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Research Article

Solution of Space-Time-Fractional Problem by Shehu Variational Iteration Method

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In this study, we deal with the problem of constructing semianalytical solution of mathematical problems including space-time-fractional linear and nonlinear differential equations. The method, called Shehu Variational Iteration Method (SVIM), applied in this study is a combination of Shehu transform (ST) and variational iteration method (VIM). First, ST is utilized to reduce the time-fractional differential equation with fractional derivative in Liouville-Caputo sense into an integer-order differential equation. Later, VIM is implemented to construct the solution of reduced differential equation. The convergence analysis of this method and illustrated examples confirm that the proposed method is one of best procedures to tackle space-time-fractional differential equations.

1. Introduction

Last couple of decades, employing fractional differential equations in modelling of processes such as dynamical systems, biology, fluid flow, signal processing, electrical networks, reaction and diffusion procedure, and advection–diffusion–reaction process [1–4] has gained great importance since these models reflect the behaviour of the processes better than integer-order differential equations.

Consequently, a great deal of methods such as [3, 4] are established to construct analytical and numerical solutions of fractional differential equations. Moreover, their existence, uniqueness, and stability have been studied by many scientists.

One of the significant integral transformations is Shehu transformation proposed by Maitama and Zhao [5]. This linear transformation is a generalization of Laplace transformation. However, the Laplace transformation is obtained by substituting $q = 1$ in Shehu transformation. By this transformation, differential equations are reduced into simpler equations.

Various methods such as the homotopy perturbation method (HPM) and VIM are utilized to establish approximate solutions of differential equations of any kind [6, 7]. As a result, it is employed widely to deal with differential equations in various branches of science [8–11]. VIM has been modified by many researchers to improve this method. By modified VIM, the approximate solutions of initial value problems can be established by making use of an initial condition.

2. Preliminaries

In this section, preliminaries, notations, and features of the fractional calculus are given [12, 13]. Riemann-Liouville time-fractional integral of a real valued function $u(x, t)$ is defined as

$$I_t^{\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(x,s) ds, \tag{1}$$

where $\alpha > 0$ denotes the order of the integral.

The α^{th}-order Liouville-Caputo time-fractional derivative operator of $u(x, t)$ is defined as

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = I_t^{m-\alpha}\left[\frac{\partial^m u(x,t)}{\partial t^m}\right] = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-y)^{m-\alpha-1} \dfrac{\partial^m u(x,y)}{\partial y^m} dy, & m-1 < \alpha < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m. \end{cases} \tag{2}$$

The function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad Re(\alpha) > 0, \quad z, \beta \in \mathbb{C}, \tag{3}$$

is called Mittag-Leffler function depending on two parameters α and β.

The following set of functions has Shehu transformation:

$$\left\{ f(t) | \exists P, \tau_1, \tau_2 > 0, |f(t)| < Pe^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0,\infty) \right\}, \tag{4}$$

and it is defined as

$$\mathbb{S}[f(t)] = F(p, q) = \int_0^{\infty} e^{-(p/q)t} f(t) dt, \tag{5}$$

which has the following property:

$$\mathbb{S}[t^{\alpha}] = \int_0^{\infty} e^{-(pt/q)} t^{\alpha} dt = \Gamma(\alpha + 1) \left(\frac{q}{p}\right)^{\alpha+1}, \quad Re(\alpha) > 0. \tag{6}$$

The inverse Shehu inverse transform of $(q/p)^{n\alpha+1}$ is defined as

$$\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{n\alpha+1}\right] = \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \quad Re(\alpha) > 0, \tag{7}$$

where $n > 0$ [5].

