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Research Article

A Tripled Fixed Point Theorem in C^*-Algebra-Valued Metric Spaces and Application in Integral Equations

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Our aim is to establish a tripled fixed and coincidence point result on generalized C^*-algebra-valued metric spaces. We present an example on matrices. At the end, we give an application on integral equations.

1. Introduction

The Banach contraction principle (BCP) was considered by Perov [1] on spaces equipped with vector-valued metrics. The result of Perov has been generalized in [2], and its related fixed point property on generalized metric spaces was investigated.

Let A be a unital algebra with the unit I and θ be its zero element. An involution on A is a conjugate linear map $\iota \mapsto \iota^*$ on A so that for all $\iota, \kappa \in A$, $\iota^{**} = \iota$ and $(\iota\kappa)^* = \kappa^*\iota^*$. The pair $(A, *)$ is named as an $*$-algebra. A Banach $*$-algebra is an $*$-algebra A with the complete submultiplicative norm so that $\|\iota^*\| = \|\iota\|$ for all $\iota \in A$. A C^*-algebra is a Banach $*$-algebra such that $\|\iota^*\iota\| = \|\iota^2\|$ for all $\iota \in A$. Let H be a Hilbert space and $B(H)$ be the family of all bounded linear operators on H; then, $B(H)$ is a C^*-algebra with the operator norm. Let A_{sa} be the family of all self-adjoint elements in A, and define the spectrum of $\iota \in A$ as $\sigma(\iota) = \{\lambda \in C : \lambda I - \iota \text{ is not invertible}\}$. An element $\iota \in A$ is positive (denoted by $\iota \geq \theta$) if $\iota \in A_{sa}$ and $\sigma(\iota) \subseteq \mathbb{R}_+$. Take $A_+ = \{\iota \in A : \iota \geq \theta\}$, then $A_+ = \{\iota^*\iota : \iota \in A\}$ (see [3]). One can define a partial ordering $\preceq$ on A_{sa} as $\iota \preceq \kappa$ iff $\kappa - \iota \succeq \theta$. If $\iota, \kappa \in A_{sa}$ and $c \in A$, then $\iota \leq \kappa \Rightarrow c^*\iota c \leq c^*\kappa c$, and if $\iota, \kappa \in A_+$ are invertible, then $\iota \leq \kappa \Longrightarrow \theta \leq \kappa^{-1} \leq \iota^{-1}$.

Definition 1 (see [4]). Let X be a nonempty set. If the function $\mho : X \times X \rightarrow A$ is so that for all $\upsilon, \tau, \eta \in X$:

(i) $\theta \leq \mho(\upsilon, \tau)$ and $\mho(\upsilon, \tau) = \theta$ iff $\upsilon = \tau$

(ii) $\mho(\upsilon, \tau) = f(\tau, \upsilon)$

(iii) $\mho(\upsilon, \tau) \leq \mho(\upsilon, \eta) + \mho(\eta, \tau)$

then $(X, A, \mho)$ is named as a C^*-algebra-valued metric space.

In this article, denote by $M_{q,q}(A)$ the set of all $q \times q$ matrices with coefficients in A. Note that $\Theta =$ the zero matrix and $I =$ the identity matrix.

Let $\mathscr{A} \in M_{q,q}(A)$, then A is said to be convergent to zero, iff A^n goes to θ as $n \longrightarrow \infty$. See [5–8] for more details.

Denote by ZM the family of all matrices $\mathscr{A} \in M_{q,q}(A)$ so that $A^n \longrightarrow \theta$. We provide the following examples.

Example 1.

$$\mathscr{A} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\quad \mathscr{B} = \frac{1}{6}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{1}$$

are in ZM. We have $(\mathscr{A}+\mathscr{B})^2(I-A^2)^{-1} \in ZM$.

Example 2.

$$\mathscr{A} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix},\quad \mathscr{B} = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix} \tag{2}$$

are in ZM. Clearly, $(\mathscr{A}+\mathscr{B})^2(I-A^2)^{-1} \in ZM$.

Example 3. $A = \alpha I$ and $B = ((I-\alpha)^3 - \alpha)I$ are in ZM. Then, for $\alpha \in \{1/4, 1/5, 1/7, 1/8\}$, one gets $(\mathscr{A}+\mathscr{B})^2(I-A^2)^{-1} \in ZM$.

Definition 2 (see [9]). An element $(\ell^1, \ell^2) \in X^2$ is named to be a *coupled fixed point* of $F : X^2 \to X$ if $F(\ell^1, \ell^2) = \ell^1$ and $F(\ell^2, \ell^1) = \ell^2$.

