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Research Article

Time-Scale Version of Generalized Birkhoffian Mechanics and Its Symmetries and Conserved Quantities of Noether Type

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The time-scale version of Noether symmetry and conservation laws for three Birkhoffian mechanics, namely, nonshifted Birkhoffian systems, nonshifted generalized Birkhoffian systems, and nonshitfed constrained Birkhoffian systems, are studied. Firstly, on the basis of the nonshifted Pfaff-Birkhoff principle on time scales, Birkhoff's equations for nonshifted variables are deduced; then, Noether's quasi-symmetry for the nonshifted Birkhoffian system is proved and time-scale conserved quantity is presented. Secondly, the nonshifted generalized Pfaff-Birkhoff principle on time scales is proposed, the generalized Birkhoff's equations for nonshifted variables are derived, and Noether's symmetry for the nonshifted generalized Birkhoffian system is established. Finally, for the nonshifted constrained Birkhoffian system, Noether's symmetry and time-scale conserved quantity are proposed and proved. The validity of the result is proved by examples.

1. Introduction

Birkhoffian mechanics is a new stage in the development of analytical dynamics. It was first proposed by Birkhoff [1] and later developed by Santilli [2] and Mei et al. [3]. In literature [4], Mei proposed and studied in detail the dynamics of the generalized Birkhoffian systems. Since then, some scholars [5–10] have carried out a series of studies on this issue.

The dynamics theory on a time scale unifies the dynamics of continuous systems, discrete systems, and quantum systems. The theory of time scale analysis can be traced back to Hilger [11], who first proposed the calculus theory on a measure chain. Time scale, as a special case of the measure chain, has strong representative, so it has attracted extensive attention. Bohner and Peterson [12] systematically studied time scale calculus and its dynamic equations. Agarwal and Bohner [13] began to study the time scale linear and nonlinear Hamiltonian systems and unify and extend the symplectic flow properties of continuous and discrete Hamiltonian system. In 2004, Bohner [14] studied the time scale variational problem for the first time. In 2008, Bartosiewicz and Torres [15] first carried out the researches about Noether's theorem on time scales. They discovered that Noether's conserved quantities can be derived without changing the time transformations. What is more, Bartosiewicz and his coworkers [16] also deduced the second Euler-Lagrange equation for variational problem on time scales. Based on the second Euler-Lagrange equations, they proposed another method to find the Noether conserved quantity. Afterwards, according to these two methods, many scholars have obtained some results have been obtained in the study of variational principle, dynamical equations, and Noether symmetries for the different mechanical systems, such as references [17–31].

With the study on time scales, scholars began to study the time-scale version of the nonshifted variational problem. Bourdin [32] found that the Euler-Lagrange has greater convergence in the discrete case of the nonshifted variational problem. Anerot et al. [33] derived the Noether theorem for the shifted and nonshifted variational problems on time scales; they pointed out that the methods of deriving Noether conserved quantities on time scales by references [15, 16] were not correct. Song and Cheng [34] researched Noether symmetry on time scales for the nonshifted Birkhoffian systems, but the work was limited to free Birkhoffian systems and to Noether symmetries. Here, we will study the Noether symmetry for more general nonshifted Birkhoffian systems, including generalized Birkhoffian systems and constrained Birkhoffian systems, not only Noether symmetry but Noether quasi-symmetry. According to the study, it was found that the shifted variational problem are not suitable for the structure-preserving algorithm, while the nonshifted variational problem on time scales is suitable for the structure-preserving algorithm for discrete systems. Therefore, the research of the paper is of great significance.

The structures of this article are as follows. In Section 2, according to nonshifted Birkhoff's equations, the Noether quasi-symmetry and time-scale conserved quantity are obtained. An example is given for discussion. In Section 3, about the nonshifted generalized systems on time scales, nonshifted generalized Pfaff-Birkhoff principle and equations are deduced. The Noether symmetries and time-scale conserved quantities are obtained. Then, an example is given for analysis. In Section 4, the equations for the nonshifted constrained Birkhoffian systems are deduced, and symmetries and time-scale conserved quantities are given. And an example is given. In Section 5, the conclusion is given.

2. Nonshifted Birkhoffian Systems on Time Scales

For the properties of calculus on a time scale, please refer to reference [12].

2.1. Nonshifted Birkhoff's Equations. On a time scale, the nonshifted Pfaff action is

$$S(a_\mu(\cdot)) = \int_{t_1}^{t_2} \left[R_\upsilon(t, a_\mu) a_\upsilon^\Delta - B(t, a_\mu)\right]\Delta t, \tag{1}$$

where the endpoint conditions are $a_\mu(t_1) = a_{\mu_1}$ and $a_\mu(t_2) = a_{\mu_2}$. a_μ^Δ is the delta derivative of a_μ with respect to t. $a_\mu \in C_{rd}^{1,\Delta}(\mathbb{T})$ for $t \in \mathbb{T}_k^k$. The Birkhoffian $B : \mathbb{T} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ and Birkhoff's functions $R_\mu : \mathbb{T} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ are of $C_{rd}^{1,\Delta}(\mathbb{T})$, where $\mu, \upsilon = 1, 2, \cdots, 2n$.

The nonshifted Birkhoff's equations on time scales are [34].

$$R_\mu^\nabla = \sigma^\nabla(t)\left[\frac{\partial R_\upsilon(t, a_\rho)}{\partial a_\mu} a_\upsilon^\Delta - \frac{\partial B(t, a_\rho)}{\partial a_\mu}\right], \tag{2}$$

where $\sigma = \sigma(t)$ is nabla differentiable on $\mathbb{T}$.

