &\equiv a^2(t) [dt^2 - f^2(r) (dr^2 + r^2 d\Phi^2)] \\ &\equiv dt^2 - a^2(t) [f^2(r) dr^2 + r^2 d\Phi^2] \\ &\equiv a^2(t) dt^2 - f^2(r) [dr^2 - r^2 d\Phi^2] \\ &\equiv dt^2 - a^2(t) [dr^2 + f^2(r) d\Phi^2] \\ &\equiv a^2(t) [dt^2 - dr^2 - f^2(r) d\Phi^2]. \end{aligned} \quad (7)$$

These equivalent forms are among the most popular models in the context of cosmological studies. Here, we use one of them which is named the Amended FRW metric [32].

$$ds^2 = -a^2(t) dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (8)$$

Dimensionless parameter $a(t)$ is a key tool to analyze the accelerated expansion of the current universe called

the scale factor. It is found that the Cotton tensor vanishes identically for the AFRW metric such that $C_{00} = 0$. The energy-momentum tensor T_{00}^{Θ} is calculated using Eq. (6)

$$T_{00}^{\Theta} = \frac{\dot{\Theta}^2}{2}. \quad (9)$$

It is worth mentioning here that for a metric to be a solution of EFEs, the Pontryagin term $*RR$ must be zero as a necessary condition, and the same has been evaluated for the Amended FRW metric identically. So, Eq. (3) reduces to

$$g^{\chi\psi}\nabla_{\chi}\nabla_{\psi}\Theta = g^{\chi\psi}\left[\partial_{\chi}\partial_{\psi}\Theta - \Gamma_{\chi\psi}^{\rho}\partial_{\rho}\Theta\right] = 0. \quad (10)$$

It is noted that the external field is a function of space-time coordinates, and for the sake of simplicity, we opt Θ a function of temporal coordinate only which reduced Eq. (10) as given below.

$$\ddot{\Theta} + 2\frac{\dot{a}}{a}\dot{\Theta} = 0. \quad (11)$$

Applying the separation method of variables from differential equations, Eq. (11) gives

$$\dot{\Theta} = Ia^{-2}, \quad (12)$$

The parameter I is integration constant. Substituting Eq. (12) in Eq. (9), one arrives at

$$\rho_{\Theta} = T_{00}^{\Theta} = \frac{I^2 a^{-4}}{2}. \quad (13)$$

4. Rényi HDE Model

The mysterious nature and dynamics of DE is a crucial issue in cosmological studies. A considerable number of models are presented to resolve, HDE model is one of them. According to this model $\rho_{\text{HDE}} = 3d^2 M_p^2 L^{-2}$, the event horizon $L = [A/4\pi]^{1/2} = H^{-1}$, d stand for numerical factor, and M_p is reduced Planck's mass. The Rényi HDE model is given by [33].

$$\rho_R = \left(\frac{3K^2 H^2}{8\pi(1 + (\pi\delta/H^2))}\right). \quad (14)$$

Since the conservation law of energy density is expressed as

$$\dot{\rho}_d + 3H(\rho_d + P_d) = 0. \quad (15)$$

Taking into account the dust case, it turned out to be

$$\rho_d = \rho_0 a^{-3}. \quad (16)$$

In the context of flat AFRW metric, CSMG equations are evaluated as

$$H^2 = \frac{8\pi G}{3}(\rho_M + \rho_R + \rho_{\Theta}), \quad (17)$$

$$H^2 + \frac{2}{3}\dot{H} = -\frac{8\pi G}{3}(P_d). \quad (18)$$

Where G is gravitational constant considered $G = 1$. Now, we use the values of ρ_M , ρ_R , and ρ_{Θ} in Eq. (17) and get

$$H^2 = \frac{8\pi}{3}\rho_0 a^{-3} + \frac{K^2 H^2}{1 + (\pi\delta/H^2)} + \frac{4\pi}{3}\rho_{\Theta} a^{-4}. \quad (19)$$

Let us consider $H(z) = \Xi(z)H_0$ and using redshift parameter $a = 1/z + 1$ in Eq. (19) which yields

$$\Xi^2(z) = [z + 1]^3 \left(\frac{\rho_0}{3H_0^2/8\pi}\right) + \frac{K^2 \Xi^2(z)}{(1 + (\pi\delta/\Xi^2(z)H_0^2))} + \left(\frac{\rho_{\Theta}}{3H_0^2/4\pi}\right)[z + 1]^4. \quad (20)$$