Definition 3 (see [10]; see also [11]). Given $F : X^2 \to X$ and $g : X \to X$. An element $(\ell^1, \ell^2) \in X^2$ so that $F(\ell^1, \ell^2) = g\ell^1$ and $F(\ell^2, \ell^1) = g\ell^2$ is named as a *coupled coincidence point* of F and g. $(g\ell^1, g\ell^2)$ is called a *coupled point of coincidence*.

Definition 4 (see [12]). An element $(\ell^1, \ell^2, \ell^3) \in X^3$ is named to be a *tripled fixed point* of $F : X^3 \to X$ if $F(\ell^1, \ell^2, \ell^3) = \ell^1$, $F(\ell^2, \ell^1, \ell^2) = \ell^2$ and $F(\ell^3, \ell^2, \ell^1) = \ell^3$.

In this manuscript, we investigate a tripled common fixed point result for a sequence of mappings $T_n : X^3 \to X$ and $g : X \to X$ in the class of complete C^*-algebra-valued metric spaces. An example and an application are presented.

2. Main Results

Our main result is as follows.

Theorem 5. *Let $(X, A, \mho)$ be a complete C^* -algebra-valued metric space. Let $g : X \to X$ and $\{T_i\}_{i\geq 0}$ be a sequence of mappings from X^3 into X so that*

$$\begin{aligned}\mho(T_i(\ell^1, \ell^2, \ell^3), T_j(u^1, u^2, u^3)) &\leq \mathscr{A}[\mho(g(\ell^1), T_i(\ell^1, \ell^2, \ell^3)) \\ &\quad + \mho(gu^1, T_j(u^1, u^2, u^3))]\mathscr{A}^* + \mathscr{B}(\mho(gu^1, g\ell^1))\mathscr{B}^*,\end{aligned} \tag{3}$$

where $I \neq \mathscr{A} = (a_{ij}) \in M_{q,q}(A)$, $I \neq \mathscr{B} = (b_{ij}) \in M_{q,q}(A_+)$ with $(\mathscr{A}+\mathscr{B})^2(I-\mathscr{A}^2)^{-1} \in ZM$. If $T_i(X^3) \subseteq g(X)$ and $g(X)$ is complete in X, then $\{T_i\}_{i\geq 0}$ and g have a tripled coincidence point. Further, if $\{T_i\}_{i\geq 0}$ and g are w-compatible, then they have a unique tripled common fixed point in X.

Proof. Take $\ell_0^1, \ell_0^2, \ell_0^3 \in X$, and let

$$\begin{aligned} g\ell_1^1 &= T_0(\ell_0^1, \ell_0^2, \ell_0^3), \\ g\ell_1^2 &= T_0(\ell_0^2, \ell_0^1, \ell_0^2), \\ g\ell_1^3 &= T_0(\ell_0^3, \ell_0^2, \ell_0^1). \end{aligned} \tag{4}$$

Continuing this technique, we get

$$\begin{aligned} g\ell_{n+1}^1 &= T_n(\ell_n^1, \ell_n^2, \ell_n^3), \\ g\ell_{n+1}^2 &= T_n(\ell_n^2, \ell_n^1, \ell_n^2), \\ g\ell_{n+1}^3 &= T_n(\ell_n^3, \ell_n^2, \ell_n^1), \\ &\text{for all } n \geq 0. \end{aligned} \tag{5}$$