2.2. Quasi-symmetry and Conserved Quantity. Introduce infinitesimal transformations

$$t^* = t + \varepsilon\xi_0(t, a_\rho), a_\mu^* = a_\mu + \varepsilon\xi_\mu(t, a_\rho), \tag{3}$$

where ε is an infinitesimal parameter and ξ_0 and ξ_μ are the generators.

Let R_υ^1 and B^1 be the other Birkhoffian and Birkhoff's functions on time scales. If accurate to a small quantity of first order, this is true

$$\begin{aligned}&\int_{t_1}^{t_2} \left[R_\upsilon(t, a_\rho) a_\upsilon^\Delta - B(t, a_\rho)\right]\Delta t \\ &= \int_{\alpha(t_1)}^{\alpha(t_2)} \left[R_\upsilon^1\left(t^*, a_\rho^*\right) a_\upsilon^{*\Delta^*} - B^1\left(t^*, a_\rho^*\right)\right]\Delta^* t^*.\end{aligned} \tag{4}$$

Then, the nonshifted Pfaff action (1) is a quasi-invariant, where $\alpha : \mathbb{T} \longrightarrow \mathbb{R}$. Obviously, R_υ^1, B^1 and R_υ, B will satisfy the same equation, so we have

$$R_\upsilon^1 = R_\upsilon + \frac{\partial G}{\partial a_\upsilon}, B^1 = B - \frac{\partial G}{\partial t}. \tag{5}$$

Thus, equation (4) can be expressed as

$$\begin{aligned}&\int_{t_1}^{t_2} \left[R_\upsilon(t, a_\rho) a_\upsilon^\Delta - B(t, a_\rho)\right]\Delta t \\ &= \int_{\alpha(t_1)}^{\alpha(t_2)} \left[R_\upsilon\left(t^*, a_\rho^*\right) a_\upsilon^{*\Delta^*} - B\left(t^*, a_\rho^*\right)\right]\Delta^* t^* \\ &+ \int_{t_1}^{t_2} \frac{\Delta}{\Delta t} G(t, a_\rho)\Delta t,\end{aligned} \tag{6}$$

where G is a small quantity of first order.

Definition 1. If the nonshifted Pfaff action (1) is a quasi-invariant, in other words, for every infinitesimal transformations (3), the following relationship

$$\Delta S = -\int_{t_1}^{t_2} \frac{\Delta}{\Delta t}(\Delta G)\Delta t, \tag{7}$$

always holds; the transformations (3) are referred to as Noether's quasi-symmetric for the nonshifted Birkhoffian system (2).

Criterion 2. If the following equation

$$\left(\frac{\partial R_\upsilon}{\partial t} a_\upsilon^\Delta - \frac{\partial B}{\partial t}\right)\xi_0 + \left(\frac{\partial R_\upsilon}{\partial a_\mu} a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu}\right)\xi_\mu - B\xi_0^\Delta + R_\upsilon \xi_\upsilon^\Delta + G^\Delta = 0 \tag{8}$$

is satisfied, the transformations (3) are quasi-symmetric. Equation (8) is called the Noether identity.

For equation (6),

$$\begin{aligned}
&\int_{t_1}^{t_2}\left[R_\upsilon(t,a_\rho)a_\upsilon^\Delta - B(t,a_\rho)\right]\Delta t \\
&= \int_{\alpha(t_1)}^{\alpha(t_2)}\left[R_\upsilon\left(t^*,a_\rho^*\right)a_\upsilon^{*\Delta^*} - B\left(t^*,a_\rho^*\right)\right]\Delta^* t^* \\
&\quad + \int_{t_1}^{t_2}\frac{\Delta}{\Delta t}G(t,a_\rho)\Delta t \\
&= \int_{t_1}^{t_2}\left[R_\upsilon\left(\alpha(t),\left(a_\rho^*\circ\alpha\right)(t)\right)a_\upsilon^{*\Delta^*}(\alpha(t))\right. \\
&\quad \left. - B\left(\alpha(t),\left(a_\rho^*\circ\alpha\right)(t)\right)\right]\alpha^\Delta(t)\Delta t + \int_{t_1}^{t_2}\frac{\Delta}{\Delta t}G(t,a_\rho)\Delta t \\
&= \int_{t_1}^{t_2}\left[R_\upsilon\left(\alpha(t),\left(a_\rho^*\circ\alpha\right)(t)\right)\frac{\left(a_\rho^*\circ\alpha\right)^\Delta(t)}{\alpha^\Delta(t)}\right. \\
&\quad \left. - B\left(\alpha(t),\left(a_\rho^*\circ\alpha\right)(t)\right)\right]\alpha^\Delta(t)\Delta t + \int_{t_1}^{t_2}\frac{\Delta}{\Delta t}G(t,a_\rho)\Delta t \\
&= \int_{t_1}^{t_2}\left[R_\upsilon\left(t+\varepsilon\xi_0(t,a_\rho),a_\rho+\varepsilon\xi_\rho(t,a_\mu)\right)\frac{\left(a_\upsilon+\varepsilon\xi_\upsilon(t,a_\rho)\right)^\Delta}{\left(t+\varepsilon\xi_0(t,a_\rho)\right)^\Delta}\right. \\
&\quad \left. -B\left(t+\varepsilon\xi_0(t,a_\rho),a_\rho+\varepsilon\xi_\rho(t,a_\mu)\right)\right]\left(t+\varepsilon\xi_0(t,a_\rho)\right)^\Delta\Delta t \\
&\quad + \int_{t_1}^{t_2}\frac{\Delta}{\Delta t}G(t,a_\rho)\Delta t.
\end{aligned} \tag{9}$$