Making it convenient to find analytic solution, we consider $(\rho_0/3H_0^2/8\pi) = \Omega_M$ and $(\rho_{\Theta}/3H_0^2/4\pi) = \Omega_{\Theta}$; therefore, Eq. (20) gets the form

$$\Xi^2(z) = \Omega_M [z + 1]^3 + \frac{K^2 \Xi^2(z)}{(1 + (\pi\delta/\Xi^2(z)H_0^2))} + \Omega_{\Theta} [z + 1]^4. \quad (21)$$

Obviously, it is fourth-order equation in $\Xi(z)$ which can be reduced in quadratic form by substituting $3.16\delta/H_0^2 = Y_z$, $\Xi^2(z) = F(z)$, and $\beta_z = \Omega_M [z + 1]^3 + \Omega_{\Theta} [z + 1]^4$ in Eq. (27)

$$F^2(z)(1 - K^2) + F(z)[Y_z - X_z] - Y_z X_z = 0. \quad (22)$$

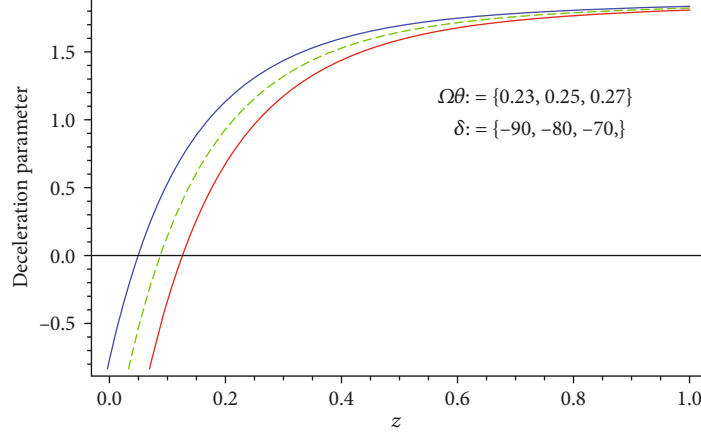
This is a quadratic equation in $F(z)$ and two solutions arise here such that

$$F(z) = \frac{(X_z - Y_z) \pm (Y_z + X_z)\sqrt{1 - 4Y_z X_z K^2 / (Y_z + X_z)^2}}{2(1 - K^2)}. \quad (23)$$

Using binomial theorem on $\sqrt{(1 - 4Y_z X_z K^2 / (Y_z + X_z)^2)}$ and neglecting the higher-order terms, one arrives at

$$\Xi^2(z) = \frac{(X_z - Y_z) \pm (Y_z + X_z)(1 - 2Y_z X_z K^2 / (Y_z + X_z)^2)}{2(1 - K^2)}. \quad (24)$$

Now, we will discuss these two solutions that arise in Eq. (30), separately.

FIGURE 1: q versus Redshift z .

4.1. Case 1. Let us consider the positive root as the first case such that

$$\Xi^2(z) = \frac{X_z}{1-K^2} \left[\frac{Y_z + X_z - Y_z K^2}{Y_z + X_z} \right]. \quad (25)$$

On backward substitution, Eq. (25) can be evaluated as

$$\Xi(z) = \left[\frac{\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4}{1-K^2} \cdot \frac{(3.16\delta/H_0^2)(1-K^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4}{(3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4} \right]^{\frac{1}{2}}. \quad (26)$$

Further, taking first-order derivative of Eq. (27) and simplifying, one arrives at

$$\begin{aligned} \frac{d}{dz} \Xi(z) = & \frac{(3.16\delta/H)^2 [3\Omega_M[z+1]^2 + 4\Omega_\Theta[z+1]^3] \sqrt{1-K^2}}{2((3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)^{3/2}} \\ & + \frac{\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4 3\Omega_M[z+1]^2 + 4\Omega_\Theta[z+1]^3 (3.16\delta/H_0^2)}{\sqrt{1-K^2} ((3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)^{3/2}} \\ & + \frac{\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4}{2\sqrt{1-K^2} ((3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)^{3/2}}. \end{aligned} \quad (27)$$

Using Eqs. (27) and (28), we explored two important cosmological parameters deceleration parameter (DP) q and equation of state (EoS) to study the nature of universe.

