Hindawi
Advances in Mathematical Physics
Volume 2021, Article ID 6672819, 9 pages
https://doi.org/10.1155/2021/6672819

Research Article

On the Exact Solitary Wave Solutions to the New $(2+1)$ and $(3+1)$-Dimensional Extensions of the Benjamin-Ono Equations

Lan Wu,[1] **Xiao Zhang**,[1] **and Jalil Manafian**[2]

[1]*School of Mathematical Sciences, Peking University, Beijing, China*
[2]*Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran*

Correspondence should be addressed to Jalil Manafian; j_manafianheris@tabrizu.ac.ir

Received 11 November 2020; Revised 26 December 2020; Accepted 20 March 2021; Published 21 April 2021

Academic Editor: Sergey Shmarev

In this paper, the Kudryashov method to construct the new exact solitary wave solutions for the newly developed $(2+1)$-dimensional Benjamin-Ono equation is successfully employed. In the same vein, also the new $(2+1)$-dimensional Benjamin-Ono equation to $(3+1)$-dimensional spaces is extended and then analyzed and investigated. Different forms of exact solitary wave solutions to this new equation were also determined. Graphical illustrations for certain solutions in both equations are provided. We alternatively offer that the determining method is general, impressive, outspoken, and powerful and can be exerted to create exact solutions of various kinds of nonlinear models originated in mathematical physics and engineering.

1. Introduction

Nonlinear evolution equations have been known for their vital roles in many fields of engineering and nonlinear sciences for long. A lot of these equations are famous in fluid flow problems and shallow water waves applications. A very good example for such equations is the Benjamin-Ono equation [1] that describes inner waves of deep-stratified fluids that reads

$$u_{tt} + \alpha\left(u^2\right)_{xx} + \beta u_{xxxx} = 0, \tag{1}$$

where α and β are nonzero constants for monitoring the nonlinear term and depth of the fluid, respectively. Further, different studies have been carried out on this important model ranging from analytical solution, numerical solution, stability, and well-posedness among others. For instance, the multisoliton solution and time-periodic solutions of the Benjamin-Ono equation were presented by Matsuno [2] and Ambrose and Wilkening [3], respectively (see also Angulo et al. [4] for the stability, Tutiya and Shiraishi [5] for discrete solutions, and [6–11] for other related studies).

Additionally, the $(2+1)$-dimensional version of Benjamin-Ono equation Eq. (1) was recently introduced by Wazwaz [12]. The new equations has the form

$$u_{tt} + \alpha\left(u^2\right)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} = 0, \tag{2}$$

where α, β, and γ are nonzero constants. Note that γ should not be zero; otherwise, we recover Eq. (1). In [12], the Hirota bilinear method and certain ansatzs methods have been used to construct a variety of multiple and complex soliton solutions and also checked the Painlevé integrality condition.

However, in this paper, we further extend the new $(2+1)$-dimensional Benjamin-Ono equation [12] given in Eq. (2) to $(3+1)$-dimensional spaces and call it the $(3+1)$-dimensional Benjamin-Ono equation given by

$$u_{tt} + \alpha\left(u^2\right)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} + \delta u_{zzzz} = 0, \tag{3}$$

where α, β, γ, and δ are nonzero constants. Furthermore, to present more new solitary wave solutions for the $(2+1)$-dimensional Benjamin-Ono equation in Eq. (2) and also to study the $(3+1)$-dimensional Benjamin-Ono equation, we developed Eq. (3), to employ the Kudryashov method [13, 14] as a powerful integration method for treating various nonlinear evolution equations (see also [15–23] for other methods). The Kudryashov method and its modified versions have been investigated by capable authors in the plenty of nonlinear models such as the nonlinear differential equations [24], higher-order local and nonlocal nonlinear equations in optical fibers [25], some $(2+1)$-dimensional nonlinear evolution equations [26], exact traveling wave solutions of the PHI-four equation, and the Fisher equation [27]. As we all know, some novel and important developments for searching the analytical solitary wave solutions for PDE were investigated. Hence, there are fascinating results on some models in which are presented in research works containing the new iterative projection method for approximating fixed point problems and variational inequality problems [28], weighted inequalities for the Dunkl fractional maximal function and Dunkl fractional integrals [29], the Painlevé analysis, soliton molecule, and lump solution of the higher-order Boussinesq equation [30], and the Darboux solutions of the classical Painlevé second equation [31]. The structure of this paper is as follows: the analysis of the method has been summed up in "Analysis of the Method." In "Applications," the applications of "Analysis of the Method" for considered equation are investigated. Also, in "Some Graphical Illustrations," the graphical illustrations for nonlinear equations will be used. In "Conclusion," the conclusions have been given.

2. Analysis of the Method

To illustrate the idea of the Kudryashov method [13, 14], we consider the following system of nonlinear differential equations:

$$F(u, T_x u, T_{xx} u, T_{tt} u, T_{xxxx} u, \cdots) = 0. \tag{4}$$

Applying the transformation

$$u(x, t) = f(\xi), \xi = ax - ct - x_0, \tag{5}$$

where a and c are nonzero constants and x_0 is arbitrary constant, converts Eq. (4) to a nonlinear ordinary differential equations as follows

$$H\left(f', f'', f''', \cdots\right) = 0, \tag{6}$$

where the derivatives are with respect to ξ. It is assumed that the solutions of Eq. (6) are presented as a finite series, say

$$f(\xi) = a_0 + \sum_{i=1}^{N} a_i \Phi^i(\xi), \tag{7}$$

where a_i, $i = 1, 2, \cdots, N$ $(a_N \neq 0)$, are constants to be computed, and $\Phi(\xi)$ is given by the following function:

$$\Phi(\xi) = \frac{1}{1 + we^{\xi}}, \tag{8}$$

which satisfies the ordinary differential equation

$$\Phi'(\xi) = \Phi(\xi)(\Phi(\xi) - 1). \tag{9}$$

Also, the value of N is determined by homogenous balancing method (see [13, 14]). Substituting Eq. (7) and its necessary derivatives like

$$\begin{aligned} f' &= \sum_{i=1}^{N} a_i \Phi^i (\Phi - 1), \\ f'' &= \sum_{i=1}^{N} a_i \Phi^i (\Phi - 1)((1 + i)\Phi - i), \\ &\vdots \end{aligned} \tag{10}$$

into Eq. (6) gives

$$P(\Phi(\xi)) = 0, \tag{11}$$

where $P(\Phi(\xi))$ is a polynomial in $\Phi(\xi)$. Equating the coefficient of each power of $\Phi(\xi)$ in Eq. (11) to zero, a system of algebraic equations will be obtained whose solution yields the exact solutions of Eq. (4).

3. Applications

In this section, some new solitary wave solutions of the $(2+1)$-dimensional and $(3+1)$-dimensional Benjamin-Ono equations are constructed using the Kudryashov method presented above.

3.1. The $(2+1)$-Dimensional Benjamin-Ono Equation. In this section, we will study the $(2+1)$-dimensional Benjamin-Ono equation given by Eq. (2)

$$u_{tt} + \alpha\left(u^2\right)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} = 0, \tag{12}$$

where $u(x, y, t)$ is a sufficiently often a differentiable function and α, β, and γ are nonzero parameters.

To determine certain solitary wave solutions, we first substitute

$$u(x, y, t) = f(\xi), \xi = ax + by - ct - x_0, \tag{13}$$

into Eq. (12) where

$$\begin{gathered} T_{tt} u = c^2 f'(\xi), T_x u = af'(\xi), T_{xx} u = a^2 f''(\xi), \\ T_{xxxx} u = a^4 f''''(\xi), T_{yyyy} u = b^4 f''''(\xi), \end{gathered} \tag{14}$$

and convert Eq. (12) to a nonlinear ordinary differential equation given below:

$$c^2 f'' + \alpha a^2 \left(f^2\right)'' + \beta a^4 f'''' + \gamma b^4 f'''' = 0. \tag{15}$$

Integrating Eq. (15) twice with respect to ξ, yields

$$c^2 f + \alpha a^2 \left(f^2\right) + \beta a^4 f'' + \gamma b^4 f'' = 0, \tag{16}$$

where the integrating constant is considered zero. Balancing

f^2 and f'' in Eq. (12) gives $2N = N + 2$, so $N = 2$. We integrate Eq. (15) with

$$f(\xi) = a_0 + a_1\Phi(\xi) + a_2\Phi^2(\xi). \tag{17}$$

Substituting Eq. (17) into Eq. (16) and equating the coefficient of each power of $\Phi(\xi)$ to zero, we get a system of algebraic equations given below:

$$\begin{gathered} c^2 a_0 + a^2\alpha a_0^2 = 0, \\ c^2 a_1 + a^4\beta a_1 + b^4\gamma a_1 + 2a^2\alpha a_0 a_1 = 0, \\ -3a^4\beta a_1 - 3b^4\gamma a_1 + a^2\alpha a_1^2 + c^2 a_2 \\ + 4a^4\beta a_2 + 4b^4\gamma a_2 + 2a^2\alpha a_0 a_2 = 0, \\ 2a^4\beta a_1 + 2b^4\gamma a_1 - 10a^4\beta a_2 - 10b^4\gamma a_2 + 2a^2\alpha a_1 a_2 = 0, \\ 6a^4\beta a_2 + 6b^4\gamma a_2 + a^2\alpha a_2^2 = 0. \end{gathered} \tag{18}$$

Solving the above nonlinear algebraic system, the following results will be concluded as follows.

Case 1.

$$a_0 = 0, a_1 = -\frac{6c^2}{a^2\alpha}, a_2 = \frac{6c^2}{a^2\alpha}, b = \mp\frac{\left(-c^2 - a^4\beta\right)^{1/4}}{\gamma^{1/4}}. \tag{19}$$

Hence, the solution is formed as

$$u_{1,2}(x, y, t) = -\frac{6c^2/a^2\alpha}{1 + we^{\xi}} + \frac{6c^2/a^2\alpha}{\left(1 + we^{\xi}\right)^2}, \ \xi = ax + by - ct - x_0. \tag{20}$$

Case 2.

$$a_0 = -\frac{c^2}{a^2\alpha}, a_1 = \frac{6c^2}{a^2\alpha}, a_2 = -\frac{6c^2}{a^2\alpha}, b = \mp\frac{\left(c^2 - a^4\beta\right)^{1/4}}{\gamma^{1/4}}. \tag{21}$$

Hence, the solution is formed as

$$u_{3,4}(x, y, t) = -\frac{c^2}{a^2\alpha} + \frac{6c^2/a^2\alpha}{1 + we^{\xi}} - \frac{6c^2/a^2\alpha}{\left(1 + we^{\xi}\right)^2}, \ \xi = ax + by - ct - x_0. \tag{22}$$

Case 3.

$$a_0 = 0, a_1 = -\frac{6c^2}{a^2\alpha}, a_2 = \frac{6c^2}{a^2\alpha}, b = \mp\frac{i\left(-c^2 - a^4\beta\right)^{1/4}}{\gamma^{1/4}}. \tag{23}$$