We have

$$\begin{aligned}
R_\upsilon(t,a_\rho)a_\upsilon^\Delta - B(t,a_\rho) &= \left[R_\upsilon\left(t+\varepsilon\xi_0(t,a_\rho),a_\rho+\varepsilon\xi_\rho(t,a_\mu)\right)\frac{\left(a_\upsilon+\varepsilon\xi_\upsilon(t,a_\rho)\right)^\Delta}{\left(t+\varepsilon\xi_0(t,a_\rho)\right)^\Delta}\right. \\
&\quad \left. -B\left(t+\varepsilon\xi_0(t,a_\rho),a_\rho+\varepsilon\xi_\rho(t,a_\mu)\right)\right]\left(t+\varepsilon\xi_0(t,a_\rho)\right)^\Delta + \frac{\Delta}{\Delta t}G(t,a_\rho)
\end{aligned} \tag{10}$$

Take the derivative of ε, on both sides of equation(10) and set $\varepsilon = 0$, we get equation (8).

Theorem 3. *If the transformations (3) satisfy Noether identity (8), then*

$$I = R_\upsilon\xi_\upsilon^\sigma + \int_{t_1}^{t}\left[\left(\frac{\partial R_\upsilon}{\partial\tau}a_\upsilon^\Delta - \frac{\partial B}{\partial\tau}\right)\xi_0 - B\xi_0^\Delta\right]\sigma^\nabla\nabla\tau + G^\sigma = const. \tag{11}$$

is the conserved quantity of this system (2).

Proof. From equation (8), we have

$$\begin{aligned}
&\left(\frac{\partial R_\upsilon}{\partial t}\xi_0 + \frac{\partial R_\upsilon}{\partial a_\mu}\xi_\mu\right)a_\upsilon^\Delta\sigma^\nabla + R_\upsilon\xi_\upsilon^\Delta\sigma^\nabla - B\xi_0^\Delta\sigma^\nabla \\
&\quad - \frac{\partial B}{\partial t}\xi_0\sigma^\nabla - \frac{\partial B}{\partial a_\mu}\xi_\mu\sigma^\nabla + G^\Delta\sigma^\nabla = 0.
\end{aligned} \tag{12}$$

Using equation (2), we get

$$\frac{\partial R_\upsilon}{\partial t}\xi_0 a_\upsilon^\Delta\sigma^\nabla + \xi_0 B^\nabla - \xi_0^\sigma B^\nabla - \frac{\partial B}{\partial t}\xi_0\sigma^\nabla + R_\upsilon\xi_\upsilon^{\sigma\nabla} + G^\Delta\sigma^\nabla = 0. \tag{13}$$

By nabla indefinite integral of equation (13), we can get conserved quantity (11).

Example 4. We can study the Hojman-Urrutia problem on time scales. This problem can be written to be a non-shifted Birkhoffian system on time scales. Let $\mathbb{T} = \{2^n : n \in \mathbb{N}\cup\{0\}\}$, it is

$$\begin{aligned}
B &= \frac{1}{2}\left\{(a_3)^2 + 2a_2a_3 - (a_4)^2\right\}, \\
R_1 &= a_2 + a_3, R_2 = 0, R_3 = a_4, R_4 = 0.
\end{aligned} \tag{14}$$

From equation (8), we get

$$\begin{aligned}
&\left(a_1^\Delta - a_3\right)\xi_2 + \left(a_1^\Delta - a_3 - a_2\right)\xi_3 + \left(a_3^\Delta + a_4\right)\xi_4 \\
&\quad - B\xi_0^\Delta + (a_3 + a_2)\xi_1^\Delta + a_4\xi_3^\Delta + G^\Delta = 0.
\end{aligned} \tag{15}$$

It is easy to solve

$$\xi_1^1 = 1, \xi_0^1 = \xi_2^1 = \xi_3^1 = \xi_4^1 = 0, G^1 = 0, \tag{16}$$

$$\xi_0^2 = 0, \xi_1^2 = t, \xi_2^2 = 0, \xi_3^2 = 1, \xi_4^2 = 0, G^2 = -a_1. \tag{17}$$

The generators (16) correspond to Noether symmetry, and generators (17) correspond to Noether quasi-symmetry.

Based on Theorem 3, we can get

$$I^1 = a_2 + a_3 = \text{const}, \tag{18}$$

$$I^2 = (a_2 + a_3)(t + \mu(t)) + a_4 - \left(a_1 + \mu(t)a_1^\Delta\right) = \text{const}. \tag{19}$$

If we take $\mathbb{T} = \mathbb{R}$, we have

$$I^1 = a_2 + a_3 = \text{const}, \tag{20}$$

$$I^2 = (a_2 + a_3)t + a_4 - a_1 = \text{const}. \tag{21}$$

3. Nonshifted Generalized Birkhoffian Systems on Time Scales

3.1. Nonshifted Generalized Birkhoff's Equations. The nonshifted generalized Pfaff-Birkhoff principle on time scales is [3].

$$\int_{t_1}^{t_2}\left\{\delta\left(R_\upsilon a_\upsilon^\Delta - B\right) + \Lambda_\upsilon\left(t,a_\rho\right)\delta a_\upsilon\right\}\Delta t = 0, \tag{22}$$

where additional term $\Lambda_\upsilon : \mathbb{T}\times\mathbb{R}^{2n}\longrightarrow\mathbb{R}$, which is of $C_{rd}^{1,\Delta}(\mathbb{T})$.