By (3), we get

$$\begin{aligned} \mho(g\ell_n^1, \ell_{n+1}^1) &= \mho(T_{n-1}(\ell_{n-1}^1, \ell_{n-1}^2, \ell_{n-1}^3), T_{n-1}(\ell_n^1, \ell_n^2, \ell_n^3)) \\ &\leq \mathscr{A}[\mho(g\ell_{n-1}^1, T_{n-1}(\ell_{n-1}^1, \ell_{n-1}^2, \ell_{n-1}^3)) \\ &\quad + \mho(g\ell_{n-1}^1, T_n(\ell_n^1, \ell_n^2, \ell_n^3))\mathscr{A}^*] \\ &\quad + \mathscr{B}(\mho(g\ell_n^1, g\ell_{n-1}^1)\mathscr{B}^* = \mathscr{A}[\mho(g\ell_{n-1}^1, g\ell_n^1) \\ &\quad + \mho(g\ell_n^1, g\ell_{n+1}^1)]\mathscr{A}^* + \mathscr{B}\mho(g\ell_n^1, g\ell_{n-1}^1)\mathscr{B}^* \\ &= \mathscr{A}\mho(g\ell_{n-1}^1, g\ell_n^1)\mathscr{A}^* + \mathscr{A}\mho(g\ell_n^1, g\ell_{n+1}^1)\mathscr{A}^* \\ &\quad + \mathscr{B}\mho(g\ell_n^1, g\ell_{n-1}^1)\mathscr{B}^*) = \mathscr{A}\mho(g\ell_{n-1}^1, g\ell_n^1)\mathscr{A}^* \\ &\quad + \mathscr{A}\mho(g\ell_n^1, g\ell_{n+1}^1)^{1/2}\mho(g\ell_n^1, g\ell_{n+1}^1)^{1/2}\mathscr{A}^* \\ &\quad + \mathscr{B}\mho(g\ell_n^1, g\ell_{n-1}^1)\mathscr{B}^* = \mathscr{A}\mho(g\ell_{n-1}^1, g\ell_n^1)\mathscr{A}^* \\ &\quad + \mathscr{A}\left|\mho(g\ell_n^1, g\ell_{n+1}^1)^{1/2}\right|^2\mathscr{A}^* + \mathscr{B}\mho(g\ell_n^1, g\ell_{n-1}^1)\mathscr{B}^* \\ &= (\mathscr{A}+\mathscr{B})\mho(g\ell_{n-1}^1, g\ell_n^1)^{1/2}\mho(g\ell_{n-1}^1, g\ell_n^1)^{1/2}(\mathscr{A}+\mathscr{B})^* \\ &\quad + \mathscr{A}\left|\mho(g\ell_n^1, g\ell_{n+1}^1)^{1/2}\right|^2\mathscr{A}^* = \left|(\mathscr{A}+\mathscr{B})\mho(g\ell_{n-1}^1, g\ell_n^1)^{1/2}\right|^2 \\ &\quad + \mathscr{A}\left|\mho(g\ell_n^1, g\ell_{n+1}^1)^{1/2}\right|^2\mathscr{A}^*. \end{aligned} \tag{6}$$

It follows that

$$\mho(g\ell_n^1, g\ell_{n+1}^1) \leq (\mathscr{A}+\mathscr{B})^2(I-\mathscr{A}^2)^{-1}\mho(g\ell_{n-1}^1, g\ell_n^1). \tag{7}$$

Similarly,

$$\mho(g\ell_n^1, g\ell_{n+1}^1) \leq (\mathscr{A}+\mathscr{B})^2(I-\mathscr{A}^2)^{-1}\mho(g\ell_{n-1}^1, g\ell_n^1), \tag{8}$$

$$\mho\left(g\ell_n^3, g\ell_{n+1}^3\right) \leq (\mathscr{A}+\mathscr{B})^2\left(I-\mathscr{A}^2\right)^{-1}\mho\left(g\ell_{n-1}^3, g\ell_n^3\right). \tag{9}$$

Adding (7), (8), and (9), we have

$$\begin{aligned}\delta_n &:= \mho\left(g\ell_n^1, g\ell_{n+1}^1\right)+\mho\left(g\ell_n^2, g\ell_{n+1}^2\right)+\mho\left(g\ell_n^3, g\ell_{n+1}^3\right)\\ &\leq (\mathscr{A}+\mathscr{B})^2\left(I-\mathscr{A}^2\right)^1\left[\mho\left(g\ell_{n-1}^1, g\ell_n^1\right)+\mho\left(g\ell_{n-1}^2, g\ell_n^2\right)\right.\\ &\quad\left.+\mho\left(g\ell_{n-1}^3, g\ell_n^3\right)\right] = \left((\mathscr{A}+\mathscr{B})^2\left(I-\mathscr{A}^2\right)^{-1}\right)\delta_{n-1}.\end{aligned} \tag{10}$$

Put $C=(\mathscr{A}+\mathscr{B})^2(I-\mathscr{A}^2)^{-1}$. Then, for all $n \geq 2$,

$$\Theta \leq \delta_n \leq C\delta_{n-1} \leq C^2\delta_{n-2} \leq \cdots \leq C^n\delta_0. \tag{11}$$