4.1.1. *Deceleration Parameter.* DP is a dimensionless quantity which explains the expansions of the universe which slows down due to self-gravity. In terms of FRW metric, the expansion of the universe given by $q = -\ddot{a}a/a^2$ where a^2 represent the derivative w.r.t. to temporal coordinate t . An expression for DP, also found in term of derivative of Hubble parameter, can be mathematically written as

$$q = -1 - \frac{\dot{H}}{H^2}. \quad (28)$$

Substituting $H(z) = \Xi(z)H_0$ and $\dot{H} = H_0(d/dz)\Xi(z)$ in Eq. (29), it reduces to

$$q = -1 + \frac{[z+1]}{\Xi(z)} \cdot \frac{d}{dz} \Xi(z). \quad (29)$$

Using Eqs. (27) and (28) in Eq. (30), one explores

$$\begin{aligned} q = & \frac{1}{2} \left[\frac{3\Omega_M + 4\Omega_\Theta[z+1]}{\Omega_M + \Omega_\Theta[z+1]} \right] + \left[\frac{1}{2} \frac{(K^2 3.16\delta/H_0) 3\Omega_M + 4\Omega_\Theta[z+1]}{[z+1]^3 [(3.16\delta/H_0^2) + \Omega_M + \Omega_\Theta[z+1]]^2} \right] \\ & \times \left[\frac{1}{(3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4} \right]. \end{aligned} \quad (30)$$

To investigate q using Rényi HDE in CSMG theory, we plotted a graph shown in Figure 1.

We opted the restrictions on parameters $\Omega_{M_0} = 0.23$, $\Omega_{\Theta_0} = 0.23, 0.25, 0.27$, $C = 2$, and $H_0 = 71 \text{ Km/s/Mpc}$. The graph illustrated decelerated phase $q < 0$ at low redshift and transit to accelerated phase $q > 0$ at high redshift. It is observed that the behavior of DP is very similar in all three cases and our graphical representation advocated the transition from deceleration to acceleration which is also predicted in [31, 34, 35, 36].

4.1.2. *Equation of State.* The EoS for perfect fluid is denoted by dimensionless parameter ω , is a ratio between pressure and energy density of the fluid, mathematically represented as

$$\omega = \frac{P}{\rho}. \quad (31)$$

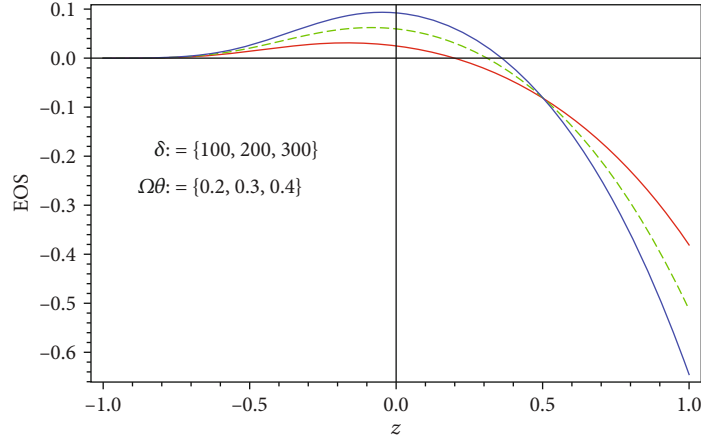
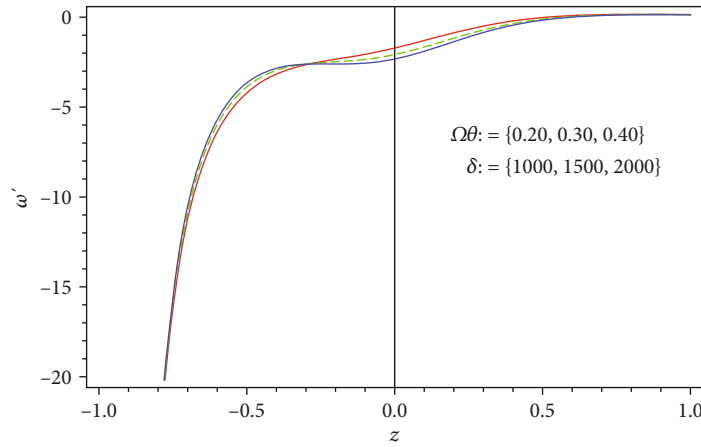
In terms of different energy components, it is expressed as

$$\omega = \left(\frac{P_d}{\rho_d + \rho_M + \rho_\Theta} \right). \quad (32)$$

The EOS can be represented in term of DP such that

$$\omega = \frac{2}{3} \left(q - \frac{1}{2} \right). \quad (33)$$

Substituting the value of DP q from Eq. (31) in Eq. (34),

FIGURE 2: The evaluation of EoS ω versus Redshift z .FIGURE 3: ω' versus Redshift z .

one arrives at

$$\omega = \frac{1}{3} \left[\frac{3\Omega_M + 4\Omega_\Theta [z+1]}{\Omega_M + \Omega_\Theta [z+1]} \right] + \frac{1+z}{2} \cdot \left[\frac{(K^2 3.16\delta/H_0) 3\Omega_M [z+1]^2 + 4\Omega_\Theta [z+1]^3}{((3.16\delta/H_0^2) + \Omega_M [z+1]^3 + \Omega_\Theta [z+1]^4)^2} \right]. \quad (34)$$

Eq. (35) represents that EoS ω is a function of z along-with dependence on some cosmological parameters. To investigate the cosmological evaluation of the universe, we plotted a graph given in Figure 2.