For the α^{th}-order of Liouville-Caputo time-fractional derivative of $f(x, t)$, the Shehu transformation has the following form [14]:

$$\mathbb{S}\left[\frac{\partial^{\alpha} f(x, t)}{\partial t^{\alpha}}\right] = \left(\frac{p}{q}\right)^{\alpha} \mathbb{S}[f(x, t)] - \sum_{k=0}^{n-1} \left[\left(\frac{p}{q}\right)^{\alpha-k-1} \frac{\partial^k f(x, 0)}{\partial t^k}\right], \quad n - 1 < \alpha \le n, n \in \mathbb{N}. \tag{8}$$

3. Main Results

3.1. Fractional Shehu Variational Iteration Method. To reveal the fundamental notions of this method, let us take the following space-time-fractional initial value problem in the Liouville-Caputo fractional derivative:

$$^{C}D_t^{\alpha} u(x, t) + R\left(u, {}^{C}D_x^{\beta} u\,; x, t\right) + N\left(u, {}^{C}D_x^{\beta} u\,; x, t\right) = g(x, t), \quad m - 1 < \alpha \le m, n - 1 < \beta \le n, m, n = 1, 2, 3, \cdots, \tag{9}$$

$$\left[\frac{\partial^{m-1} u(x, t)}{\partial t^{m-1}}\right]_{t=0} = g_{m-1}(x), \tag{10}$$

where N, R, and $g(x, t)$ denote the nonlinear, linear part of the differential equation, and the source function, respectively.

Utilizing Shehu transformation for Equation (9), we have

$$\mathbb{S}[u(x, t)] = \sum_{k=0}^{m-1} \left[\left(\frac{q}{p}\right)^{k+1} \frac{\partial^k u(x, 0)}{\partial t^k}\right] - \left(\frac{q}{p}\right)^{\alpha} \mathbb{S}\left[R\left(u, {}^{C}D_x^{\beta} u\,; x, t\right) + N\left(u, {}^{C}D_x^{\beta} u\,; x, t\right)\right] + \left(\frac{q}{p}\right)^{\alpha} \mathbb{S}[g(x, t)]. \tag{11}$$

Employing the inverse Shehu transformation for Equation (11) leads to

$$u(x, t) = k(x, t) - \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\left[\mathbb{S}\left[R\left(u, {}^{C}D_x^{\beta} u\,; x, t\right) + N\left(u, {}^{C}D_x^{\beta} u\,; x, t\right)\right]\right]\right], \tag{12}$$

where $k(x, t) = \mathbb{S}^{-1}[(q/p)^{\alpha}[\mathbb{S}[\sum_{k=0}^{m-1}[(p/q)^{k+1}(\partial^k u(x, 0)/\partial t^k)]]] + (q/p)^{\alpha}\mathbb{S}[g(x, t)]]$, and so

$$\frac{\partial u(x, t)}{\partial t} + \frac{\partial}{\partial t}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\left[\mathbb{S}\left[R\left(u, {}^{C}D_x^{\beta} u\,; x, t\right) + N\left(u, {}^{C}D_x^{\beta} u\,; x, t\right)\right]\right]\right] - \frac{\partial}{\partial t}k(x, t) = 0. \tag{13}$$

The following recurrence relation is established by VIM:

$$u_{m+1}(x, t) = u_m(x, t) - \int_0^t \left[\frac{\partial u_m(x, \tau)}{\partial \tau} + \frac{\partial}{\partial \tau}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\left[\mathbb{S}\left[R\left(u_m, {}^{C}D_x^{\beta} u_m\,; x, \tau\right) + N\left(u_m, {}^{C}D_x^{\beta} u_m\,; x, \tau\right)\right]\right]\right] - \frac{\partial}{\partial \tau}k(x, \tau)\right] d\tau. \tag{14}$$

Alternately,

$$u_{m+1}(x, t) = k(x, t\,; \beta) - \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\left[\mathbb{S}\left[R\left(u_m, {}^{C}D_x^{\beta} u_m\,; x, t\right) + N\left(u_m, {}^{C}D_x^{\beta} u_m\,; x, t\right)\right]\right]\right] \tag{15}$$

is called the $(m + 1)^{\text{th}}$-order of truncated solution.

