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Research Article

Finite-Time Stability of Solutions for Nonlinear q-Fractional Difference Coupled Delay Systems

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In this paper, we investigate and prove a new discrete q-fractional version of the coupled Gronwall inequality. By applying this result, the finite-time stability criteria of solutions for a class of nonlinear q-fractional difference coupled delay systems are obtained. As an application, an example is provided to demonstrate the effectiveness of our result.

1. Introduction

The q-difference equations have numerous applications in diverse fields in recent years and have gained intensive interest [1–4]. For more details on q-calculus, we recommend the readers to [5]. In the last two decades, the fractional difference equations have recently received considerable attention in many fields of science and engineering, see [6–9] and the references therein. We know that the q-fractional difference equations can be used as a bridge between fractional difference equations and q-difference equations, and there are many papers on this research direction which have been appeared in [10–16]. And, we recommend [17] and the papers cited therein.

For $0 < q < 1$, we define the time scale $\mathbb{T}_q = \{q^n: n \in \mathbb{Z}\} \cup \{0\}$, where $\mathbb{Z}$ is the set of integers. For $a = q^{n_0}$ and $n_0 \in \mathbb{Z}$, we denote $\mathbb{T}_a = [a, \infty)_q = \{q^{-i}a: i = 0, 1, 2, \ldots\}$.

In [18], Abdeljawad and Alzabut established a discrete q-fractional version of the Gronwall-type inequality as follows:

Theorem 1 (see [18]). *Let $\alpha > 0$, u and μ be nonnegative real valued functions such that $0 \le \mu(t) < 1/t^{\alpha}(1-q)^{\alpha}$, for all $t \in \mathbb{T}_a$, and*

$$u(t) \le u(a) + {}_q\nabla_a^{-\alpha} u(t)\mu(t), \tag{1}$$

Then,

$$u(t) \le u(a) \sum_{k=0}^{\infty} {}_qE_{\mu}^{k} 1, \tag{2}$$

where ${}_qE_{\mu}^{k}1 = \mu^k (t-a)_q^{k\alpha}/\Gamma_q(k\alpha+1)$.

Abdeljawad et al., in [19], extended the above inequality and obtained the following generalized q-fractional Gronwall-type inequality.

Theorem 2 (see [19]). *Let $\alpha > 0$, u and ν be nonnegative functions, and $w(t)$ be nonnegative and nondecreasing function, for $t \in [a, \infty)_q$, such that $w(t) \le M$, where M is a constant. If*

$$u(t) \le \nu(t) + w(t) {}_q\nabla_a^{-\alpha} u(t), \tag{3}$$

then

$$u(t) \le \nu(t) + \sum_{k=1}^{\infty} \left(w(t)\Gamma_q(\alpha)\right)^k {}_q\nabla_a^{-k\alpha} \nu(t). \tag{4}$$

Based on the above result, Abdeljawad et al. investigated the following nonlinear delay q-fractional difference system:

$$\begin{cases} {}_qC_a^{\alpha}x(t) = A_0x(t) + A_1x(\tau t) + f(t, x(t), x(\tau t)), & t \in [a, \infty)_q, \\ x(t) = \phi(t), & t \in \mathbb{I}_\tau, \end{cases} \tag{5}$$

where ${}_qC_a^{\alpha}$ means the Caputo fractional difference of order α, $\mathbb{I}_\tau = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, \ldots, a\}$, $\tau = q^d \in \mathbb{T}_q$, with $d \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

An interesting topic in control theory is finite-time control, and the objective of finite-time control is to design a control law, making the system state converge to the origin in finite time. In [20], Sun et al. developed the finite-time state feedback stabilisation scheme for nonlinear time-delay systems with high-order and low-order nonlinearities. Recently, Wang and Xiang [21, 22] presented a finite-time output feedback control scheme for a class of nonlinear systems and nonlinear time-delay in the p-normal form, respectively. In [23], Modiri and Mobayen studied the synchronization of fractional-order uncertain chaotic systems in the finite time. Mofid et al., in [24], considered the sliding mode disturbance observer control of a class of fractional-order chaotic systems by using adaptive synchronization. Moreover, the observer-based state feedback stabilizer design for a class of chaotic systems and the fixed-time attitude control for a flexible spacecraft in the presence of actuator faults, external disturbances, and coupling effect of flexible modes have been considered in [25, 26], respectively.

On the contrary, finite-time stability is a method which is much valuable to analyze the transient behavior of nature of a system within a finite interval of time. In recent decades, the finite-time stability analysis of fractional differential systems have recently considerable attention, see, for instance, [27–31] and the references therein. However, till now, few researchers focus on finite-time stability of fractional delay difference systems.