Hence, the solution is formed as

$$u_{5,6}(x, y, t) = -\frac{6c^2/a^2\alpha}{1 + we^{\xi}} + \frac{6c^2/a^2\alpha}{\left(1 + we^{\xi}\right)^2}, \ \xi = ax + by - ct - x_0. \tag{24}$$

Case 4.

$$a_0 = -\frac{c^2}{a^2\alpha}, a_1 = \frac{6c^2}{a^2\alpha}, a_2 = -\frac{6c^2}{a^2\alpha}, b = \mp\frac{i\left(c^2 - a^4\beta\right)^{1/4}}{\gamma^{1/4}}. \tag{25}$$

Hence, the solution is formed as:

$$u_{7,8}(x, y, t) = -\frac{c^2}{a^2\alpha} + \frac{6c^2/a^2\alpha}{1 + we^{\xi}} - \frac{6c^2/a^2\alpha}{\left(1 + we^{\xi}\right)^2}, \ \xi = ax + by - ct - x_0. \tag{26}$$

The corresponding dynamic characteristics of the periodic wave solution are plotted in Figures 1 and 2 and arise at spaces $y = -1$, $y = 0$, and $y = 1$, in Figure 3, they arise at spaces $y = -10$, $y = -7$, and $y = 1$, and also in Figure 4, they arise at spaces $y = -10$, $y = 0$, and $y = 1$ with the following special parameters:

$$a = 2, \alpha = 2, \beta = -3, c = 2, \gamma = 2, r = 1, w = 0.3, t = 20, \tag{27}$$

with considering time $t = 20$.

3.2. The (3 + 1)-Dimensional Benjamin-Ono Equation. In this section, we will study the (3 + 1)-dimensional Benjamin-Ono equation which we give as

$$u_{tt} + \alpha\left(u^2\right)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} + \delta u_{zzzz} = 0, \tag{28}$$

where $u(x, y, z, t)$ is a sufficiently often differentiable function and α, β, γ and δ are nonzero parameters. Also to determine some soliton solutions, we first substitute the transformation

$$u(x, y, z, t) = f(\xi), \xi = ax + by + dz - ct - x_0, \tag{29}$$

into Eq. (28) where

$$\begin{gathered} T_{tt}u = c^2 f'(\xi), T_x u = af'(\xi), T_{xx}u = a^2 f''(\xi), \\ T_{xxxx}u = a^4 f''''(\xi), T_{yyyy}u = b^4 f''''(\xi), T_{zzzz}u = d^4 f''''(\xi), \end{gathered} \tag{30}$$

which converts Eq. (28) into a nonlinear ordinary differential equation as follows:

$$c^2 f'' + \alpha a^2\left(f^2\right)'' + \beta a^4 f'''' + \gamma b^4 f'''' + \delta d^4 f'''' = 0. \tag{31}$$

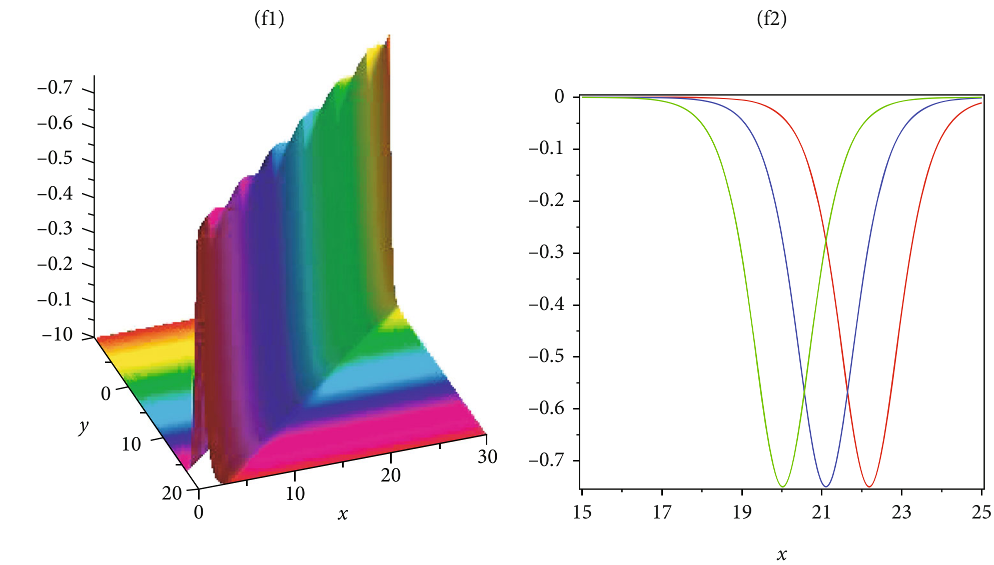

Figure 1: Graph of Eq. (20) for the (2 + 1)-dimensional Benjamin-Ono equation at $a = 2$, $\alpha = 2$, $\beta = -3$, $c = 2$, $\gamma = 2$, $r = 1$, $w = 0.3$, $t = 20$ and for 2 plot spaces $y = -1, 0, 1$.