The particular restrictions are imposed on the parameters such as $\Omega_{M_0} = 0.23$, $\Omega_{\Theta_0} = 0.2, 0.3, 0.4$, $C = 3$, $H_0 = 71 \text{ Km/s/Mpc}$. Actually, different values of EoS illustrates the dominance era of the universe by different components. For instance, $\omega = 0$, $\omega = 1/3$, and $\omega = 1$ indicate that the universe is influenced by dust, radiation, and stiff fluid, respectively. On the other hand, ($\omega = -1/3$, $\omega = -1$, and $\omega < -1$) are conditions of quintes-

For the particular values of parameters $\Omega_{M_0} = 0.23$, $\Omega_{\Theta_0} = 0.25$, $K^2 = 3$, $H_0 = 71 \text{ Km/s/Mpc}$, and three different values of δ , a graph of ω' is plotted in Figure 3.

sence DE, Λ CDM, and Phantom eras, respectively. The graphical behavior showed that the universe is under the influence of DE as the EoS predicted accelerated expansion phase.

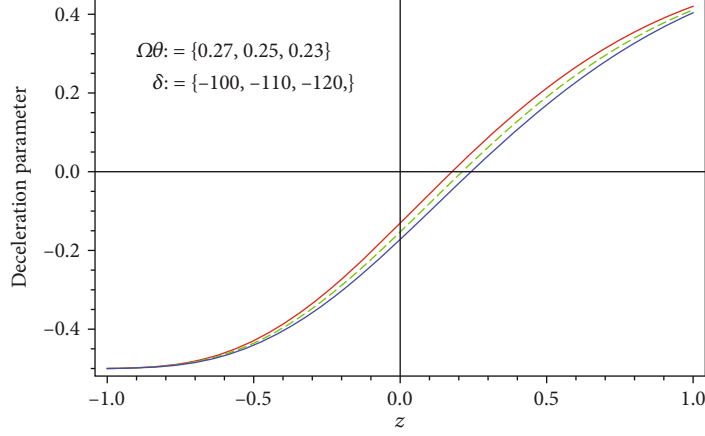
4.1.3. $\omega - \omega'$ Plane. Caldwell and Linder [37] introduced $\omega - \omega'$ plane to explore the cosmic evolution of the quintessence DE model. They found a result which support the assumption that any region occupied by a DE model is subdivided into freezing ($\omega < 0$, $\omega' < 0$) and thawing ($\omega < 0$, $\omega' > 0$) regions, respectively. It is also found that the cosmic expansion is more accelerating in the freezing region as compared to thawing.

Taking first order derivative of Eq. (39), we obtained

$$\omega' = \left[\frac{2[z+1]^4 (\Omega_M^2 (z-8) + 3\Omega_M \Omega_\Theta [z+1]^2 - 8[z+1] - 14\Omega_\Theta^2 [z+1]^2)}{(\Omega_M [z+1]^3 + \Omega_\Theta [z+1]^4)^2} \right]. \quad (35)$$

Derivative of (EOS) representing that ω' is a function of redshift z .

The graphical representation $\omega < 0$, $\omega' < 0$ advocated that the Rényi HDE model is in freezing region and cosmic expansion is more accelerating in the context of CSMG theory.

FIGURE 4: The evaluation of DP q versus Redshift z .

$$\begin{aligned}
\omega - \omega' &= \frac{1+z}{3} \left[\frac{3\Omega_M[z+1]^2 + 4[z+1]^3}{\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4} \right] \\
&+ \frac{1+z}{2} \left[\frac{(K^2 3.16\delta/H_0) 3\Omega_M[z+1]^2 + 4[z+1]^3}{((3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)} \right] \\
&\times \left[\frac{1}{((3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)} \right] \\
&- \left[\frac{2[z+1]^4 (\Omega_M^2(z-8) + 3\Omega_M\Omega_\Theta[z+1]^2 - 8[z+1] - 14\Omega_\Theta^2[z+1]^2)}{(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)^2} \right] \\
&- \left[\frac{[z+1]^5 [(3.16\delta/H_0^2)K^2(\Omega_M^2 + 16\Omega_\Theta^2[z+1]^2) + 8\Omega_M\Omega_\Theta[z+1]]}{((3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)((3.16\delta/H_0^2)(1-K^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)} \right] \\
&+ 2[z+1] \left[\frac{(3.16\delta/H_0^2)K^2\Omega_M + 2\Omega_\Theta[z+1]}{((3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)((3.16\delta/H_0^2)(1-K^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)} \right] \\
&- \frac{[z+1]^5}{3} \left[\frac{(3.16\delta/H_0^2)K^2(9\Omega_M^2 + 16\Omega_\Theta^2[z+1]^2 + 24[z+1])}{((3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)((3.16\delta/H_0^2)(1-K^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)} \right] \\
&+ \frac{[z+1]^4}{3} \left[\frac{3\Omega_M + 4[z+1]}{\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4} \right] \\
&+ \frac{[z+1]^4}{3} \left[\frac{(3.16\delta/H_0^2)K^2(3\Omega_M + 4\Omega_\Theta[z+1])}{((3.16\delta/H_0^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)((3.16\delta/H_0^2)(1-K^2) + \Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)} \right]. \tag{36}
\end{aligned}$$