Motivated by the above works, we will to extend the q-fractional Gronwall-type inequality (Theorem 2) to coupled q-fractional Gronwall inequality. As an application, we establish a finite-time stability criterion of the following nonlinear coupled delay q-fractional difference system:

$$\begin{cases} {}_qC_a^{\alpha}x(t) = A_0y(t) + A_1y(\tau t) + f(t, y(t), y(\tau t)), & t \in [a, 1]_q, \\ {}_qC_a^{\alpha}y(t) = B_0x(t) + B_1x(\tau t) + g(t, x(t), x(\tau t)), & t \in [a, 1]_q, \\ x(t) = \phi(t), \quad y(t) = \psi(t), & t \in \mathbb{I}_\tau, \end{cases} \tag{6}$$

where $[a, 1]_q = [a, 1] \cap \mathbb{T}_a$, $\mathbb{I}_\tau = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, \ldots, a\}$, $\tau = q^d \in \mathbb{T}_q$ with $d \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, ${}_qC_a^{\alpha}$ and ${}_qC_a^{\beta}$ mean the Caputo fractional difference of order $\alpha \in (0, 1)$ and order $\beta \in (0, 1)$, respectively, and the constant matrices A_0, A_1, B_0, and B_1 are of appropriate dimensions.

In this paper, the coupled q-fractional Gronwall inequality is studied and given for the first time, which is a powerful tool and method to deal with finite-time stability and other stability of nonlinear coupled delay q-fractional difference systems. And, we studied the finite-time stability of a class of nonlinear coupled delay q-fractional difference system by using this inequality.

The organization of this paper is given as follows. In Section 2, we give some notations, definitions, and preliminaries. Section 3 is devoted to proving a coupled q-fractional Gronwall inequality. In Section 4, the finite-time stability theorem of nonlinear coupled delay q-fractional difference system is proved. In Section 5, an example is given to illustrate our theoretical result. Finally, the paper is concluded in Section 6.

2. Preliminaries

In this section, we provided some basic definitions and lemmas which are used in the sequel.

Let $f\colon \mathbb{T}_q \longrightarrow \mathbb{R}$, and we define the nabla q-derivative of f by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}, \quad t \in \mathbb{T}_q \setminus \{0\}. \tag{7}$$

The nabla q-integral of f has the following form:

$$\int_0^t f(s)\nabla_q s = (1 - q)t \sum_{i=0}^{\infty} q^i f\left(tq^i\right), \tag{8}$$

and for $0 \le a \in \mathbb{T}_q$,

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\nabla_q s - \int_0^a f(s)\nabla_q s. \tag{9}$$

The definition of the q-factorial function for a nonpositive integer α is given by

$$(t - s)_q^{\alpha} = t^{\alpha} \prod_{i=0}^{\infty} \frac{1 - (s/t)q^i}{1 - (s/t)q^{i+\alpha}}. \tag{10}$$

For a function $f\colon \mathbb{T}_q \longrightarrow \mathbb{R}$, the left q-fractional integral ${}_q\nabla_a^{-\alpha}$ of order $\alpha \ne 0, -1, -2, \ldots$ and starting at $0 < a \in \mathbb{T}_q$ is defined by

$${}_q\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - \text{qs})_q^{\alpha-1} f(s)\nabla_q s, \tag{11}$$

where

$$\Gamma_q(\alpha + 1) = \frac{1 - q^{\alpha}}{1 - q}\Gamma_q(\alpha), \quad \Gamma_q(1) = 1,\ \alpha > 0. \tag{12}$$

Definition 1 (see [11]). Let $0 < \alpha \notin \mathbb{N}$. Then, the Caputo left q-fractional derivative of order α of a function f defined on $\mathbb{T}_q$ is defined by

$$\begin{aligned} {}_qC_a^{\alpha} f(t) &:= {}_q\nabla_a^{-(n-\alpha)}\nabla_q^n f(t) \\ &= \frac{1}{\Gamma_q(n - \alpha)} \int_a^t (t - \text{qs})_q^{n-\alpha-1} \nabla_q^n f(s)\nabla_q s, \end{aligned} \tag{13}$$

where $n = [\alpha] + 1$.