From the principle (22), we have

$$\begin{aligned}
&\int_{t_1}^{t_2}\left\{\delta\left(R_\upsilon\left(t,a_\rho\right)a_\upsilon^\Delta - B\left(t,a_\rho\right)\right) + \Lambda_\upsilon\left(t,a_\rho\right)\delta a_\upsilon\right\}\Delta t \\
&\quad = \int_{t_1}^{t_2}\left(\frac{\partial R_\upsilon\left(t,a_\rho\right)}{\partial a_\mu}a_\upsilon^\Delta\delta a_\mu + R_\upsilon\left(t,a_\rho\right)\delta a_\upsilon^\Delta \right.\\
&\qquad \left. - \frac{\partial B\left(t,a_\rho\right)}{\partial a_\mu}\delta a_\mu + \Lambda_\upsilon\left(t,a_\rho\right)\delta a_\upsilon\right)\Delta t \\
&\quad = \int_{t_1}^{t_2}\left\{\left[-\int_{t_1}^{\sigma(t)}\left(\frac{\partial R_\upsilon\left(\tau,a_\rho\right)}{\partial a_\mu}a_\upsilon^\Delta\right)\Delta\tau + R_\mu\left(\tau,a_\rho\right)\right.\right.\\
&\qquad \left.\left. + \int_{t_1}^{\sigma(t)}\frac{\partial B\left(\tau,a_\rho\right)}{\partial a_\mu}\Delta\tau - \int_{t_1}^{\sigma(t)}\Lambda_\mu\left(\tau,a_\rho\right)\Delta\tau\right]\left(\delta a_\mu\right)^\Delta\right\}\Delta t = 0
\end{aligned} \tag{23}$$

We get

$$R_\mu - \int_{t_1}^{\sigma(t)}\left[\frac{\partial R_\upsilon\left(\tau,a_\rho\right)}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B\left(\tau,a_\rho\right)}{\partial a_\mu} + \Lambda_\mu\left(\tau,a_\rho\right)\right]\Delta\tau = \text{const.} \tag{24}$$

Let $h^\sigma(t) = \int_{t_1}^{\sigma(t)}[(\partial R_\upsilon(\tau,a_\rho)/\partial a_\mu)a_\upsilon^\Delta - (\partial B(\tau,a_\rho)/\partial a_\mu) + \Lambda_\mu(\tau,a_\rho)]\Delta\tau$, we have

$$\begin{aligned}
&\left\{R_\mu - \int_{t_1}^{\sigma(t)}\left[\frac{\partial R_\upsilon\left(\tau,a_\rho\right)}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B\left(\tau,a_\rho\right)}{\partial a_\mu} + \Lambda_\mu\left(\tau,a_\rho\right)\right]\Delta\tau\right\}^\nabla \\
&\quad = R_\mu^\nabla - \left(h^\sigma\right)^\nabla(t) = R_\lambda^\nabla - \sigma^\nabla(t)h^\Delta(t) \\
&\quad = R_\mu^\nabla - \sigma^\nabla(t)\left\{\int_a^{\sigma(t)}\left[\frac{\partial R_\upsilon}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} + \Lambda_\mu\right]\Delta\tau\right\}^\Delta \\
&\quad = R_\mu^\nabla - \sigma^\nabla(t)\left[\frac{\partial R_\upsilon}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} + \Lambda_\mu\right] = 0.
\end{aligned} \tag{25}$$

Therefore, we get

$$\sigma^\nabla(t)\left[\frac{\partial R_\upsilon}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} + \Lambda_\mu\right] = R_\mu^\nabla \quad (\mu,\upsilon = 1,2,\cdots,2n). \tag{26}$$

Equation (26) is called nonshifted generalized Birkhoff's equations. When $\Lambda_\mu = 0$, equation (26) becomes nonshifted Birkhoff's equation (2).

3.2. Quasi-symmetry and Conserved Quantity

Definition 5. If nonshifted Pfaff action (1) is a generalized quasi-invariant, that is, for every infinitesimal transformations (3), the following relationship

$$\Delta S = -\int_{t_1}^{t_2}\left[\frac{\Delta}{\Delta t}(\Delta G) + \Lambda_\upsilon\delta a_\upsilon\right]\Delta t, \tag{27}$$

always holds, the transformations (3) are referred to as generalized quasi-symmetric for nonshifted generalized Birkhoffian system (26).

Criterion 6. If the following Noether identity

$$\begin{aligned}
&\left(\frac{\partial R_\upsilon}{\partial t}a_\upsilon^\Delta - \frac{\partial B}{\partial t}\right)\xi_0 + \left(\frac{\partial R_\upsilon}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu}\right)\xi_\mu \\
&\quad - B\xi_0^\Delta + R_\upsilon\xi_\upsilon^\Delta + G^\Delta + \Lambda_\upsilon\left(\xi_\upsilon - a_\upsilon^\Delta\xi_0\right) = 0,
\end{aligned} \tag{28}$$

is satisfied, then transformations (3) are generalized quasi-symmetry.