4.2. Case 2. Taking into account the other root of the equation, we worked on same lines to explore the relations for DP and EOS in the context of CSMG theory.

$$\Xi^2(z) = \frac{(X_z - Y_z) - (Y_z + X_z) - 2Y_z X_z K^2 / (Y_z + X_z)}{2(1-K^2)}. \tag{37}$$

Putting values of Y_z and X_z in Eq. (42) and simplifying

we obtained the value of $\Xi(z)$ in terms of redshift z .

$$\Xi(z) = \left[\frac{(3.16\delta/H_0^2)}{1-K^2} \frac{K^2(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4)}{\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4 + (3.16\delta/H_0^2)} \right]^{\frac{1}{2}}. \tag{38}$$

Substituting Eq. (43) in Eq. (35) is evaluated as

$$q = -\frac{-2c^4(3.16\delta/H_0)[z+1]^3\Omega_M^2 + [z+1]\Omega_\Theta(-2)(3.16\delta/H_0) + [z+1]^3(\Omega_M + 2[z+1]\Omega_\Theta)}{2[c^2(3.16\delta/H_0)\Omega_M + [z+1]\Omega_\Theta][(3.16\delta/H_0) + [z+1]^3((1-c^2)\Omega_M) + [z+1]^4\Omega_\Theta]} + \frac{c^2\Omega_M[-(3.16\delta/H_0)^2 - [z+1]^4\Omega_\Theta + (3.16\delta/H_0)[z+1]^3(2\Omega_M + 3[z+1]\Omega_\Theta)]}{2[c^2(3.16\delta/H_0)\Omega_M + [z+1]\Omega_\Theta][(3.16\delta/H_0) + [z+1]^3((1-c^2)\Omega_M) + [z+1]^4\Omega_\Theta]}. \quad (39)$$

To investigate q using Rényi HDE in CSMG theory, we plotted a graph shown in Figure 4. We opted the restrictions on parameters $\Omega_{M_0} = 0.23, \Omega_{\Theta_0} = 0.23, 0.25, 0.27, C = 3$, and $H_0 = 67 \text{ Km/s/Mpc}$.

It is noted that q is negative for an accelerating universe and positive for a decelerating universe. Figure 4 represented a flip of sign for q from negative to positive which gives the

best match with the observational data collected by Riess et al. [1], Perlmutter et al. [2], and [20, 26, 38]. It is concluded that Rényi HDE model predicts a deceleration to acceleration transition compatible with observational data.

Furthermore, in Eq. (39), it is obvious that the EoS ω is a function of z along with dependence on some cosmological parameters.

$$\omega = \frac{2}{3} \left[-\frac{-2c^4(3.16\delta/H_0)[z+1]^3\Omega_M^2 + [z+1]\Omega_\Theta(-2)(3.16\delta/H_0) + [z+1]^4(\Omega_M + 2[z+1]\Omega_\Theta)}{2[c^2(3.16\delta/H_0)\Omega_M + [z+1]\Omega_\Theta][(3.16\delta/H_0) + [z+1]^3((1-c^2)\Omega_M) + [z+1]^4\Omega_\Theta]} + \frac{c^2\Omega_M[-(3.16\delta/H_0)^2 - [z+1]^4\Omega_\Theta + (3.16\delta/H_0)[z+1]^3(2\Omega_M + 3[z+1]\Omega_\Theta)]}{2[c^2(3.16\delta/H_0)\Omega_M + [z+1]\Omega_\Theta][(3.16\delta/H_0) + [z+1]^3((1-c^2)\Omega_M) + [z+1]^4\Omega_\Theta]} - \frac{1}{2} \right]. \quad (40)$$