Lemma 1 (see [11]). *Let $\alpha > 0$ and f be defined in a suitable domain. Thus,*

$$ {}_q\nabla_a^{-\alpha}{}_qC_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a), \tag{14} $$

and if $0 < \alpha \leq 1$, *we have*

$$ {}_q\nabla_a^{-\alpha}{}_qC_a^{\alpha} f(t) = f(t) - f(a). \tag{15} $$

The following identity plays a crucial role in solving the linear q-fractional equations:

$$ {}_q\nabla_a^{-\alpha}(x-a)_q^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)}(x-a)_q^{\mu+\alpha} \quad 0 < a < x < b, \tag{16} $$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1,\infty)$. The q-analog of the Mittag–Leffler function with double index (α,β) is introduced as follows.

Definition 2 (see [11]). For $z, z_0 \in \mathbb{C}$ and $\Re(\alpha) > 0$, the q-Mittag–Leffler function is defined by

$$ {}_qE_{\alpha,\beta}(\lambda, z-z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z-z_0)_q^{\alpha k}}{\Gamma_q(\alpha k+\beta)}. \tag{17} $$

In the case $\beta = 1$, we utilize ${}_qE_{\alpha}(\lambda, z-z_0) = {}_qE_{\alpha,1}(\lambda, z-z_0)$.

Moreover, the modified q-Mittag–Leffler function is used in [19] as follows:

$$ {}_qe_{\alpha,\beta}(\lambda, z-z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z-z_0)_q^{\alpha k+(\beta-1)}}{\Gamma_q(\alpha k+\beta)}. \tag{18} $$

3. A Generalized Coupled q-Fractional Gronwall Inequality

In this section, we give and prove the following a generalized coupled q-fractional Gronwall inequality, which extend a generalized q-fractional Gronwall inequality in Theorem 2.

Theorem 3. *Assume that* $u(t)$, $v(t)$, *and* $g_i(t)$ $(i = 1,2)$ *are nonnegative functions for* $t \in \mathbb{T}_a$. *Let* $w_i(t)$ $(i = 1,2)$ *be nonnegative and nondecreasing functions for* $t \in \mathbb{T}_a$ *with* $w_i(t) \leq M_i$, *where* M_i *are constants* $(i = 1,2)$. *If*

$$ \begin{cases} u(t) \leq g_1(t) + w_1(t)\,{}_q\nabla_a^{-\alpha} v(t), & t \in [a,1]_q, \\ v(t) \leq g_2(t) + w_2(t)\,{}_q\nabla_a^{-\beta} u(t), & t \in [a,1]_q, \end{cases} \tag{19} $$

and

$$ M_1 M_2 (1-q)^{\alpha+\beta} < 1, \tag{20} $$

hold, then

$$ \begin{aligned} u(t) \leq{}& g_1(t) + \frac{w_1(t)}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{q}s)_q^{\alpha-1} g_2(s) \nabla_q s \\ &+ \sum_{k=1}^{\infty} w_1(t)^k w_2(t)^k\, {}_q\nabla_a^{-k(\alpha+\beta)} \left(g_1(t) + \frac{w_1(t)}{\Gamma_q(\alpha)} \int_q^t (t-\mathrm{q}s)_q^{\alpha-1} g_2(s) \nabla_q s \right), \end{aligned} \tag{21} $$

and

$$ \begin{aligned} v(t) \leq{}& g_2(t) + \frac{w_2(t)}{\Gamma_q(\beta)} \int_a^t (t-\mathrm{q}s)_q^{\beta-1} g_1(s) \nabla_q s \\ &+ \sum_{k=1}^{\infty} w_1(t)^k w_2(t)^k\, {}_q\nabla_a^{-k(\alpha+\beta)} \left(g_2(t) + \frac{w_2(t)}{\Gamma_q(\beta)} \int_a^t (t-\mathrm{q}s)_q^{\beta-1} g_1(s) \nabla_q s \right). \end{aligned} \tag{22} $$

Proof. Let

$$ \mathrm{A}v(t) = \frac{w_1(t)}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{q}s)_q^{\alpha-1} v(s) \nabla_q s, \quad t \in \mathbb{T}_a, \tag{23} $$

and

$$ \mathrm{B}u(t) = \frac{w_2(t)}{\Gamma_q(\beta)} \int_a^t (t-\mathrm{q}s)_q^{\beta-1} u(s) \nabla_q s, \quad t \in \mathbb{T}_a. \tag{24} $$

According to (19), one has

$$ u(t) \leq g_1(t) + \mathrm{A}v(t), \; v(t) \leq g_2(t) + \mathrm{B}u(t). \tag{25} $$