Table 1: Comparison of the exact solution with the truncated solutions by SVIM for various β and α for Example 1.

t	x	$\alpha = 0.5$, $\beta = 0.5$, u_{our}	$\alpha = 0.75$, $\beta = 0.75$, u_{our}	$\alpha = 1$, $\beta = 1$, u_{our}	u_{exact}
	-5	-0,0728794142905183	-0,0556900475433218	0,00822974704902003	0,00822974704902003
0,2	0	0,464049675672513	0,865372126175867	1,22140275816017	1,22140275816017
	5	272,995384456584	235,378704685654	181,272241875151	181,272241875151
	-5	-0,102568289434788	-0,0717844261439092	0,0100518357446336	0,0100518357446336
0,4	0	0,653089516564494	1,11546396921542	1,49182469764127	1,49182469764127
	5	384,205469814638	303,402959554171	221,406416204187	221,406416204187
	-5	-0,140869106433447	-0,0919274211102744	0,0122773399030684	0,0122773399030684
0,6	0	0,896964716156101	1,42846758746576	1,82211880039051	1,82211880039051
	5	527,674601164535	388,539034541079	270,426407426153	270,426407426153
	-5	-0,190052579642384	-0,117094221086887	0,0149955768204777	0,0149955768204777
0,8	0	1,21013373669836	1,81953651567707	2,22554092849247	2,22554092849247
	5	711,908534824601	494,908647082167	330,299559909649	330,299559909649
	-5	-0,252961550186961	-0,148489952829492	0,0183156388887342	0,0183156388887342
1	0	1,61069797918433	2,30739731539736	2,71828182845905	2,71828182845905
	5	947,556128411521	627,605367523717	403,428793492735	403,428793492735

If

$$\lim_{m\to\infty} u_m(x,t) \tag{16}$$

exists, then the analytical solution $u(x,t) = \lim_{m\to\infty} u_m(x,t)$.

3.2. Convergence Theorem. Now, the convergence of VIM is investigated and required conditions and error estimate [15] are established for Equation (9).

The operator V is introduced as

$$V = -\int_0^t \left[\frac{\partial u_m(x,\tau)}{\partial\tau} + \frac{\partial}{\partial\tau}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\left[\mathbb{S}\left[R\left(u_m, {}^{C}D_x^{\beta}u_m\,; x,\tau\right) + N\left(u_m, {}^{C}D_x^{\beta}u_m\,; x,\tau\right)\right]\right]\right] - \frac{\partial}{\partial\tau}k(x,\tau)\right]d\tau, \tag{17}$$

where v_k, $k = 0, 1, 2, \cdots$, denote the components of the solution satisfying

$$u(x,t) = \lim_{m\to\infty} u_m(x,t) = \sum_{k=0}^{\infty} v_k. \tag{18}$$

Theorem 1 [16]. *Let V, defined in (5∗), be an operator from a Banach space BS to BS. The series solution $u(x,t) = \lim_{m\to\infty} u_m(x,t) = \sum_{k=0}^{\infty} v_k$ as defined in (6∗) converges if $0 < p < 1$ exists such that $\|V[v_0 + v_1 + v_2 + \cdots + v_{k+1}]\| \le p\|V[v_0 + v_1 + v_2 + \cdots + v_k]\|$, (i.e., $\|v_{k+1}\| \le \|v_k\|$), $\forall k \in \mathbb{N}\cup\{0\}$.*

Theorem 1, obtained from the Banach fixed-point theorem, is utilized to establish a sufficient condition for the convergence of fractional VIM.

Theorem 2 [16]. *The exact solution of nonlinear problem (9) exists under the condition that the series solution $u(x,t) = \sum_{k=0}^{\infty} v_k$ defined in (18) converges.*

Theorem 3 [16]. *Suppose that the series solution $\sum_{k=0}^{\infty} v_k$ defined in (18) converges to the solution $u(x,t)$. The maximum error $E_j(x,t)$ for the approximate solution $\sum_{k=0}^{j} v_k$ satisfies the following inequality:*

$$E_j(x,t) \le \frac{1}{1-p}p^{j+1}\|v_0\|. \tag{19}$$

The series solution $\sum_{k=0}^{\infty} v_k$ of problem (9) is convergent to an exact solution $u(x,t)$, if the conditions

$$0 < \chi_i \le 1, \tag{20}$$

$\forall i \in \mathbb{N}\cup\{0\}$, hold where the parameters χ_i for $i \in \mathbb{N}\cup\{0\}$ are introduced as

$$\chi_i = \begin{cases} \dfrac{\|v_{i+1}\|}{\|v_i\|}, & \|v_i\| \ne 0, \\ 0, & \|v_i\| = 0. \end{cases} \tag{21}$$

Furthermore, the maximum absolute truncation error satisfies the inequality

$$\left\|u(x,t) - \sum_{k=0}^{\infty} v_k\right\| \le \frac{1}{1-\chi}\chi^{j+1}\|v_0\|, \tag{22}$$

where $\chi = \max\{\chi_i, i = 0, 1, 2, \cdots, j\}$.