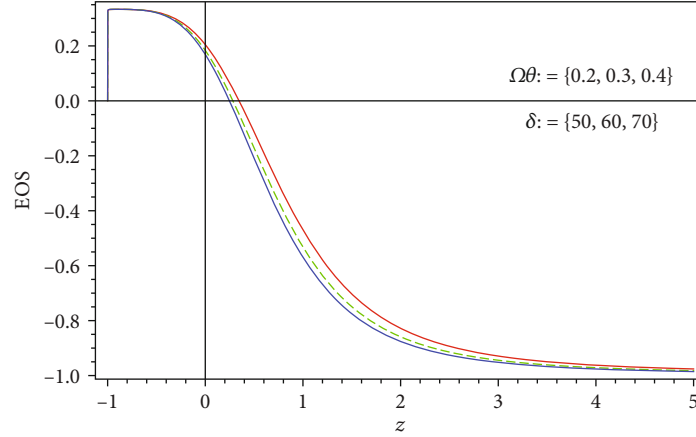
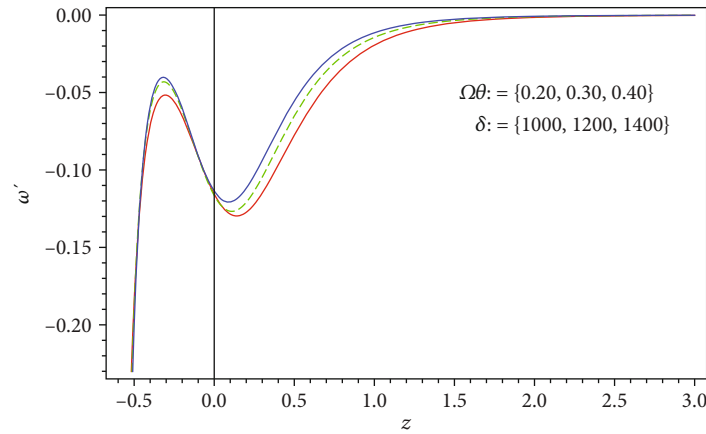
To understand about the cosmological evaluation of the universe, we plotted a graph given in Figure 5.

Particular restrictions are imposed on the parameters such as $\Omega_{M_0} = 0.23, \Omega_{\Theta_0} = 0.2, 0.3, 0.4, C = 3$, and $H_0 = 71$

Km/s/Mpc. The graphical representation shown that the universe is under the influence of DE as the EoS predicted accelerated expansion phase.

$$\omega' = \left[\frac{2[z+1](6\Omega_M + 12\Omega_\Theta[z+1]^4) + 3\Omega_M[z+1]^2 + 4\Omega_\Theta[z+1]^3}{3(K^2-1)(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2} \right] - \left[\frac{2(K^2-1)[z+1]^3(3\Omega_M + 4\Omega_\Theta[z+1]^2)^2}{3((K^2-1)(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2)^2(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2} \right] - \left[\frac{2[z+1]^5(3\Omega_M + 4\Omega_\Theta[z+1]^3)^2}{3((K^2-1)(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2)^2(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2} \right].$$

$$\omega - \omega' = -1 + \left[\frac{2[z+1]^3(3\Omega_M + 4\Omega_\Theta[z+1])}{\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4 + (3.16\delta/H_0^2)} \right] \times \left[\frac{1}{3(K^2-1)(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2} \right] - \left[\frac{2[z+1](6\Omega_M + 12\Omega_\Theta[z+1]^4) + 3\Omega_M[z+1]^2 + 4\Omega_\Theta[z+1]^3}{3(K^2-1)(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2} \right] - \left[\frac{2(K^2-1)[z+1]^3(3\Omega_M + 4\Omega_\Theta[z+1]^2)^2}{3((K^2-1)(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2)^2(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2} \right] - \left[\frac{2[z+1]^5(3\Omega_M + 4\Omega_\Theta[z+1]^3)^2}{3((K^2-1)(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2)^2(\Omega_M[z+1]^3 + \Omega_\Theta[z+1]^4) - 3.16\delta/H_0^2} \right]. \quad (41)$$

FIGURE 5: ω vs. z .FIGURE 6: ω' versus Redshift z .

The graph of Eq. (41) is plotted under the restrictions on parameters $\Omega_{M_0} = 0.23$, $\Omega_{\Theta_0} = 0.25$, $K^2 = 20$, $H_0 = 71 \text{ Km/s/Mpc}$, and δ shown in Figure 6. In this case, $\omega < 0$, $\omega' < 0$ indicated that the Rényi HDE model is also in freezing region and cosmic expansion will be more accelerating in the context of CSMG theory.