For equation (27), we have

$$\begin{aligned}
&\int_{t_1}^{t_2}\left[R_\upsilon\left(t,a_\rho\right)a_\upsilon^\Delta - B\left(t,a_\rho\right)\right]\Delta t \\
&\quad = \int_{\alpha(t_1)}^{\alpha(t_2)}\left[R_\upsilon\left(t^*,a_\rho^*\right)a_\upsilon^{*\Delta^*} - B\left(t^*,a_\rho^*\right)\right]\Delta^* t^* \\
&\qquad + \int_{t_1}^{t_2}\left[\frac{\Delta}{\Delta t}G\left(t,a_\rho\right) + \Lambda_\upsilon\left(t,a_\rho\right)\delta a_\upsilon\right]\Delta t \\
&\quad = \int_{t_1}^{t_2}\left[R_\upsilon\left(\alpha(t),\left(a_\rho^*\circ\alpha\right)(t)\right)a_\upsilon^{*\Delta^*}(\alpha(t))\right.\\
&\qquad \left. - B\left(\alpha(t),\left(a_\rho^*\circ\alpha\right)(t)\right)\right]\alpha^\Delta(t)\Delta t \\
&\qquad + \int_{t_1}^{t_2}\left[\frac{\Delta}{\Delta t}G\left(t,a_\rho\right) + \Lambda_\upsilon\left(t,a_\rho\right)\delta a_\upsilon\right]\Delta t \\
&\quad = \int_{t_1}^{t_2}\left[R_\upsilon\left(\alpha(t),\left(a_\rho^*\circ\alpha\right)(t)\right)\frac{\left(a_\rho^*\circ\alpha\right)^\Delta(t)}{\alpha^\Delta(t)}\right.\\
&\qquad \left. - B\left(\alpha(t),\left(a_\rho^*\circ\alpha\right)(t)\right)\right]\alpha^\Delta(t)\Delta t \\
&\qquad + \int_{t_1}^{t_2}\left[\frac{\Delta}{\Delta t}G\left(t,a_\rho\right) + \Lambda_\upsilon\left(t,a_\rho\right)\delta a_\upsilon\right]\Delta t \\
&\quad = \int_{t_1}^{t_2}\left[R_\upsilon\left(t+\varepsilon\xi_0\left(t,a_\rho\right),a_\rho+\varepsilon\xi_\rho\left(t,a_\mu\right)\right)\frac{\left(a_\upsilon+\varepsilon\xi_\upsilon\left(t,a_\rho\right)\right)^\Delta}{\left(t+\varepsilon\xi_0\left(t,a_\rho\right)\right)^\Delta}\right.\\
&\qquad \left. - B\left(t+\varepsilon\xi_0\left(t,a_\rho\right),a_\rho+\varepsilon\xi_\rho\left(t,a_\mu\right)\right)\right]\left(t+\varepsilon\xi_0\left(t,a_\rho\right)\right)^\Delta\Delta t \\
&\qquad + \int_{t_1}^{t_2}\left[\frac{\Delta}{\Delta t}G\left(t,a_\rho\right) + \Lambda_\upsilon\left(t,a_\rho\right)\delta a_\upsilon\right]\Delta t.
\end{aligned} \tag{29}$$

We have

$$R_\upsilon(t,a_\rho)a_\upsilon^\Delta - B(t,a_\rho) = \left[R_\upsilon\left(t+\varepsilon\xi_0(t,a_\rho),a_\rho+\varepsilon\xi_\rho(t,a_\rho)\right)\frac{\left(a_\upsilon+\varepsilon\xi_\upsilon(t,a_\rho)\right)^\Delta}{\left(t+\varepsilon\xi_0(t,a_\rho)\right)^\Delta} - B\left(t+\varepsilon\xi_0(t,a_\theta),a_\rho+\varepsilon\xi_\rho(t,a_\rho)\right)\right]\left(t+\varepsilon\xi_0(t,a_\rho)\right)^\Delta + \frac{\Delta}{\Delta t}G(t,a_\rho) + \Lambda_\upsilon(t,a_\rho)\delta a_\upsilon. \tag{30}$$

Take the derivative of ε on both sides of equation (30) and set $\varepsilon = 0$, we obtain equation (28).

Theorem 7. *If the transformations (3) satisfy Noether identity (28), then*

$$I = R_\upsilon\xi_\upsilon^\sigma + \int_{t_1}^{t}\left[\left(\frac{\partial R_\upsilon}{\partial\tau}a_\upsilon^\Delta - \frac{\partial B}{\partial\tau} - \Lambda_\upsilon a_\upsilon^\Delta\right)\xi_0 - B\xi_0^\Delta\right]\sigma^\nabla\nabla\tau + G^\sigma = const. \tag{31}$$

is the conserved quantity for the system (26).

Proof. By equation (28), we have

$$\left(\frac{\partial R_\upsilon}{\partial t}\xi_0 + \frac{\partial R_\upsilon}{\partial a_\mu}\xi_\mu\right)a_\upsilon^\Delta\sigma^\nabla + R_\upsilon\xi_\upsilon^\Delta\sigma^\nabla - B\xi_0^\Delta\sigma^\nabla - \frac{\partial B}{\partial t}\xi_0\sigma^\nabla - \frac{\partial B}{\partial a_\mu}\xi_\mu\sigma^\nabla + G^\Delta\sigma^\nabla + \Lambda_\upsilon\left(\xi_\upsilon - a_\upsilon^\Delta\xi_0\right)\sigma^\nabla = 0. \tag{32}$$

Using equation (26), we have

$$\frac{\partial R_\upsilon}{\partial t}\xi_0 a_\upsilon^\Delta\sigma^\nabla + \xi_0 B^\nabla - \left(\xi_0^\sigma B\right)^\nabla - \frac{\partial B}{\partial t}\xi_0\sigma^\nabla + R_\upsilon\xi_\upsilon^\sigma + G^\Delta\sigma^\nabla + \Lambda_\upsilon\left(\xi_\upsilon - a_\upsilon^\Delta\xi_0\right)\sigma^\nabla = 0. \tag{33}$$

By nabla indefinite integral of equation (33), we can get conserved quantity (31).