By (25) and the monotonicity of the operators A and B, we obtain

$$\begin{aligned} u(t) &\le g_1(t) + A(g_2(t) + \mathrm{Bu}(t)) = g_1(t) + \mathrm{Ag}_2(t) + \mathrm{ABu}(t) \\ &\le g_1(t) + Ag_2(t) + \mathrm{AB}\left[g_1(t) + \mathrm{Ag}_2(t) + \mathrm{ABu}(t)\right] \\ &= g_1(t) + \mathrm{ABg}_1(t) + \mathrm{Ag}_2(t) + \mathrm{ABAg}_2(t) + (\mathrm{AB})^2 u(t) \\ &\le \sum_{k=0}^{n-1} (\mathrm{AB})^k g_1(t) + \sum_{k=0}^{n-1} (\mathrm{AB})^k \mathrm{Ag}_2(t) + (\mathrm{A})^n u(t), \quad t \in \mathbb{T}_a. \end{aligned} \tag{26}$$

Similarly, we obtain

$$v(t) \le \sum_{k=0}^{n-1} (\mathrm{BA})^k g_2(t) + \sum_{k=0}^{n-1} (\mathrm{BA})^k \mathrm{Bg}_1(t) + (\mathrm{BA})^n v(t), \quad t \in \mathbb{T}_a, \tag{27}$$

where $(\mathrm{AB})^0 g_1(t) = g_1(t)$ and $(\mathrm{BA})^0 g_2(t) = g_2(t)$.

In the following, we will prove that

$$(\mathrm{AB})^n u(t) \le w_1(t)^n w_2(t)^n {}_q\nabla_a^{-n(\alpha+\beta)} u(t), \tag{28}$$

$$(\mathrm{BA})^n v(t) \le w_1(t)^n w_2(t)^n {}_q\nabla_a^{-n(\alpha+\beta)} v(t), \tag{29}$$

where $t \in \mathbb{T}_a$ and

$$\lim_{n\longrightarrow\infty} (\mathrm{AB})^n u(t) = 0, \quad \lim_{n\longrightarrow\infty} (\mathrm{BA})^n v(t) = 0. \tag{30}$$

We know that (28) and (29) are true for $n = 1$. In fact, one has

$$\begin{aligned} \mathrm{ABu}(t) &= A(\mathrm{Bu}(t)) = \frac{w_1(t)}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} \mathrm{Bu}(s) \nabla_q s \\ &= \frac{w_1(t)}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} w_2(s) \int_a^s (t-\mathrm{qr})_q^{\beta-1} u(r) \nabla_q r \nabla_q s \\ &\le \frac{w_1(t) w_2(t)}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} \int_a^s (t-\mathrm{qr})_q^{\beta-1} u(r) \nabla_q r \nabla_q s \\ &= \frac{w_1(t) w_2(t)}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_a^t \int_r^t (t-\mathrm{qs})_q^{\alpha-1} (s-\mathrm{qr})_q^{\beta-1} u(r) \nabla_q r \nabla_q s \\ &= \frac{w_1(t) w_2(t)}{\Gamma_q(\beta)} \int_a^t \left[\frac{1}{\Gamma_q(\alpha)} \int_r^t (t-\mathrm{qs})_q^{\alpha-1} (s-\mathrm{qr})_q^{\beta-1} \nabla_q s\right] u(r) \nabla_q r \\ &= \frac{w_1(t) w_2(t)}{\Gamma_q(\beta)} \int_a^t {}_q\nabla_{qr}^{-\alpha} (t-\mathrm{qr})_q^{\beta-1} u(r) \nabla_q r, \end{aligned} \tag{31}$$

where ${}_q\nabla_{qr}^{-\alpha} u(t) = 1/\Gamma_q(\alpha) \int_{qr}^t (t-\mathrm{qs})_q^{\alpha-1} u(s) \nabla_q s$ has been used. By (16), we have

$${}_q\nabla_a^{-\alpha} (x-a)_q^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)_q^{\mu+\alpha} \quad (0 < a < x < b),\ \mu > -1. \tag{32}$$

Combining (31) with (32), one has

$$\begin{aligned} \mathrm{ABu}(t) &\le \frac{w_1(t) w_2(t)}{\Gamma_q(\beta)} \int_a^t \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} (t-\mathrm{qr})_q^{\alpha+\beta-1} u(r) \nabla_q r \\ &= \frac{w_1(t) w_2(t)}{\Gamma_q(\alpha+\beta)} \int_a^t (t-\mathrm{qr})_q^{\alpha+\beta-1} u(r) \nabla_q r \\ &= w_1(t) w_2(t) {}_q\nabla_a^{-(\alpha+\beta)} u(t). \end{aligned} \tag{33}$$