5. Conclusions

This article is devoted to studying the Rényi HDE model considering the Amended FRW model in the background of CSMG theory. We explored the EoS, DP, and cosmological plane in interacting scenarios. There were two different solutions evaluated and discussed separately. In the first case, Figure 1 illustrated the decelerated phase $q > 0$ at low redshift and transit to accelerated phase $q < 0$ at high redshift. Also, it is observed that the behavior of DP is very similar for all values of δ of Rényi HDE model and our graphical representation advocated the transition from deceleration to acceleration phase of the universe which is fully consistent with the observational data [34, 35] [31]. In fact, EoS illustrates the era of dominance of the universe by different components. For example, $\omega = 0$, $\omega = 1/3$, and $\omega = 1$ indicate that

the universe is influenced by dust, radiation, and stiff fluid, respectively. On the other hand, ($\omega = -1/3$, $\omega = -1$, and $\omega < -1$) stand for quintessence DE, Λ CDM, and Phantom eras, respectively. The graphical behavior showed that the universe is under the influence of DE as the EoS predicted accelerated expansion phase Figure 2. The graphical behavior of Figure 3 $\omega < 0$, $\omega' < 0$ indicated that the Rényi HDE model is in freezing region, and cosmic expansion is more accelerating in the context of CSMG theory. In the second case, Figure 4 represents that the universe is in a decelerated phase of expansion as $q < 0$ for each value of the redshift parameter z . Further, EoS predicted that the universe is under the influence of Λ CDM. Finally, $\omega - \omega'$ plane indicated that the Rényi HDE model is also in the freezing region and cosmic expansion will be more accelerating in the context of CSMG theory. At the end, it is concluded that the Rényi HDE model is supported by the results of general relativity in the framework of CSMG theory.