Example 8. Let $\mathbb{T} = \{2^n : n \in \mathbb{N}\cup\{0\}\}$, the nonshifted generalized Birkhoffian system on time scales is

$$\begin{cases} B = \dfrac{1}{2}(a_3)^2 + a_2, \\ R_1 = a_3, R_2 = a_4, R_3 = 0, R_4 = 0, \\ \Lambda_1 = 0, \Lambda_2 = 0, \Lambda_3 = 0, \Lambda_4 = -a_4. \end{cases} \tag{34}$$

From equation (28), we get

$$-\xi_2 - a_3\xi_3 - B\xi_0^\Delta + a_3\xi_1^\Delta + a_4\xi_2^\Delta - a_4\left(\xi_4 - a_4^\Delta\xi_0\right) + G^\Delta = 0. \tag{35}$$

Equation (35) has the following solution:

$$\xi_0^1 = 0, \xi_1^1 = 0, \xi_2^1 = 1, \xi_3^1 = 0, \xi_4^1 = 0, G^1 = t, \tag{36}$$

$$\xi_0^2 = 0, \xi_1^2 = 1, \xi_2^2 = 0, \xi_3^2 = 0, \xi_4^2 = 0, G^2 = 0. \tag{37}$$

By Theorem 7, the conserved quantities corresponding to the generators (36) and (37) are

$$I^1 = a_4 + 2t = \text{const}, \tag{38}$$

$$I^2 = a_3 = \text{const}. \tag{39}$$

4. Nonshifted Constrained Birkhoffian Systems on Time Scales

4.1. Nonshifted Constrained Birkhoff's Equations. If the variables in nonshifted Birkhoffian system are not independent of each other on time scales, but subject to some constraints, these constraints are shown as

$$f_\beta(t,a_\mu) = 0(\beta = 1,2,\cdots,2n). \tag{40}$$

To calculate the isochronous variation of equation (40), we have

$$\frac{\partial f_\beta}{\partial a_\mu}\delta a_\mu = 0. \tag{41}$$

From equation (41), we can get

$$\int_{t_1}^{t_2}\lambda_\beta\frac{\partial f_\beta}{\partial a_\mu}\delta a_\mu\Delta t = 0. \tag{42}$$

By integrating by parts on time scales with equation (42), we get

$$\int_{t_1}^{t_2}\lambda_\beta\frac{\partial f_\beta}{\partial a_\mu}\delta a_\mu\Delta t = \left(\delta a_\mu\int_{t_1}^{t}\lambda_\beta\frac{\partial f_\beta}{\partial a_\mu}\nabla\tau\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2}\left(\int_{t_1}^{\sigma(t)}\lambda_\beta\frac{\partial f_\beta}{\partial a_\mu}\nabla\tau\right)\left(\delta a_\mu\right)^\Delta\Delta t = 0, \tag{43}$$

i.e.,

$$\int_{t_1}^{t_2}\left(\int_{t_1}^{\sigma(t)}\lambda_\beta\frac{\partial f_\beta}{\partial a_\mu}\nabla\tau\right)\left(\delta a_\mu\right)^\Delta\Delta t = 0. \tag{44}$$

According to nonshifted Pfaff-Birkhoff principle $\delta S = 0$,we get [34].

$$\int_{t_1}^{t_2}\left\{\left[-\int_{t_1}^{\sigma(t)}\left(\frac{\partial R_\upsilon(\tau,a_\rho)}{\partial a_\mu}a_\upsilon^\Delta\right)\Delta\tau + R_\mu(\tau,a_\rho) + \int_{t_1}^{\sigma(t)}\frac{\partial B(\tau,a_\rho)}{\partial a_\mu}\Delta\tau\right](\delta a_\mu)^\Delta\right\}\Delta t = 0. \tag{45}$$

Add equations (44) to (45), we have

$$\int_{t_1}^{t_2}\left\{\left[-\int_{t_1}^{\sigma(t)}\left(\frac{\partial R_\upsilon(\tau,a_\rho)}{\partial a_\mu}a_\upsilon^\Delta\right)\Delta\tau + R_\mu(\tau,a_\rho) + \int_{t_1}^{\sigma(t)}\frac{\partial B(\tau,a_\rho)}{\partial a_\mu}\Delta\tau + \int_{t_1}^{\sigma(t)}\lambda_\beta\frac{\partial f_\beta(\tau,a_\rho)}{\partial a_\mu}\Delta\tau\right](\delta a_\mu)^\Delta\right\}\Delta t = 0 \tag{46}$$

We get

$$R_\mu - \int_{t_1}^{\sigma(t)}\left[\frac{\partial R_\upsilon(\tau,a_\rho)}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B(\tau,a_\rho)}{\partial a_\mu} - \lambda_\beta\frac{\partial f_\beta(\tau,a_\rho)}{\partial a_\mu}\right]\Delta\tau = \text{const.} \tag{47}$$