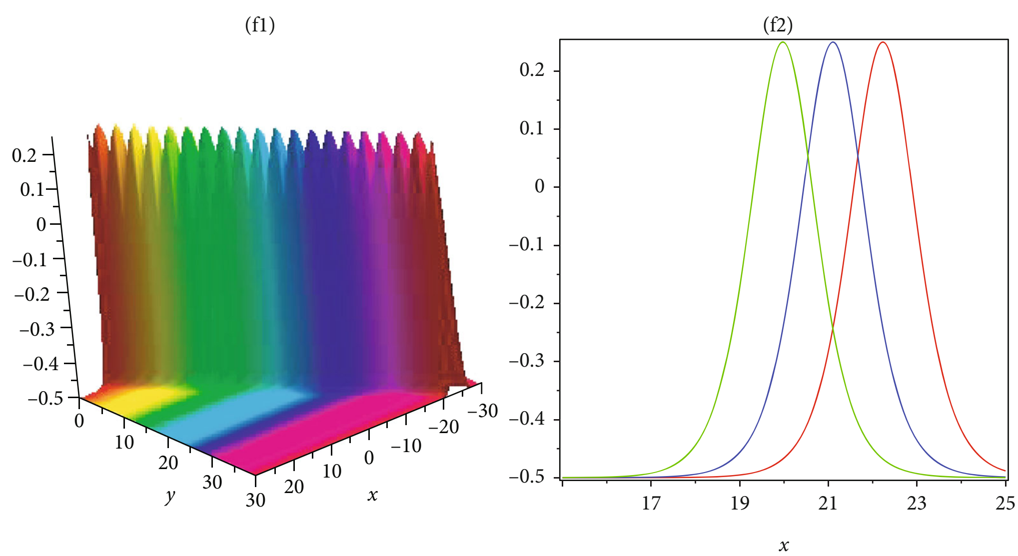

Figure 2: Graph of Eq. (22) for the (2 + 1)-dimensional Benjamin-Ono equation at $a = 2$, $\alpha = 2$, $\beta = -3$, $c = 2$, $\gamma = 2$, $r = 1$, $w = 0.3$, $t = 20$ and for 2 plot spaces $y = -1, 0, 1$.

Integrating (31) once with respect to ξ and setting the integrating constant zero yield

$$c^2 f + \alpha a^2 \left(f^2\right) + \beta a^4 f'' + \gamma b^4 f'' + \delta d^4 f'' = 0. \tag{32}$$

Balancing f^2 and f'' in Eq. (32) results to $2N = N + 2$, so $N = 2$. This offers a truncated series as the following form:

$$f(\xi) = a_0 + a_1 \Phi(\xi) + a_2 \Phi^2(\xi). \tag{33}$$

Substituting Eq. (33) into Eq. (32) and equating the coefficient of each power of $\Phi(\xi)$ to zero, we get the following system of algebraic equations:

$$\begin{gathered}
c^2 a_0 + a^2 \alpha a_0^2 = 0, \\
c^2 a_1 + a^4 \beta a_1 + b^4 \gamma a_1 + d^4 \delta a_1 + 2a^2 \alpha a_0 a_1 = 0, \\
-3a^4 \beta a_1 - 3b^4 \gamma a_1 - 3d^4 \delta a_1 + a^2 \alpha a_1^2 + c^2 a_2 + 4a^4 \beta a_2 \\
+ 4b^4 \gamma a_2 + 4d^4 \delta a_2 + 2a^2 \alpha a_0 a_2 = 0, \\
2a^4 \beta a_1 + 2b^4 \gamma a_1 + 2d^4 \delta a_1 - 10a^4 \beta a_2 - 10b^4 \gamma a_2 \\
- 10d^4 \delta a_2 + 2a^2 \alpha a_1 a_2 = 0, \\
6a^4 \beta a_2 + 6b^4 \gamma a_2 + 6d^4 \delta a_2 + a^2 \alpha a_2^2 = 0.
\end{gathered} \tag{34}$$

Solving the above system, yields the following.

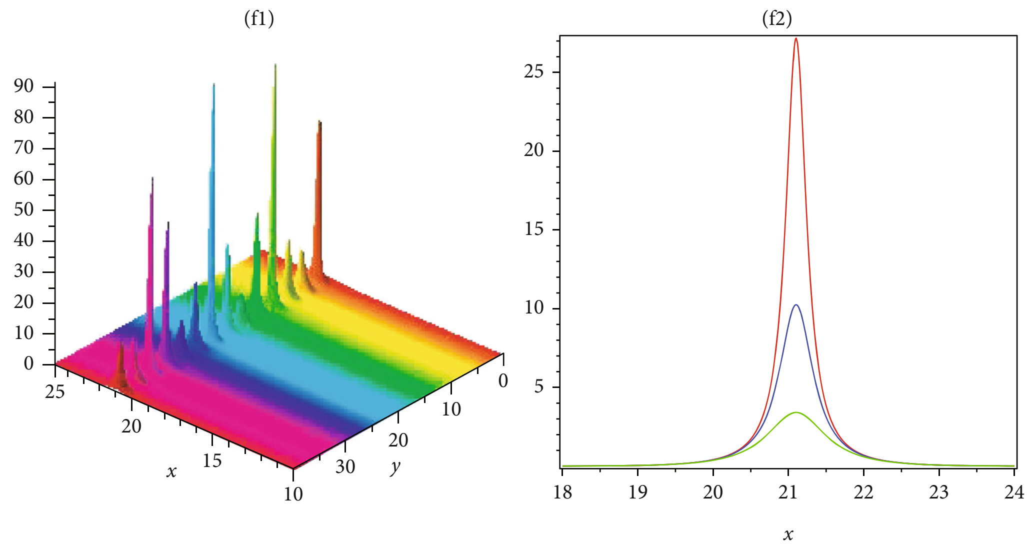

Figure 3: Graph of the absolute value of Eq. (24) for the $(2+1)$-dimensional Benjamin-Ono equation at $a=2, \alpha=2, \beta=-3, c=2, \gamma=2, r=1, w=0.3, t=20$ and for 2 plot spaces $y=-10,-7,1$.