Data Availability

No data is available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] A. G. Riess, A. V. Filippenko, P. Challis et al., "Observational evidence from supernovae for an accelerating universe and a cosmological constant," *The Astronomical Journal*, vol. 116, no. 3, pp. 1009–1038, 1998.
- [2] S. Perlmutter, G. Aldering, G. Goldhaber et al., "Measurements of Ω and Λ from 42 high-redshift supernovae," *The Astrophysical Journal*, vol. 517, no. 2, pp. 565–586, 1999.
- [3] B. P. Schmidt, N. B. Suntzeff, M. M. Phillips et al., "The high- z supernova search: measuring cosmic deceleration and global curvature of the universe using type Ia supernovae," *The Astrophysical Journal*, vol. 507, no. 1, pp. 46–63, 1998.
- [4] L. Amendola and S. Tsujikawa, *Dark Energy: Theory and Observations*, Cambridge University Press, 2010.
- [5] E. J. Copeland, M. Sami, and S. Tsujikawa, "Dynamics of dark energy," *International Journal of Modern Physics D*, vol. 15, no. 11, pp. 1753–1935, 2006.
- [6] A. Joyce, L. Lombriser, and F. Schmidt, "Dark energy versus modified gravity," *Annual Review of Nuclear and Particle Science*, vol. 66, no. 1, pp. 95–122, 2016.
- [7] R. Jackiw and S. Y. Pi, "Chern-simons modification of general relativity," *Physical Review D*, vol. 68, no. 10, article 104012, 2003.
- [8] S. Alexander and N. Yunes, "Chern-Simons modified general relativity," *Physics Reports*, vol. 480, no. 1-2, pp. 1–55, 2009.
- [9] T. L. Smith, A. L. Erickcek, R. R. Caldwell, and M. Kamionkowski, "Effects of Chern-Simons gravity on bodies orbiting the Earth," *Physical Review D*, vol. 77, no. 2, article 024015, 2008.
- [10] M. Li, "A model of holographic dark energy," *Physics Letters B*, vol. 603, no. 1-2, pp. 1–5, 2004.
- [11] A. Sheykhi, "Holographic scalar field models of dark energy," *Physical Review D*, vol. 84, no. 10, article 107302, 2011.
- [12] B. Hu and Y. Ling, "Interacting dark energy, holographic principle and coincidence problem," *Physical Review D*, vol. 73, no. 12, p. 123510, 2006.
- [13] Y. Z. Ma, Y. Gong, and X. Chen, "Features of holographic dark energy under combined cosmological constraints," *European Physical Journal C*, vol. 60, no. 2, pp. 303–315, 2009.
- [14] S. D. H. Hsu, "Entropy bounds and dark energy," *Physics Letters B*, vol. 594, no. 1-2, pp. 13–16, 2004.
- [15] H. Wei and S. N. Zhang, "Age problem in the holographic dark energy model," *Physical Review D*, vol. 76, no. 6, article 063003, 2007.
- [16] C. Gao, F. Wu, X. Chen, and Y.-G. Shen, "Holographic dark energy model from Ricci scalar curvature," *Physical Review D*, vol. 79, no. 4, article 043511, 2009.
- [17] L. N. Granda and A. Oliveros, "Infrared cut-off proposal for the holographic density," *Physics Letters B*, vol. 669, no. 5, pp. 275–277, 2008.
- [18] L. N. Granda and A. Oliveros, "New infrared cut-off for the holographic scalar fields models of dark energy," *Physics Letters B*, vol. 671, no. 2, pp. 199–202, 2009.
- [19] K. Karami and J. Fehri, "New holographic scalar field models of dark energy in non-flat universe," *Physics Letters B*, vol. 684, no. 2-3, pp. 61–68, 2010.
- [20] S. Wang, Y. Wang, and M. Li, "Holographic dark energy," *Physics Reports*, vol. 696, pp. 1–57, 2017.
- [21] C.-H. Chou and K.-W. Ng, "Decaying superheavy dark matter and subgalactic structure of the Universe," *Physics Letters B*, vol. 594, no. 1-2, pp. 1–7, 2004.
- [22] A. G. Cohen, D. B. Kaplan, and A. E. Nelson, "Effective field theory, black holes, and the cosmological constant," *Physical Review Letters*, vol. 82, no. 25, pp. 4971–4974, 1999.
- [23] F. Adabi, K. Karami, and M. Mousivand, "Ghost Chaplygin scalar field model of dark energy," *Canadian Journal of Physics*, vol. 91, no. 1, pp. 54–59, 2013.
- [24] K. Karami and J. Fehri, "Interacting entropy-corrected new agegraphic dark energy in the non-flat universe," *Physica Scripta*, vol. 82, no. 5, article 059802, 2010.
- [25] A. Pasqua, R. da Rocha, and S. Chattopadhyay, "Holographic dark energy models and higher order generalizations in dynamical Chern-Simons modified gravity," *The European Physical Journal C*, vol. 75, no. 2, p. 44, 2015.
- [26] S. Ali and M. J. Amir, "A study of holographic dark energy models in Chern-Simon modified gravity," *International Journal of Theoretical Physics*, vol. 55, no. 12, pp. 5095–5105, 2016.
- [27] S. Ali and M. J. Amir, "Cosmological analysis of modified holographic Ricci dark energy in Chern-Simons modified gravity," *Advances in High Energy Physics*, vol. 2019, Article ID 3709472, 9 pages, 2019.
- [28] M. Tavayef, A. Sheykhi, K. Bamba, and H. Moradpour, "Tsallis holographic dark energy," *Physics Letters B*, vol. 781, pp. 195–200, 2018.
- [29] H. Moradpour, S. A. Moosavi, I. P. Lobo, J. P. Moraes Graa, A. Jawad, and I. G. Salako, "Thermodynamic approach to holographic dark energy and the Rényi entropy," *The European Physical Journal C*, vol. 78, no. 10, p. 829, 2018.
- [30] A. S. Jahromi, S. A. Moosavi, H. Moradpour et al., "Generalized entropy formalism and a new holographic dark energy model," *Physics Letters B*, vol. 780, pp. 21–24, 2018.
- [31] M. Younas, A. Jawad, S. Qummer, H. Moradpour, and S. Rani, "Cosmological implications of the generalized entropy based holographic dark energy models in dynamical Chern-Simons modified gravity," *Advances in High Energy Physics*, vol. 2019, Article ID 1287932, 9 pages, 2019.
- [32] J. Hunter, "Redshifts and scale factors: a new cosmological model," *Journal of Cosmology*, vol. 6, pp. 1485–1497, 2010.
- [33] J. Almeida and J. Pereira, "Holographic dark energy and the universe expansion acceleration," *Physics Letters B*, vol. 636, no. 2, pp. 75–79, 2006.
- [34] Y. Z. Ma, "Variable cosmological constant model: the reconstruction equations and constraints from current observational data," *Nuclear Physics B*, vol. 804, no. 1-2, pp. 262–285, 2008.
- [35] R. A. Daly, S. G. Djorgovski, K. A. Freeman et al., "Improved constraints on the acceleration history of the universe and the properties of the dark energy," *The Astrophysical Journal*, vol. 677, no. 1, pp. 1–11, 2008.
- [36] A. Tripathi and A. Sangwan, "Dark energy equation of state parameter and its evolution at low redshift," *Journal of Cosmology and Astroparticle Physics*, vol. 2017, no. 6, p. 12, 2017.
- [37] R. R. Caldwell and E. V. Linder, "Limits of Quintessence," *Physical Review Letters*, vol. 95, no. 14, article 141301, 2005.
- [38] M. J. Amir and S. Ali, "Ricci dark energy of amended FRW universe in chern-simon modified gravity," *International Journal of Theoretical Physics*, vol. 54, no. 4, pp. 1362–1369, 2015.