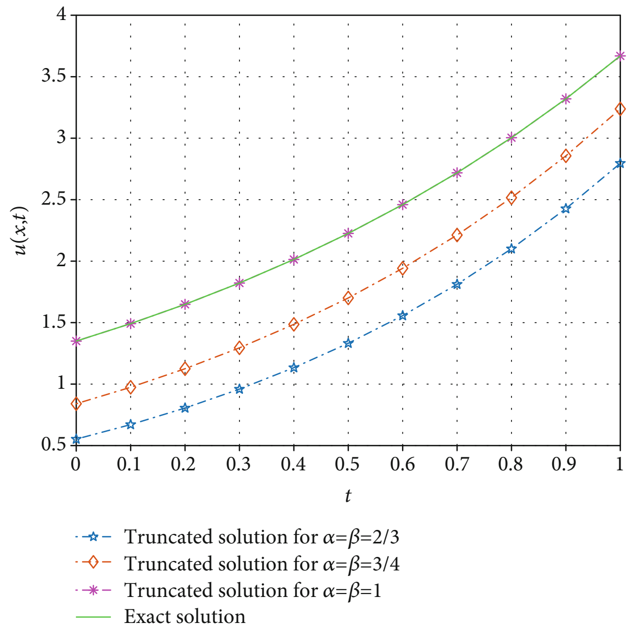

FIGURE 1: 6th order of truncated solutions for various values of α and β and exact solution at $x = 0.3$ for Example 1.

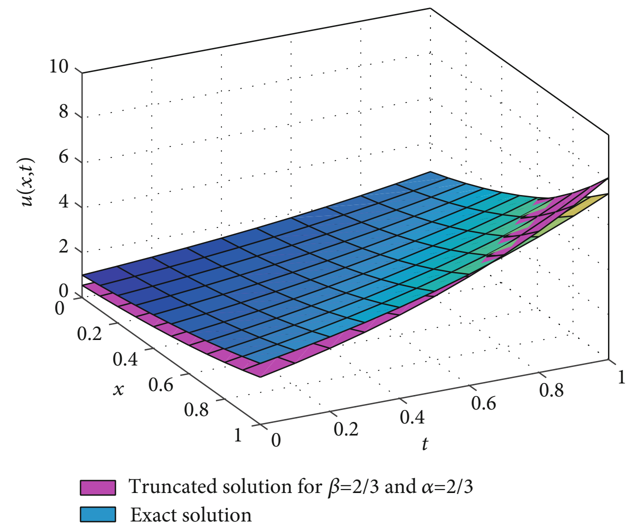

FIGURE 2: 6th order of truncated solutions for $\alpha = \beta = 2/3$ and exact solution for Example 1.

4. Illustrative Examples

Example 1. Let us consider following space-time-fractional initial value problem

$$ {}^{C}D_t^{\alpha}u(x,t) = {}^{C}D_t^{2\beta}u(x,t), \quad 0<\alpha\le 1,\ 1<2\beta\le 2,\ 0\le x\le l,\ t>0, \tag{23}$$

$$u(x,0) = E_{\beta,1}\left(x^{\beta}\right). \tag{24}$$

Step 1. Implementing Shehu transform for (23), we have

$$\begin{aligned}\mathbb{S}[u(x,t)] &= \left(\frac{q}{p}\right)E_{\beta,1}\left(x^{\beta}\right)\\ &\quad + \left(\frac{q}{p}\right)^{\alpha}\mathbb{S}\left[{}^{C}D_x^{2\beta}u(x,t)\right].\end{aligned} \tag{25}$$