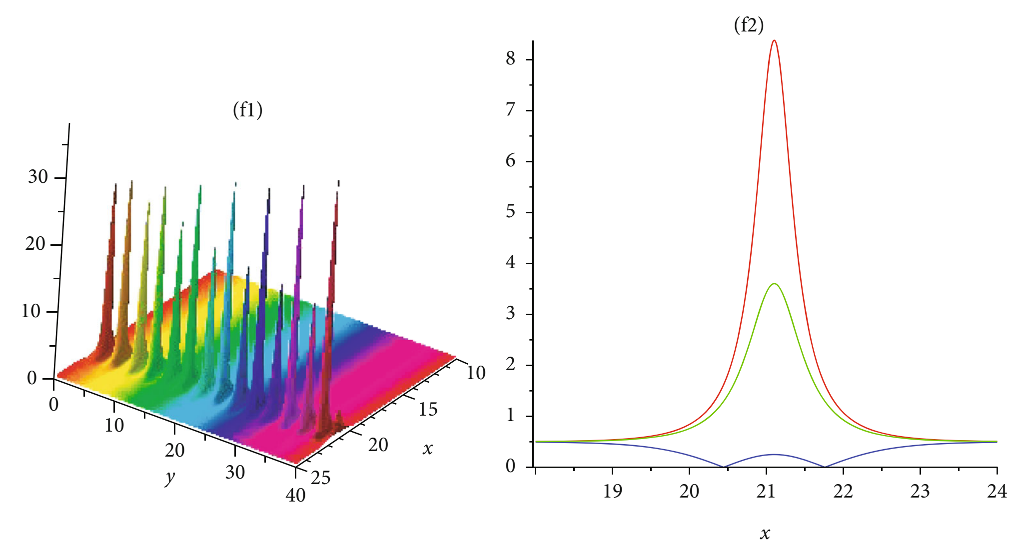

Figure 4: Graph of the absolute value of Eq. (42) for the $(2+1)$-dimensional Benjamin-Ono equation at $a=2, \alpha=2, \beta=-3, c=2, \gamma=2, r=1, w=0.3, t=20$ and for 2 plot spaces $y=-10,0,1$.

Case 1.

$$a_0=0,\ a_1=-\frac{6c^2}{a^2\alpha},\ a_2=\frac{6c^2}{a^2\alpha},\ b=\mp\frac{\left(-c^2-a^4\beta-d^4\delta\right)^{1/4}}{\gamma^{1/4}}. \tag{35}$$

Hence, the solution is formed as

$$u_{1,2}(x,y,z,t)=-\frac{6c^2/a^2\alpha}{1+we^{\xi}}+\frac{6c^2/a^2\alpha}{\left(1+we^{\xi}\right)^2},\ \xi=ax+by+dz-ct-x_0. \tag{36}$$

Case 2.

$$a_0=-\frac{c^2}{a^2\alpha},\ a_1=\frac{6c^2}{a^2\alpha},\ a_2=-\frac{6c^2}{a^2\alpha},\ b=\mp\frac{\left(c^2-a^4\beta-d^4\delta\right)^{1/4}}{\gamma^{1/4}}. \tag{37}$$

Hence, the solution is formed as

$$u_{3,4}(x,y,t)=-\frac{c^2}{a^2\alpha}+\frac{6c^2/a^2\alpha}{1+we^{\xi}}-\frac{6c^2/a^2\alpha}{\left(1+we^{\xi}\right)^2},\ \xi=ax+by+dz-ct-x_0. \tag{38}$$

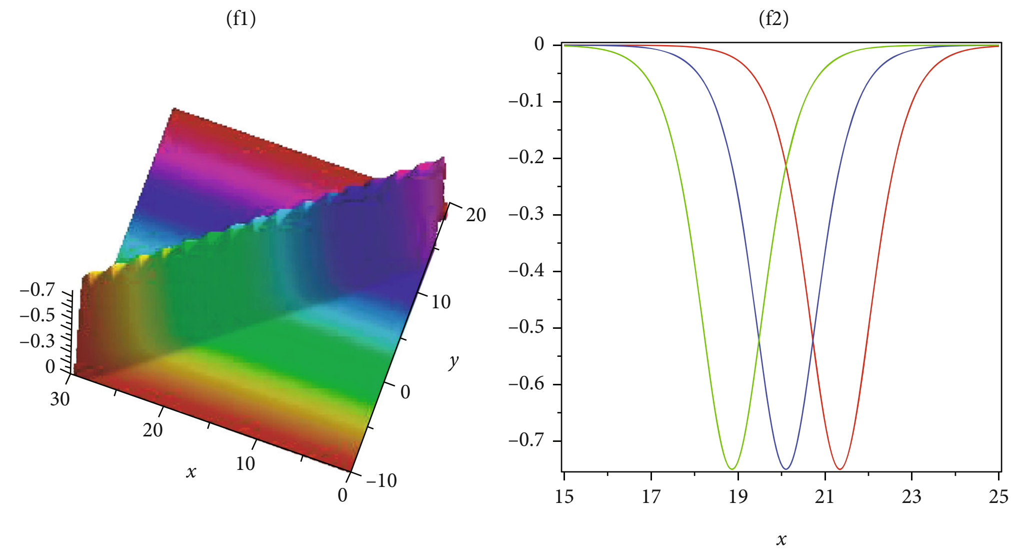

Figure 5: Graph of Eq. (36) for the (2 + 1)-dimensional Benjamin-Ono equation at $a = 2, \alpha = 2, \beta = -3, c = 2, \gamma = 2, r = 1, w = 0.3, d = 2, \delta = -2, z = 1, t = 20$ and for 2 plot spaces $y = -1, 0, 1$.