Using the triangle inequality, for all $p \geq 1$,

$$\begin{aligned}&\mho\left(g\ell_n^1, g\ell_{n+p}^1\right)+\mho\left(g\ell_n^2, g\ell_{n+p}^2\right)+\mho\left(g\ell_n^3, g\ell_{n+p}^3\right)\\ &\leq \mho\left(g\ell_n^1, g\ell_{n+1}^1\right)+\mho\left(g\ell_n^2, g\ell_{n+1}^2\right)+\mho\left(g\ell_n^3, g\ell_{n+1}^3\right)\\ &\quad+\mho\left(g\ell_{n+1}^1, g\ell_{n+2}^1\right)+\mho\left(g\ell_{n+1}^2, g\ell_{n+2}^2\right)+\mho\left(g\ell_{n+1}^3, g\ell_{n+2}^3\right)\\ &\quad+\cdots+\mho\left(g\ell_{n+p-1}^1, g\ell_{n+p}^1\right)+\mho\left(g\ell_{n+p-1}^2, g\ell_{n+p}^2\right)\\ &\quad+\mho\left(g\ell_{n+p-1}^3, g\ell_{n+p}^3\right)=\delta_n+\delta_{n+1}+\cdots\delta_{n+p-1}\\ &\leq \left(C^n+C^{n+1}+\cdots+C^{n+p-1}\right)\delta_0 \leq C^n\left(I+C+\cdots+C^{p-1}+\cdots\right)\delta_0\\ &= C^n(I-C)^{-1}\delta_0.\end{aligned} \tag{12}$$

We have

$$\begin{aligned}&\left\|\mho\left(g\ell_n^1, g\ell_{n+p}^1\right)+\mho\left(g\ell_n^2, g\ell_{n+p}^2\right)+\mho\left(g\ell_n^3, g\ell_{n+p}^3\right)\right\|\\ &\leq \left\|C^n(I-C)^{-1}\right\|\delta_0 = \left\|\left((\mathscr{A}+\mathscr{B})^2\left(I-\mathscr{A}^2\right)^{-1}\right)^n\right.\\ &\quad\left.\cdot\left(I-(\mathscr{A}+\mathscr{B})^2\left(I-\mathscr{A}^2\right)^{-1}\right)^{-1}\right\|\delta_0.\end{aligned} \tag{13}$$

Now, taking the limit as $n \to +\infty$, we conclude

$$\begin{aligned}&\left\|\mho\left(g\ell_n^1, g\ell_{n+p}^1\right)+\mho\left(g\ell_n^2, g\ell_{n+p}^2\right)+\mho\left(g\ell_n^3, g\ell_{n+p}^3\right)\right\|\\ &\leq \left\|\left((\mathscr{A}+\mathscr{B})^2\left(I-\mathscr{A}^2\right)^{-1}\right)^n\left(I-(\mathscr{A}+\mathscr{B})^2\left(I-\mathscr{A}^2\right)^{-1}\right)^{-1}\right\|\\ &\quad\cdot\delta_0 \to 0.\end{aligned} \tag{14}$$

This implies that

$$\begin{aligned}\left\|\mho\left(g\ell_n^1, g\ell_{n+p}^1\right)\right\| &= \left\|\mho\left(g\ell_n^2, g\ell_{n+p}^2\right)\right\|\\ &= \left\|\mho\left(g\ell_n^3, g\ell_{n+p}^3\right)\right\| = 0,\end{aligned} \tag{15}$$

and $\{g\ell_n^1\}$, $\{g\ell_n^2\}$, and $\{g\ell_n^3\}$ are Cauchy sequences in $g(X)$, which is complete; there are $(\wp^1, \wp^2, \wp^3) \in X^3$ so that

$$\begin{aligned}\lim_{n\to+\infty}\left\{g\ell_n^1\right\} &= g\wp^1 := \ell^1,\\ \lim_{n\to+\infty}\left\{g\ell_n^2\right\} &= g\wp^2 := \ell^2,\\ \lim_{n\to+\infty}\left\{g\ell_n^3\right\} &= g\wp^3 := \ell^3.\end{aligned} \tag{16}$$

We have

$$\begin{aligned}\mho\left(T_i\left(\ell^1,\ell^2,\ell^3\right), g\ell^1\right) &\leq \mho\left(T_i\left(\ell^1,\ell^2,\ell^3\right), g\ell_{n+1}^1\right)+\mho\left(g\ell_{n+1}^1, g\ell^1\right)\\ &= \mho\left(T_i\left(\ell^1,\ell^2,\ell^3\right), T_n\left(g\ell_n^1, g\ell_n^2, g\ell_n^3\right)\right)+\mho\left(g\ell_{n+1}^1, g\ell^1\right)\\ &\leq \mathscr{A}\left[\mho\left(g\ell^1, T_i\left(\ell^1,\ell^2,\ell^3\right)\right)+\mho\left(g\ell_n^1, T_n\left(g\ell_n^1, g\ell_n^2, g\ell_n^3\right)\right)\right]\mathscr{A}^*\\ &\quad+\mathscr{B}\mho\left(g\ell_n^1, g\ell^1\right)\mathscr{B}^*+\mho\left(g\ell_{n+1}^1, g\ell^1\right).\end{aligned} \tag{17}$$

Taking the limit as $n \to +\infty$ in the above relation, we obtain $g\ell^1 = T_i(\ell^1, \ell^2, \ell^3)$. Similarly, $g\ell^2 = T_i(\ell^2, \ell^1, \ell^2)$ and $g\ell^3 = T_i(\ell^3, \ell^2, \ell^1)$. Therefore, (ℓ^1, ℓ^2, ℓ^3) is a tripled coincidence point of $\{T_i\}_{i\in\mathbb{N}}$ and g.