Step 2. Taking the inverse Shehu transform of (25), we get

$$\begin{aligned}u(x,t) &= \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)E_{\beta,1}\left(x^{\beta}\right)\right] + \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\mathbb{S}\left[{}^{C}D_x^{2\beta}u(x,t)\right]\right],\\ u(x,t) &= E_{\beta,1}\left(x^{\beta}\right) + \mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\mathbb{S}\left[{}^{C}D_x^{2\beta}u(x,t)\right]\right],\end{aligned} \tag{26}$$

and so

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \frac{\partial}{\partial t}E_{\beta,1}\left(x^{\beta}\right) + \frac{\partial}{\partial t}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\mathbb{S}\left[{}^{C}D_x^{2\beta}u(x,t)\right]\right],\\ \frac{\partial u(x,t)}{\partial t} &- \frac{\partial}{\partial t}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\mathbb{S}\left[{}^{C}D_x^{2\beta}u(x,t)\right]\right] = 0.\end{aligned} \tag{27}$$

Step 3. Employing the variational iteration method, we obtain

$$\begin{aligned}u_{m+1}(x,t) = u_m(x,t) - \int_0^t \Bigg[\frac{\partial u_m(x,\tau)}{\partial \tau}\\ - \frac{\partial}{\partial \tau}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^{\alpha}\mathbb{S}\left[{}^{C}D_x^{2\beta}u(x,t)\right]\right]\Bigg]d\tau.\end{aligned} \tag{28}$$

Based on the iteration formula (28), we have

$$\begin{aligned}u_0(x,t) &= E_{\beta,1}\left(x^{\beta}\right),\\ u_1(x,t) &= E_{\beta,1}\left(x^{\beta}\right) + E_{\beta,1}\left(x^{\beta}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\ u_2(x,t) &= E_{\beta,1}\left(x^{\beta}\right)\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right],\\ u_3(x,t) &= E_{\beta,1}\left(x^{\beta}\right)\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right].\end{aligned} \tag{29}$$

By using the recurrence relation, we obtained the m^{th} approximate solution of (23) as follows:

$$u_m(x,t) = E_{\beta,1}\left(x^{\beta}\right)\sum_{k=0}^{m}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad m = 0,1,2,\cdots. \tag{30}$$

TABLE 2: Comparison of the exact solution with the truncated solutions by SVIM for various β and α for Example 2.

t	x	$\alpha = 0.5$, $\beta = 0.5$, u_{our}	$\alpha = 0.75$, $\beta = 0.75$, u_{our}	$\alpha = 1$, $\beta = 1$, u_{our}	u_{exact}
	-5	2,97723189881641	2,95468850942983	3,00924465929121	3,00924465929121
0,2	0	3,14497276182554	3,70409889494893	4,37202909023507	4,37202909023507
	5	88,2859013271367	194,512854240003	206,627171662420	206,627171662420
	-5	3,01929874786381	2,98503015564532	3,00507357659074	3,00507357659074
0,4	0	2,87711786964444	3,23261761498140	3,75298552978051	3,75298552978051
	5	-69,2902227453216	66,2713155938424	114,752961233252	114,752961233252
	-5	3,03346340884022	2,99928895133517	3,00278443786961	3,00278443786961
0,6	0	2,78692633344582	3,01104904237009	3,41324722055397	3,41324722055397
	5	-122,348923979388	6,00530743066654	64,3313254927734	64,3313254927734
	-5	3,03543594562253	3,00548196320081	3,00152813190282	3,00152813190282
0,8	0	2,77436647600195	2,91481533308062	3,22679488322353	3,22679488322353
	5	-129,737751709177	-20,1699819673740	36,6593450875042	36,6593450875042
	-5	3,03265269180591	3,00773966021300	3,00083865656976	3,00083865656976
1	0	2,79208846297849	2,87973279769261	3,12446767091966	3,12446767091966
	5	-119,312099237844	-29,7123296891665	21,4726402473266	21,4726402473266

As a result, the analytical solution of (23) is reached by taking the limit of (30):

$$u(x,t)=\lim_{m\to\infty}u_m(x,t)=E_{\beta,1}\left(x^\beta\right)E_{\alpha,1}(t^\alpha), \tag{31}$$

where $E_{\alpha,1}(t^\alpha)$ is the two-parameter Mittag-Leffler function.

Notice from Table 1 and Figure 1 that the values of the solution for $\alpha=\beta=1$ and exact solution are the same which implies that the method implemented in this study is one of the best one for the solution of space-time-fractional differential equations of any order. Moreover, it is clear from Figure 1 that as α and β tend to 1, the corresponding solutions tend to exact solution. Three-dimensional graphs of exact solution and a truncated solution are given in Figure 2.