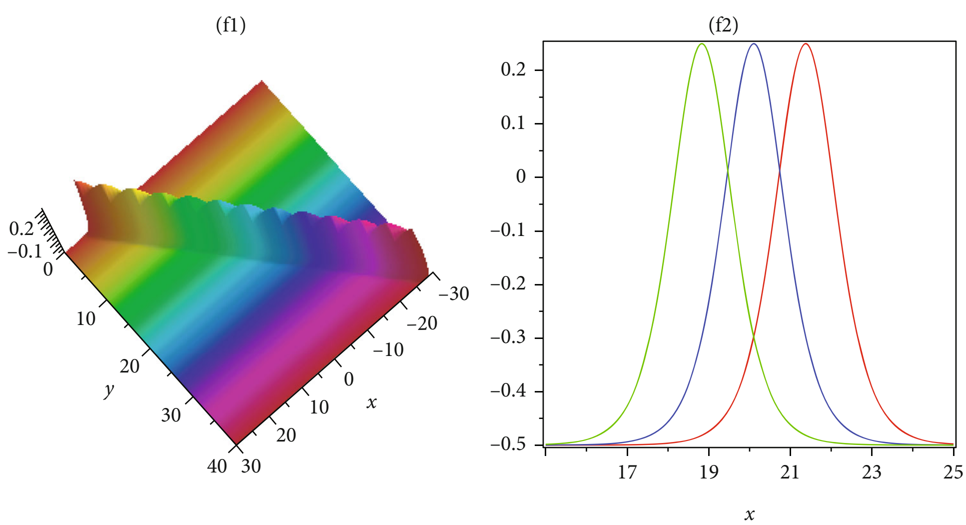

Figure 6: Graph of Eq. (38) for the (2 + 1)-dimensional Benjamin-Ono equation at $a = 2, \alpha = 2, \beta = -3, c = 2, \gamma = 2, r = 1, w = 0.3, d = 2, \delta = -2, z = 1, t = 20$ and for 2 plot spaces $y = -1, 0, 1$.

Case 3.

$$a_0 = 0, a_1 = -\frac{6c^2}{a^2\alpha}, a_2 = \frac{6c^2}{a^2\alpha}, b = \mp\frac{i\left(-c^2 - a^4\beta - d^4\delta\right)^{1/4}}{\gamma^{1/4}}. \tag{39}$$

Hence, the solution is formed as

$$u_{5,6}(x, y, t) = -\frac{6c^2/a^2\alpha}{1 + we^{\xi}} + \frac{6c^2/a^2\alpha}{\left(1 + we^{\xi}\right)^2}, \xi = ax + by + dz - ct - x_0. \tag{40}$$

Case 4.

$$a_0 = -\frac{c^2}{a^2\alpha}, a_1 = \frac{6c^2}{a^2\alpha}, a_2 = -\frac{6c^2}{a^2\alpha}, b = \mp\frac{i\left(c^2 - a^4\beta - d^4\delta\right)^{1/4}}{\gamma^{1/4}}. \tag{41}$$

Hence, the solution is formed as

$$u_{7,8}(x, y, t) = -\frac{c^2}{a^2\alpha} + \frac{6c^2/a^2\alpha}{1 + \mathrm{we}(\xi)} - \frac{6c^2/a^2\alpha}{\left(1 + we^{\xi}\right)^2}, \xi = ax + by + dz - ct - x_0. \tag{42}$$

The corresponding dynamic characteristics of the periodic wave solution are plotted in Figures 5 and 6 and arise at spaces $y = -1$, $y = 0$, and $y = 1$, in Figure 7, they arise at

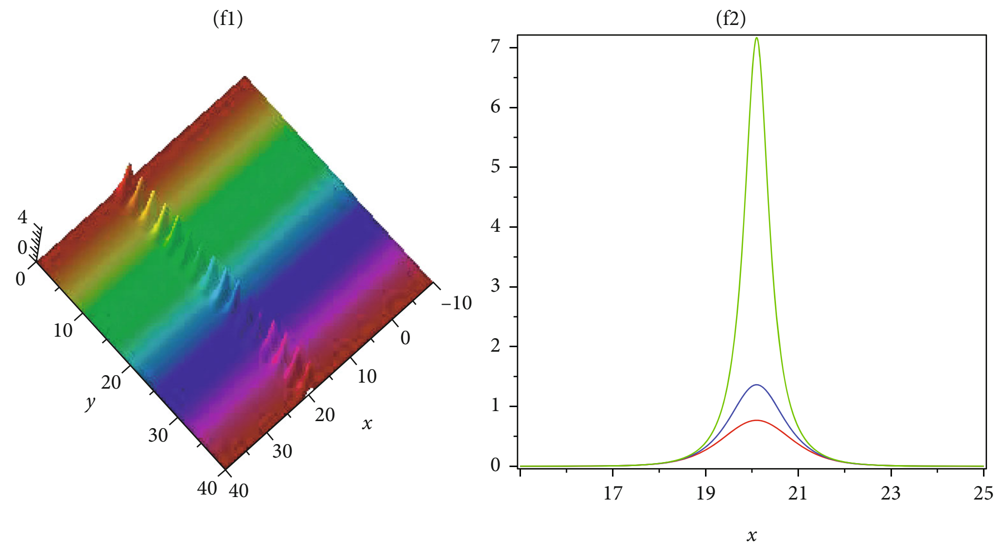

FIGURE 7: Graph of the absolute value of Eq. (40) for the $(2+1)$-dimensional Benjamin-Ono equation at $a=2, \alpha=2, \beta=-3, c=2, \gamma=2, r=1, w=0.3, d=2, \delta=-2, z=1, t=20$ and for 2 plot spaces $y=-10, -7, 1$.