Let (ℓ^1, ℓ^2, ℓ^3) and $(\wp^1, \wp^2, \wp^3)$ be tripled coincidence points, then

$$\begin{aligned}\mho\left(g\ell^1, g\wp^1\right) &= \mho\left(T_i\left(\ell^1,\ell^2,\ell^3\right), T_j\left(\wp^1,\wp^2,\wp^3\right)\right)\\ &\leq \mathscr{A}\left[\mho\left(g\ell^1, T_i\left(\ell^1,\ell^2,\ell^3\right)\right)+\mho\left(g\wp^1, T_j\left(\wp^1,\wp^2,\wp^3\right)\right)\right]\mathscr{A}^*\\ &\quad+\mathscr{B}\mho\left(g\wp^1, g\ell^1\right)\mathscr{B}^*.\end{aligned} \tag{18}$$

That is,

$$\begin{aligned}\left(I-\mathscr{B}^2\right)\mho\left(g\ell^1, g\wp^1\right) &\leq \mathscr{A}^2\left[\mho\left(g\ell^1, T_i\left(\ell^1,\ell^2,\ell^3\right)\right)\right.\\ &\quad\left.+\mho\left(g\wp^1, T_j\left(\wp^1,\wp^2,\wp^3\right)\right)\right],\end{aligned} \tag{19}$$

so

$$\begin{aligned}\mho\left(g\ell^1, g\wp^1\right) &\leq \mathscr{A}^*\left(I-\mathscr{B}^2\right)^{-1}\left[\mho\left(g\ell^1, T_i\left(\ell^1,\ell^2,\ell^3\right)\right)\right.\\ &\quad\left.+\mho\left(g\wp^1, T_j\left(\wp^1,\wp^2,\wp^3\right)\right)\right],\end{aligned} \tag{20}$$

which further induces that

$$\begin{aligned}\left\|\mho\left(g\ell^1, g\wp^1\right)\right\| &\leq \left\|\mathscr{A}^2\right\|\left\|\left(I-\mathscr{B}^2\right)^{-1}\right\|\left\|\mho\left(g\ell^1, T_i\left(\ell^1,\ell^2,\ell^3\right)\right)\right.\\ &\quad\left.+\mho\left(g\wp^1, T_j\left(\wp^1,\wp^2,\wp^3\right)\right)\right\|.\end{aligned} \tag{21}$$

Therefore, $\mho(g\ell^1, g\wp^1) = \Theta$, that is, $g\ell^1 = g\wp^1$. Similarly, we can prove that $g\ell^2 = g\wp^2$ and $g\ell^3 = g\wp^3$. So, $g\ell^1 = g\ell^2 = g\ell^3 = g\wp^1 = g\wp^2 = g\wp^3$. Therefore, $\{T_i\}_{i\in\mathbb{N}}$ and g have a unique tripled coincidence point. $(g\ell^1, g\ell^1, g\ell^1)$. Now, set $g\ell^1 = u$, then $u = g\ell^1 = T_i(\ell^1, \ell^1, \ell^1)$. By w-compatibility of $\{T_i\}_{i\in\mathbb{N}}$ and g,

$$\begin{aligned}gu = gg\ell^1 &= g\left(T_i\left(\ell^1,\ell^1,\ell^1\right)\right) = T_i\left(g\ell^1, g\ell^1, g\ell^1\right)\\ &= T_i(u,u,u) = g\ell^1.\end{aligned} \tag{22}$$

Then, (gu, gu, gu) is a tripled coincidence point of $\{T_i\}_{i\in\mathbb{N}}$ and g. By the uniqueness, we know $gu = g\ell^1$, which yields that $u = gu = T_i(u, u)$. Hence, (u, u, u) is a unique tripled common fixed point of $\{T_i\}_{i\in\mathbb{N}}$ and g.

Letting $g = Id_X$ in Theorem 5, we have the following.