Example 2. Let us consider the space-time-fractional equation

$$\begin{aligned} {}^C D_t^\alpha u(x,t) &= \left({}^C D_x^\beta u(x,t)\right)^2 \\ &\quad - u(x,t)\,{}^C D_x^\beta u(x,t), \quad 0<\alpha,\beta\le 1, 0\le x\le l, t>0, \end{aligned} \tag{32}$$

with the condition at $t=0$.

$$u(x,0)=3+\frac{5}{2}E_{\beta,1}\left(x^\beta\right). \tag{33}$$

Step 1. Carrying out Shehu transform of (32), we have

$$\begin{aligned} \mathbb{S}[u(x,t)] &= \left(\frac{q}{p}\right)u(x,0)+\left(\frac{q}{p}\right)^\alpha \mathbb{S}\left[\left({}^C D_x^\beta u(x,t)\right)^2\right. \\ &\quad \left.- u(x,t)\,{}^C D_x^\beta u(x,t)\right]. \end{aligned} \tag{34}$$

Step 2. Enforcing inverse Shehu transform of (34), we obtain

$$\begin{aligned} u(x,t) &= u(x,0)+\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^\alpha \mathbb{S}\left[\left({}^C D_x^\beta u(x,t)\right)^2\right.\right. \\ &\quad \left.\left.- u(x,t)\,{}^C D_x^\beta u(x,t)\right]\right], \end{aligned} \tag{35}$$

and so

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &- \frac{\partial}{\partial t}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^\alpha \mathbb{S}\left[\left({}^C D_x^\beta u(x,t)\right)^2\right.\right. \\ &\quad \left.\left.- u(x,t)\,{}^C D_x^\beta u(x,t)\right]\right]=0. \end{aligned} \tag{36}$$

Step 3. Utilizing the variational iteration method, we have

$$\begin{aligned} u_{m+1}(x,t) &= u_m(x,t)-\int_0^t\left[\frac{\partial u_m(x,\tau)}{\partial\tau}\right. \\ &\quad - \frac{\partial}{\partial\tau}\mathbb{S}^{-1}\left[\left(\frac{q}{p}\right)^\alpha \mathbb{S}\left[\left({}^C D_x^\beta u_m(x,\tau)\right)^2\right.\right. \\ &\quad \left.\left.\left.- u_m(x,\tau)\,{}^C D_x^\beta u_m(x,\tau)\right]\right]\right]d\tau. \end{aligned} \tag{37}$$

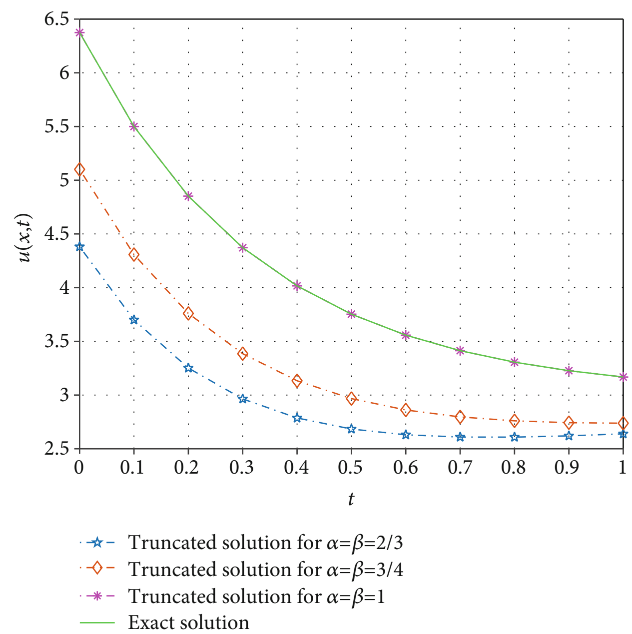

FIGURE 3: 6th order of truncated solutions for various values of α and β and exact solution at $x=0.3$ for Example 2.