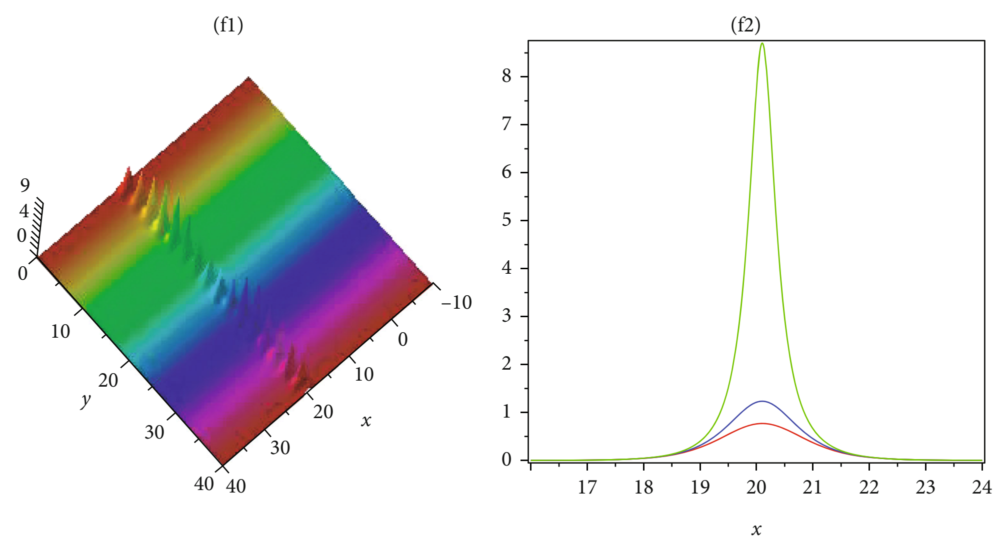

FIGURE 8: Graph of the absolute value of Eq. (42) for the $(2+1)$-dimensional Benjamin-Ono equation at $a=2, \alpha=2, \beta=-3, c=2, \gamma=2, r=1, w=0.3, d=2, \delta=-2, z=1, t=20$ and for 2 plot spaces $y=-10, -3, 1$.

spaces $y=-10$, $y=-7$, and $y=1$, and also in Figure 4, they arise at spaces $y=-10$, $y=0$, and $y=1$ with the following special parameters:

$$a=2, \alpha=2, \beta=-3, c=2, \gamma=2, r=1, w=0.3, d=2, \delta=-2, \tag{43}$$

considering space and time $z=1, t=20$.

4. Some Graphical Illustrations

We depict in this section some graphical illustrations of the obtained solutions for the $(2+1)$- and $(3+1)$-dimensional extensions of the Benjamin-Ono equations, both the two and three dimensional plots for the solutions are plotted. Figures 1 and 4 show the graph of the solutions (20)–(26) for the $(2+1)$-dimensional Benjamin-Ono equation, respectively. Figures 5 and 8 show the behavior of the solutions (36)–(42) for the $(3+1)$-dimensional Benjamin-Ono equation, respectively.

5. Conclusion

In conclusion, we have presented new solitary wave solutions for the $(2+1)$-dimensional Benjamin-Ono equation introduced recently by Wazwaz and also extended it to $(3+1)$-dimensional spaces called the $(3+1)$-dimensional Benjamin-Ono equation. While constructing the solitary wave solutions, we make use of the Kudryashov method being one of the powerful integration methods for treating various nonlinear evolution equations and construct various exponential solutions to both equations. The development of offered method may allow the extensions of the Benjamin-Ono equations to be used in more general configurations. The solutions are all verified by putting them back into the original equations with the aid of the Maple symbolic computation package 18.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Acknowledgments

Lan Wu and Xiao Zhang contributed equally to this work.

References

[1] T. B. Benjamin, "Internal waves of permanent form in fluids of great depth," *Journal of Fluid Mechanics*, vol. 29, no. 3, pp. 559–592, 1967.

[2] Y. Matsuno, "Exact multi-soliton solution of the Benjamin-Ono equation," *Journal of Physics A: Mathematical and General*, vol. 12, no. 4, pp. 619–621, 1979.

[3] D. M. Ambrose and J. J. Wilkening, "Computation of time-periodic solutions of the Benjamin–Ono equation," *Journal of Nonlinear Science*, vol. 20, no. 3, pp. 277–308, 2010.

[4] J. Angulo, M. Scialom, and C. Banquet, "The regularized Benjamin-Ono and BBM equations: well-posedness and nonlinear stability," *Journal of Differential Equations*, vol. 250, no. 11, pp. 4011–4036, 2011.

[5] Y. Tutiya and J. Shiraishi, "On some special solutions to periodic Benjamin-Ono equation with discrete Laplacian," *Mathematics and Computers in Simulation*, vol. 82, no. 7, pp. 1341–1347, 2012.

[6] L. Song, W. Chen, Z. Xu, and H. Chen, "Rogue wave for the Benjamin Ono equation," *Advances in Pure Mathematics*, vol. 5, no. 2, pp. 82–87, 2015.

[7] Y. K. Liu and B. Li, "Dynamics of rogue waves on multisoliton background in the Benjamin Ono equation," *Pramana-Journal of Physics*, vol. 88, no. 4, p. 57, 2017.