Corollary 6. *Let $(X, A, \mho)$ be a complete generalized C^*-algebra-valued metric space. Suppose that $\{T_i\}_{i\geq 0}$ is a sequence of mappings from X^3 into X so that*

$$\begin{aligned}\mho\left(T_i\left(\ell^1,\ell^2,\ell^3\right), T_j\left(u^1,u^2,u^3\right)\right) \leq \mathscr{A}\left[\mho\left(\ell^1, T_i\left(\ell^1,\ell^2,\ell^3\right)\right)\right.\\ \left. + \mho\left(u^1, T_j\left(u^1,u^2,u^3\right)\right)\right]\mathscr{A}^* + \mathscr{B}\mho\left(u^1,\ell^1\right)\mathscr{B}^*,\end{aligned} \tag{23}$$

where $\mathscr{A}, \mathscr{B} \in A$ with $I \neq \mathscr{A} = (a_{ij})$, $I \neq \mathscr{B} = (b_{ij}) \in M_{m,m}(A_+)$, $(\mathscr{A} + \mathscr{B})^2(I - \mathscr{A}^2)^{-1} \in ZM$. Then, $\{T_i\}_{i\geq 0}$ has a unique tripled fixed point.

Example 4. Take $X = [0, 1]$. Given

$$\mho\left(\ell^1,\ell^2\right) = \begin{pmatrix} \left|\ell^1-\ell^2\right| & 0 \\ 0 & \left|\ell^1-\ell^2\right| \end{pmatrix}. \tag{24}$$

Then, $(X, A, \mho)$ is a complete generalized C^*-algebra-valued metric space.

Consider $T_i : X^3 \to X$ and $g : X \to X$ as

$$\begin{aligned} T_i\left(\ell^1,\ell^2,\ell^3\right) &= \frac{\ell^1,\ell^2,\ell^3}{3^i}, \\ g\left(\ell^1\right) &= 9\ell^1. \end{aligned} \tag{25}$$

Choose

$$\begin{aligned} \mathscr{A} &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \\ \mathscr{B} &= \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}. \end{aligned} \tag{26}$$

By induction, (3) holds for all $\ell^1, \ell^2, \ell^3 \in X$. Set $x = (\ell^1 + \ell^2 + \ell^3)/3$ and $u = (u^1, u^2, u^3)/3^j$. Here, for $i = 1$, we have

$$\begin{aligned}\begin{pmatrix} |x-u| & 0 \\ 0 & |x-u| \end{pmatrix} &\leq \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \left|9\ell^1-x\right| + \left|9u^1-u\right| & 0 \\ 0 & \left|9\ell^1-x\right| + \left|9u^1-u\right| \end{pmatrix} \\ &\cdot \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} \left|9\left(u^1-\ell^1\right)\right| & 0 \\ 0 & \left|9\left(u^1-\ell^1\right)\right| \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}.\end{aligned} \tag{27}$$

Also,

$$\begin{aligned}\alpha &:= \begin{pmatrix} \left|x-\frac{1}{3}u\right| & 0 \\ 0 & \left|x-\frac{1}{3}u\right| \end{pmatrix} \leq \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \\ &\cdot \begin{pmatrix} \left|9\ell^1-x\right| + \left|3u^1-\frac{1}{3}u\right| & 0 \\ 0 & \left|9\ell^1-x\right| + \left|3u^1-\frac{1}{3}u\right| \end{pmatrix} \\ &\cdot \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} + \sqrt{9}\begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} \left|\left(\frac{u^1}{3}-\ell^1\right)\right| \\ \left|\left(\frac{u^1}{3}-\ell^1\right)\right| \end{pmatrix} \\ &+ \sqrt{9}\begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix} := \beta.\end{aligned} \tag{28}$$

So,

$$\begin{aligned}\alpha &\leq \begin{pmatrix} |x-u| & 0 \\ 0 & |x-u| \end{pmatrix} + \frac{2}{3}\begin{pmatrix} |x| & 0 \\ 0 & |x| \end{pmatrix} \leq \frac{1}{3}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \\ &\cdot \begin{pmatrix} \left|9\ell^1-x\right| + \left|9u^1-u\right| & 0 \\ 0 & \left|9\ell^1-x\right| + \left|9u^1-u\right| \end{pmatrix} \\ &\cdot \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} \left|u^1-\ell^1\right| & 0 \\ 0 & \left|u^1-\ell^1\right| \end{pmatrix} \\ &\cdot \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} |x| & 0 \\ 0 & |x| \end{pmatrix} \leq \beta.\end{aligned} \tag{29}$$

Clearly, g and $\{T_i\}_{i\geq 0}$ are w-compatible. Therefore, all conditions in Theorem 5 hold, and $(0, 0, 0)$ is the unique tripled common fixed point of g and $\{T_i\}_{i\geq 0}$.