Let $h^\sigma(t) = \int_{t_1}^{\sigma(t)}[(\partial R_\upsilon(\tau,a_\rho)/\partial a_\mu)a_\upsilon^\Delta - (\partial B(\tau,a_\rho)/\partial a_\mu) - \lambda_\beta(\partial f_\beta(\tau,a_\rho)/\partial a_\mu)]\Delta\tau$, then

$$\begin{aligned}&\left\{R_\mu - \int_{t_1}^{\sigma(t)}\left[\frac{\partial R_\upsilon(\tau,a_\rho)}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B(\tau,a_\rho)}{\partial a_\mu} - \lambda_\beta\frac{\partial f_\beta(\tau,a_\rho)}{\partial a_\mu}\right]\Delta\tau\right\}^\nabla \\ &= R_\mu^\nabla - (h^\sigma)^\nabla(t) = R_\lambda^\nabla - \sigma^\nabla(t)h^\Delta(t) \\ &= R_\mu^\nabla - \sigma^\nabla(t)\left\{\int_a^{\sigma(t)}\left[\frac{\partial R_\upsilon}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} - \lambda_\beta\frac{\partial f_\beta}{\partial a_\mu}\right]\Delta\tau\right\}^\Delta \\ &= R_\mu^\nabla - \sigma^\nabla(t)\left[\frac{\partial R_\upsilon}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} - \lambda_\beta\frac{\partial f_\beta}{\partial a_\mu}\right] = 0.\end{aligned} \tag{48}$$

Therefore, we get

$$\sigma^\nabla(t)\left[\frac{\partial R_\upsilon}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} - \lambda_\beta\frac{\partial f_\beta}{\partial a_\mu}\right] - R_\mu^\nabla = 0(\mu,\upsilon = 1,2,\cdots,2n\,;\beta = 1,2,\cdots,g). \tag{49}$$

Equation (49) is called nonshifted constrained Birkhoff's equations. If the system is nonsingular, by using equations (40) and (49), we can solve $\lambda_\beta = \lambda_\beta(t,a_\rho)$. Equation (51) can be written as

$$\sigma^\nabla(t)\left[\frac{\partial R_\upsilon}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} - P_\mu\right] = R_\mu^\nabla, \tag{50}$$

where

$$P_\mu = \lambda_\beta\frac{\partial f_\beta}{\partial a_\mu}. \tag{51}$$

We call the system determined by equation (50) as the corresponding free Birkhoffian system.

4.2. Quasi-symmetry and Conserved Quantity

Definition 9. If nonshifted Pfaff action (1) is a generalized quasi-invariant, in other words, for every infinitesimal transformations (3), the following relationship

$$\Delta S = -\int_{t_1}^{t_2}\left[\frac{\Delta}{\Delta t}(\Delta G) + P_\upsilon\delta a_\upsilon\right]\Delta t, \tag{52}$$

always holds, the transformations (3) are referred to as generalized quasi-symmetric for the corresponding free Birkhoffian system (50).

Criterion 10. If the Noether identity

$$\begin{aligned}&\left(\frac{\partial R_\upsilon}{\partial t}a_\upsilon^\Delta - \frac{\partial B}{\partial t}\right)\xi_0 + \left(\frac{\partial R_\upsilon}{\partial a_\mu}a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu}\right)\xi_\mu - B\xi_0^\Delta + R_\upsilon\xi_\upsilon^\Delta \\ &+ G^\Delta - P_\upsilon(\xi_\upsilon - a_\upsilon^\Delta\xi_0) = 0,\end{aligned} \tag{53}$$

is satisfied, then transformations (3) are generalized quasi-symmetric for the corresponding free Birkhoffian system (50). If the restriction equation

$$\frac{\partial f_\beta}{\partial a_\mu}\left(\xi_\mu - a_\mu^\Delta\xi_0\right) = 0(\beta = 1,2,\cdots,g), \tag{54}$$

is also satisfied, then transformations (3) are generalized quasi-symmetric for the nonshifted constrained Birkhoffian system (40) and (49).

Similar to the derivation of equation (29), from equation (52), we have

$$\begin{aligned}&\int_{t_1}^{t_2}[R_\upsilon(t,a_\rho)a_\upsilon^\Delta - B(t,a_\rho)]\Delta t \\ &= \int_{t_1}^{t_2}\left[R_\upsilon(t+\varepsilon\xi_0(t,a_\rho), a_\rho + \varepsilon\xi_\rho(t,a_\mu))\frac{(a_\upsilon + \varepsilon\xi_\upsilon(t,a_\rho))^\Delta}{(t+\varepsilon\xi_0(t,a_\rho))^\Delta}\right. \\ &\left. - B(t+\varepsilon\xi_0(t,a_\rho), a_\rho + \varepsilon\xi_\rho(t,a_\mu))\right](t+\varepsilon\xi_0(t,a_\rho))^\Delta\Delta t \\ &+ \int_{t_1}^{t_2}\left[\frac{\Delta}{\Delta t}G(t,a_\rho) + P_\upsilon(t,a_\rho)\delta a_\upsilon\right]\Delta t\end{aligned} \tag{55}$$

From equation (55), we have

$$R_v(t,a_\rho)a_v^\Delta - B(t,a_\rho) = \Big[R_v(t+\varepsilon\xi_0(t,a_\rho), a_\rho+\varepsilon\xi_\rho(t,a_\mu))\frac{(a_v+\varepsilon\xi_v(t,a_\rho))^\Delta}{(t+\varepsilon\xi_0(t,a_\rho))^\Delta} - B(t+\varepsilon\xi_0(t,a_\rho), a_\rho+\varepsilon\xi_\rho(t,a_\mu))\Big](t+\varepsilon\xi_0(t,a_\rho))^\Delta + \frac{\Delta}{\Delta t}G(t,a_\rho) - P_v(t,a_\rho)\delta a_v. \tag{56}$$

Take the derivative of ε on both sides of equation (56) and set $\varepsilon = 0$, we obtain equation (53).