[8] M. J. Ablowitz, A. Demirci, and Y. P. Ma, "Dispersive shock waves in the Kadomtsev-Petviashvili and two dimensional Benjamin-Ono equations," *Physica D: Nonlinear Phenomena*, vol. 333, pp. 84–98, 2016.

[9] A. Esfahani and S. Levandosky, "Solitary waves of the rotation-generalized Benjamin-Ono equation," 2011, https://arxiv.org/abs/1105.5369.

[10] C. Muñoz and G. Ponce, "On the asymptotic behavior of solutions to the Benjamin-Ono equation," *Proceedings of the American Mathematical Society*, vol. 147, pp. 5303–5312, 2019.

[11] W. Tan and Z. Dai, "Spatiotemporal dynamics of lump solution to the (1+1)-dimensional Benjamin-Ono equation," *Nonlinear Dynamics*, vol. 89, no. 4, pp. 2723–2728, 2017.

[12] A. Wazwaz, "A $(2+1)$-dimensional extension of the Benjamin-Ono equation," *International Journal of Numerical Methods Heat Fluid and Flow*, vol. 28, no. 11, pp. 2681–2687, 2018.

[13] A. A. Gaber, A. F. Aljohani, A. Ebaid, and J. T. Machado, "The generalized Kudryashov method for nonlinear space-time fractional partial differential equations of Burgers type," *Nonlinear Dynamics*, vol. 95, no. 1, pp. 361–368, 2019.

[14] S. Tuluce Demiray, Y. Pandir, and H. Bulut, "Generalized Kudryashov method for time-fractional differential equations," *Abstract and Applied Analysis*, vol. 2014, Article ID 901540, 13 pages, 2014.

[15] J. Manafian and M. Lakestani, "Optical solitons with Biswas-Milovic equation for Kerr law nonlinearity," *The European Physical Journal Plus*, vol. 130, article 61, 2015.

[16] J. Manafian and S. Heidari, "Periodic and singular kink solutions of the Hamiltonian amplitude equation," *Advanced Mathematical Models & Applications*, vol. 4, pp. 134–149, 2019.

[17] H. M. Baskonus, D. A. Koç, and H. Bulut, "New travelling wave prototypes to the nonlinear Zakharov-Kuznetsov equation with power law nonlinearity," *Nonlinear Science Letters A*, vol. 7, pp. 67–76, 2016.

[18] M. Dehghan, J. Manafian, and A. Saadatmandi, "Solving nonlinear fractional partial differential equations using the homotopy analysis method," *Numerical Methods for Partial Differential Equations*, vol. 26, no. 2, pp. 448–479, 2010.

[19] M. Dehghan, J. Manafian, and A. Saadatmandi, "Analytical treatment of some partial differential equations arising in mathematical physics by using the Exp-function method," *International Journal of Modern Physics B*, vol. 25, pp. 2965–2981, 2011.

[20] J. Manafian, "Novel solitary wave solutions for the (3+1)-dimensional extended Jimbo-Miwa equations," *Computers & Mathematics with Applications*, vol. 76, no. 5, pp. 1246–1260, 2018.

[21] A. R. Seadawy and J. Manafian, "New soliton solution to the longitudinal wave equation in a magneto-electro-elastic circular rod," *Results in Physics*, vol. 8, pp. 1158–1167, 2018.

[22] J. Manafian, S. A. Mohammed, A. Alizadeh, H. M. Baskonus, and W. Gao, "Investigating lump and its interaction for the third-order evolution equation arising propagation of long waves over shallow water," *European Journal of Mechanics - B/Fluids*, vol. 84, pp. 289–301, 2020.

[23] M. Eslami and H. Rezazadeh, "The first integral method for Wu-Zhang system with conformable time-fractional derivative," *Calcolo*, vol. 53, no. 3, pp. 475–485, 2016.

[24] N. A. Kudryashov, "One method for finding exact solutions of nonlinear differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 6, pp. 2248–2253, 2012.

[25] S. Garai, A. G. Choudhury, and J. Dan, "On the solution of certain higher-order local and nonlocal nonlinear equations in optical fibers using Kudryashov's approach," *Optik*, vol. 222, p. 165312, 2020.

[26] M. M. Hassan, M. A. Abdel-Razek, and A. A. H. Shoreh, "New exact solutions of some $(2+1)$-dimensional nonlinear evolution equations via extended Kudryashov method," *Reports on Mathematical Physics*, vol. 74, no. 3, pp. 347–358, 2014.

[27] F. Mahmud, M. Samsuzzoh, and M. A. Akbar, "The generalized Kudryashov method to obtain exact traveling wave

solutions of the PHI-four equation and the Fisher equation," *Results in Physics*, vol. 7, pp. 4296–4302, 2017.

[28] I. Karahan and M. Ozdemir, "A new iterative projection method for approximating fixed point problems and variational inequality problems," *Proceedings of the Institute of Mathematics and Mechanics*, vol. 40, pp. 68–79, 2014.

[29] Y. Y. Mammadov, "Weighted inequalities for Dunkl fractional maximal function and Dunkl fractional integrals," *Proceedings of the Institute of Mathematics and Mechanics*, vol. 40, pp. 93–103, 2014.

[30] B. Ren, "Painlevé analysis, soliton molecule, and lump solution of the higher-order Boussinesq equation," *Advances in Mathematical Physics*, vol. 2021, Article ID 6687632, 6 pages, 2021.

[31] I. Mahmood and M. Waseem, "Lax representation and Darboux solutions of the classical Painlevé second equation," *Advances in Mathematical Physics*, vol. 2021, Article ID 8851043, 5 pages, 2021.