3. Application

Consider the following sequence of the integral equations:

$$\begin{aligned} x(t) = \int_E &\left(\Upsilon_1(r,t,s) + \Upsilon_2(r,t,s)\right)\left(f_i(t,s,x(s)) + g_i(t,s,x(s))\right. \\ &+ h_i(t,s,x(s))ds + h(t), \end{aligned} \tag{30}$$

for all $r, t, s \in E$, where E is a Lebesgue measurable set and $m(E) < \infty$.

Denote by $X = L^{\infty}(E)$ the set of essentially bounded measurable functions on E. We consider the following assumptions:

(i) $f_i, g_i, h_i : E \times E \times \mathbb{R} \to \mathbb{R}$, $\Upsilon_1 : E \times E \times E \to 0,\infty)$, $\Upsilon_2 : E \times E \times E \to -\infty,0)$ are integrable, and $h \in L^{\infty}(E)$

(ii) There is $k \in (0, 1/2)$ so that for all $\ell^1, \ell^2 \in \mathbb{R}$

$$\begin{aligned} &0 \le \left|f_i\left(t, s, \ell^1(s)\right) - f_i\left(t, s, \ell^2(s)\right)\right| \le k\left(\ell^1 - \ell^2\right), \\ &-k\left(\ell^1 - \ell^2\right) \le \left|g_i\left(t, s, \ell^1(s)\right) - g_i\left(t, s, \ell^2(s)\right)\right| \le 0, \\ &0 \le \left|h_i\left(t, s, \ell^1(s)\right) - h_i\left(t, s, \ell^2(s)\right)\right| \le k\left(\ell^1 - \ell^2\right), \end{aligned} \tag{31}$$

for all $s, t \in E$ with

$$\begin{aligned} k &\le \mathscr{A}^2 = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}, \\ k &\le \mathscr{B}^2 = \begin{pmatrix} 0 & \frac{1}{9} \\ \frac{1}{9} & 0 \end{pmatrix} \end{aligned} \tag{32}$$

(iii) $\sup_{s,t\in E}\int_E (\Upsilon_1(r,s,t) - \Upsilon_2(r,s,t)ds \le 1$

Theorem 7. *Suppose that assumptions (i)–(iii) hold. Then, (30) has a unique solution in $L^{\infty}(E)$.*

Proof. Let $X = L^{\infty}(E)$ and $B(L^2(E))$ be the set of bounded linear operators on the Hilbert space $L^2(E)$. We endow X with the cone metric $\mho : X \times X \to B(L^2(E))$ defined by $\mho(f, g) = M_{|f-g|}$, where $M_{|f-g|}$ is the multiplication operator on $L^2(E)$. It is clear that $(X, B(L^2(E)), \mho)$ is a complete C^*-algebra-valued metric space. Define the self-mapping $T_i : X^3 \to X$ by

$$\begin{aligned} T\left(\ell^1, \ell^2, \ell^3\right)(t) = T_i\left(\ell^1, \ell^2, \ell^3\right)(t) = \int_E (\Upsilon_1(r,s,t) + \Upsilon_2(r,s,t)) \\ \cdot \left(f_i\left(t, s, \ell^1(s)\right) + g_i\left(t, s, \ell^2(s)\right) + h_i\left(t, s, \ell^3(s)\right)\right)ds + h(t), \end{aligned} \tag{33}$$

for all $\ell^1, \ell^2, \ell^3 \in X$ and $r, s, t \in E$.

Now, we have

$$\mho\left(T_i\left(\ell^1, \ell^2, \ell^3\right), T_j\left(u^1, u^2, u^3\right)\right) = M_{\left|T_i\left(\ell^1,\ell^2,\ell^3\right), T_j\left(u^1,u^2,u^3\right)\right|}. \tag{34}$$