Based on the iteration formula (37), we have

$$\begin{aligned} u_0(x,t) &= 3+\frac{5}{2}E_{\beta,1}\left(x^\beta\right), \\ u_1(x,t) &= 3+\frac{5}{2}E_{\beta,1}\left(x^\beta\right)-\frac{15}{2}E_{\beta,1}\left(x^\beta\right)\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2(x,t) &= 3+\frac{5}{2}E_{\beta,1}\left(x^\beta\right)-\frac{15}{2}E_{\beta,1}\left(x^\beta\right)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \frac{45}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}E_{\beta,1}\left(x^\beta\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3(x,t) &= 3+\frac{5}{2}E_{\beta,1}\left(x^\beta\right)-\frac{15}{2}E_{\beta,1}\left(x^\beta\right)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \frac{45}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}E_{\beta,1}\left(x^\beta\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad - \frac{135}{2}E_{\beta,1}\left(x^\beta\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ u_4(x,t) &= 3+\frac{5}{2}E_{\beta,1}\left(x^\beta\right)-\frac{15}{2}E_{\beta,1}\left(x^\beta\right)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \frac{45}{2}\frac{t^\alpha}{\Gamma(\alpha+1)}E_{\beta,1}\left(x^\beta\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad - \frac{135}{2}E_{\beta,1}\left(x^\beta\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &\quad + \frac{405}{2}E_{\beta,1}\left(x^\beta\right)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}. \end{aligned} \tag{38}$$

By using the recurrence relation, the m^{th} approximate solution of (32) is obtained as follows:

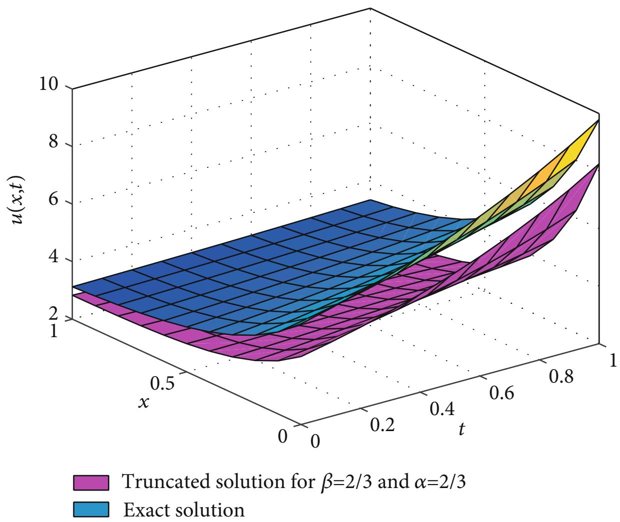

FIGURE 4: 6th order of truncated solutions for $\alpha = \beta = 2/3$ and exact solution for Example 2.

$$u_m(x,t) = 3 + \left[\frac{5}{2} \sum_{k=0}^{m} \frac{(-3t^{\alpha})^k}{\Gamma(k\alpha+1)} \right] \sum_{l=0}^{m} \frac{(x^{\beta})^l}{\Gamma(l\beta+1)}. \tag{39}$$

Hence, the analytical solution of (32) and (33) is reached by taking the limit of (39):

$$u(x,t) = \lim_{m\to\infty} u_m(x,t) = 3 + \left[\frac{5}{2} E_{\alpha,1}(-3t^{\alpha}) \right] E_{\beta,1}\left(x^{\beta}\right), \tag{40}$$

which is the same as obtained in [17].

As in Example 1, it is obvious from Table 2 and Figure 3 that the values of the solution for $\alpha = \beta = 1$ and exact solution are the same and as α and β tend to 1, the corresponding solutions tend to exact solution which indicates that the method employed in this research is a good choice for the solution of space-time-fractional differential equations of any order in Figure 4, 3-dimensional graphs of exact solution and a truncated solution are presented.

5. Conclusions

In this research, the targeted goal is to construct truncated solutions of linear/nonlinear space-time-fractional initial value problem by employing SVIM, the combination of the Shehu transform and variational iteration method. The main advantage of this method is that its implementation is straightforward and fruitful. Moreover, the illustrated examples reveal that the obtained approximate solutions with high precision converge swiftly to exact analytical solutions.

In the future study, this method and its improved modifications are applied to initial value problems including space-time-fractional linear and nonlinear differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to this work.

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