Theorem 11. *If the transformations (3) satisfy the Noether identity (53) and the restriction equation (54), then*

$$I = R_v\xi_v^\sigma + \int_{t_1}^{t}\left[\left(\frac{\partial R_v}{\partial \tau}a_v^\Delta - \frac{\partial B}{\partial \tau} + P_v a_v^\Delta\right)\xi_0 - B\xi_0^\Delta\right]\sigma^\nabla \nabla\tau + G^\sigma = const. \tag{57}$$

is the conserved quantity of the system (40) and (49).

Proof. From equation (53), we have

$$\left(\frac{\partial R_v}{\partial t}\xi_0 + \frac{\partial R_v}{\partial a_\mu}\xi_\mu\right)a_v^\Delta\sigma^\nabla + R_v\xi_v^\Delta\sigma^\nabla - B\xi_0^\Delta\sigma^\nabla - \frac{\partial B}{\partial t}\xi_0\sigma^\nabla - \frac{\partial B}{\partial a_\mu}\xi_\mu\sigma^\nabla + G^\Delta\sigma^\nabla - P_v(\xi_v - a_v^\Delta\xi_0)\sigma^\nabla = 0. \tag{58}$$

Using equation (50), we have

$$\frac{\partial R_v}{\partial t}\xi_0 a_v^\Delta\sigma^\nabla + \xi_0 B^\nabla - (\xi_0^\sigma B)^\nabla - \frac{\partial B}{\partial t}\xi_0\sigma^\nabla + (R_v\xi_v^\sigma)^\nabla + G^\Delta\sigma^\nabla - P_v(\xi_v - a_v^\Delta\xi_0)\sigma^\nabla = 0. \tag{59}$$

By nabla indefinite integral of equation (59), we can get conserved quantity (57).

Example 12. Let $\mathbb{T} = \{2^n : n \in \mathbb{N}\cup\{0\}\}$, the nonshifted constrained Birkhoffian system on time scales is

$$B = \frac{1}{2}\left[(a_1)^2 + (a_3)^2 + (a_4)^2\right], R_1 = a_3, R_2 = a_4, R_3 = R_4 = 0. \tag{60}$$

The constraint equations are

$$\begin{aligned} f_1 &= a_1a_3 - (c_1)^2 = 0,\\ f_2 &= a_1 + a_4 - c_2 = 0. \end{aligned} \tag{61}$$

According to equation (49), we have

$$\begin{cases} a_3^\nabla = 2(-\lambda_1 a_3 - \lambda_2 - a_1),\\ a_4^\nabla = 0,\\ a_1^\Delta - a_3 = \lambda_1 a_1,\\ a_2^\Delta - a_4 = \lambda_2. \end{cases} \tag{62}$$

From equations (61) and (62), we have

$$\lambda_1 = -\frac{a_3}{a_1}, \lambda_2 = -a_1 + \frac{(a_3)^2}{a_1}. \tag{63}$$

Hence, we get

$$P_1 = -a_1, P_2 = 0, P_3 = -a_3, P_4 = -a_1 + \frac{(a_3)^2}{a_1}. \tag{64}$$

According to equation (53), we have

$$(a_1^\Delta - a_3)\xi_3 + (a_2^\Delta - a_4)\xi_4 - a_1\xi_1 - B\xi_0^\Delta + a_3\xi_1^\Delta + a_4\xi_2^\Delta - P_1(\xi_1 - a_1^\Delta\xi_0) - P_3(\xi_3 - a_3^\Delta\xi_0) - P_4(\xi_4 - a_4^\Delta\xi_0) + G^\Delta = 0. \tag{65}$$

Equation (65) has the following solutions:

$$\xi_0^1 = 0, \xi_1^1 = 0, \xi_2^1 = 1, \xi_3^1 = 0, \xi_4^1 = 0, G^1 = 0, \tag{66}$$

$$\xi_0^2 = 0, \xi_1^2 = 1, \xi_2^2 = 0, \xi_3^2 = 0, \xi_4^2 = 0, G^2 = 0. \tag{67}$$

According to Theorem 11, the conserved quantities corresponding to the generators (66) and (67) are

$$I^1 = a_4 = \text{const}, \tag{68}$$

$$I^2 = a_3 = \text{const}. \tag{69}$$

5. Conclusions

Time scale has been widely used in many fields. At present, most of the researches on time scales are about the shifted case. In this article, we studied the time-scale version of the nonshifted variational problem for three types of Birkhoffian systems. We proposed the nonshifted generalized Pfaff-Birkhoff principle, derived nonshifted generalized and constrained Birkhoff's equation, studied Noether quasi-symmetries for these nonshifted Birkhoffian systems, and gave the condition of the symmetry resulting in conserved quantity and obtained conserved quantities for these nonshifted Birkhoffian systems on time scales. According to this passage, we also will research symmetries and time-scale conserved quantities for other nonshifted dynamical systems, including Lie and Mei symmetries.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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