Using (31), we have

$$\begin{aligned} &\left|T_i\left(\ell^1, \ell^2, \ell^3\right)(t) - T_j\left(u^1, u^2, u^3\right)(t)\right| = \left|\int_E (\Upsilon_1(r,s,t) + \Upsilon_2(r,s,t))\right. \\ &\quad \cdot \left(f_i\left(t, s, \ell^1(s)\right)\right) + g_i\left(t, s, \ell^2(s)\right) + h_i\left(t, s, \ell^3(s)\right)\big)ds \\ &\quad - \int_E (\Upsilon_1(r,s,t) + \Upsilon_2(r,s,t))\left(f_i\left(t, s, u^1(s)\right)\right) + g_i\left(t, s, u^2(s)\right) \\ &\quad + h_i\left(t, s, u^3(s)\right)\big)ds\Big| = \left|\int_E (\Upsilon_1(r,s,t) + \Upsilon_2(r,s,t))\left(f_i\left(t, s, \ell^1(s)\right)\right.\right. \\ &\quad - f_i\left(t, s, u^1(s)\right) + g_i\left(t, s, \ell^2(s)\right) - g_i\left(t, s, u^2(s)\right) + h_i\left(t, s, \ell^3(s)\right) \\ &\quad - h_i\left(t, s, u^3(s)\right)\big)ds\Big| \le \int_E |(\Upsilon_1(r,s,t) + \Upsilon_2(r,s,t))| \\ &\quad \cdot \left|\left(f_i\left(t, s, \ell^1(s)\right) - f_i\left(t, s, u^1(s)\right)\right) + g_i\left(t, s, \ell^2(s)\right) - g_i\left(t, s, u^2(s)\right)\right) \\ &\quad + h_i\left(t, s, \ell^3(s)\right) - h_i\left(t, s, u^3(s)\right)\big)|ds \le \sup_{s,t\in E}\int_E |(\Upsilon_1(r,s,t) + \Upsilon_2(r,s,t))| \\ &\quad \cdot ds.k\left(\left|\ell^1 - u^1\right| + \left|\ell^2 - u^2\right| + \left|\ell^3 - u^3\right|\right) \\ &\le k\left(\left\|\ell^1 - u^1\right\|_{\infty} + \left\|\ell^2 - u^2\right\|_{\infty} + \left\|\ell^3 - u^3\right\|_{\infty}\right), \end{aligned} \tag{35}$$

for all $r, s, t \in E$.

Therefore, for any $\varphi \in L^2(E)$, we have

$$\begin{aligned} &\left\|T_i\left(\ell^1, \ell^2, \ell^3\right) - T_j\left(u^1, u^2, u^3\right)\right\| = \left\|M_{\left|T_i\left(\ell^1,\ell^2,\ell^3\right) - T_j\left(u^1,u^2,u^3\right)\right|}\right\| \\ &= \sup_{\|\varphi\|=1}\left(M_{\left|T_i\left(\ell^1,\ell^2,\ell^3\right) - T_j\left(u^1,u^2,u^3\right)\right|}\right) = \sup_{\|\varphi\|=1}\int_E \left|T_i\left(\ell^1, \ell^2, \ell^3\right)\right. \\ &\quad - T_j\left(u^1, u^2, u^3\right)(t)|\varphi(t)\bar{\varphi}(t)dt \le \sup_{\|\varphi\|=1}\int_E |\varphi(t)|^2 dt \\ &\quad \cdot \left(k\left(\left\|\ell^1 - u^1\right\|_{\infty} + \left\|\ell^2 - u^2\right\|_{\infty} + \left\|\ell^3 - u^3\right\|_{\infty}\right)\right) \\ &\le k\left(\left\|\ell^1 - u^1\right\|_{\infty} + \left\|\ell^2 - u^2\right\|_{\infty} + \left\|\ell^3 - u^3\right\|_{\infty}\right) \\ &\le |\mathscr{B}|^2\left(\left\|\ell^1 - u^1\right\|_{\infty} + \left\|\ell^2 - u^2\right\|_{\infty} + \left\|\ell^3 - u^3\right\|_{\infty}\right) \\ &= \mathscr{B}\left(\left\|\ell^1 - u^1\right\|_{\infty} + \left\|\ell^2 - u^2\right\|_{\infty} + \left\|\ell^3 - u^3\right\|_{\infty}\right)\mathscr{B}*. \end{aligned} \tag{36}$$

Consequently,

$$\begin{aligned} \mho\left(T_i\left(\ell^1, \ell^2, \ell^3\right), T_j\left(u^1, u^2, u^3\right)\right) \le \mathscr{A}\left[\mho\left(\ell^1, \ell^2, \ell^3\right)\right. \\ \left. + \mho\left(u^1, T_j, \left(u^1, u^2, u^3\right)\right)\right]\mathscr{A}^* + \mathscr{B}\mho\left(u^1, \ell^1\right)\mathscr{B}^*. \end{aligned} \tag{37}$$

Hence, all hypotheses of Corollary 6 hold. Hence, (30) possesses a unique solution in $L^{\infty}(E)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no competing interest regarding the publication of this manuscript.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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