Similarly, one has

$$\mathrm{BA}v(t) \leq w_1(t) w_2(t)_q \nabla_a^{-(\alpha+\beta)} v(t). \tag{34}$$

Thus, (28) and (29) are valid for $n = 1$. Assume that (28) and (29) are true, for $n = k$, which are

$$(\mathrm{AB})^k u(t) \leq w_1(t)^k w_2(t)^k {}_q\nabla_a^{-k(\alpha+\beta)} u(t), \tag{35}$$

$$(\mathrm{BA})^k v(t) \leq w_1(t)^k w_2(t)^k {}_q\nabla_a^{-k(\alpha+\beta)} v(t). \tag{36}$$

For $n = k+1$ and $t \in \mathbb{N}_{a+1+k(\nu+\mu)}$, by using (33) and (35) and the nondecreasing functions $w_1(t)$ and $w_2(t)$, we obtain

$$\begin{aligned}
(\mathrm{AB})^{k+1} u(t) &= \mathrm{AB}\left((\mathrm{AB})^k u(t)\right) \\
&\leq \frac{w_1(t) w_2(t)}{\Gamma_q(\alpha+\beta)} \int_a^t (t-\mathrm{q}s)_q^{\alpha+\beta-1} \int_a^s \frac{w_1(s)^k w_2(s)^k}{\Gamma_q(k(\alpha+\beta))} (t-\mathrm{q}r)_q^{k(\alpha+\beta)-1} u(r) \nabla_q r \nabla_q s \\
&\leq \frac{w_1(t)^{k+1} w_2(t)^{k+1}}{\Gamma_q(k(\alpha+\beta))} \int_a^t \left[\int_r^t \frac{1}{\Gamma_q(\alpha+\beta)} (t-\mathrm{q}s)_q^{\alpha+\beta-1} (s-\mathrm{q}r)_q^{k(\alpha+\beta)-1} \nabla_q s\right] u(r) \nabla_q r \\
&= \frac{w_1(t)^{k+1} w_2(t)^{k+1}}{\Gamma_q(k(\alpha+\beta))} \int_a^t {}_q\nabla_{qr}^{-(\alpha+\beta)} (t-\mathrm{q}r)_q^{k(\alpha+\beta)-1} u(r) \nabla_q r,
\end{aligned} \tag{37}$$

where ${}_q\nabla_{qr}^{-(\alpha+\beta)} u(t) = 1/\Gamma_q(\alpha) \int_{qr}^t (t-\mathrm{q}s)_q^{\alpha+\beta-1} u(s) \nabla_q s$ has been used. By using (16) and (37), we obtain

$$\begin{aligned}
(\mathrm{AB})^{k+1} u(t) &\leq \frac{w_1(t)^{k+1} w_2(t)^{k+1}}{\Gamma_q(k(\alpha+\beta))} \int_a^t (t-\mathrm{q}r)_q^{(k+1)(\alpha+\beta)-1} \frac{\Gamma_q(k(\alpha+\beta))}{\Gamma_q(k+1)(\alpha+\beta)} u(r) \nabla_q r \\
&= \frac{w_1(t)^{k+1} w_2(t)^{k+1}}{\Gamma_q((k+1)(\alpha+\beta))} \int_a^t (t-\mathrm{q}r)_q^{(k+1)(\alpha+\beta)-1} u(r) \nabla_q r \\
&= w_1(t)^{k+1} w_2(t)^{k+1} {}_q\nabla_a^{-(k+1)(\alpha+\beta)} u(t).
\end{aligned} \tag{38}$$

Similarly, we can obtain

$$(\mathrm{BA})^{k+1} v(t) \leq w_1(t)^{k+1} w_2(t)^{k+1} {}_q\nabla_a^{-(k+1)(\alpha+\beta)} v(t). \tag{39}$$

Thus, (28) and (29) are proved.

Using Stirling's formula of the q-gamma function [32] yields that

$$\Gamma_q(x) = [2]_q^{1/2} \Gamma_{q^2}\left(\frac{1}{2}\right)(1-q)^{(1/2)-x} e^{\theta q^x/(1-q)-q^x}, \quad 0<\theta<1, \tag{40}$$

that is,

$$\Gamma_q(x) \sim D(1-q)^{(1/2)-x}, \quad x \longrightarrow \infty, \tag{41}$$

where $D = [2]_q^{1/2} \Gamma_{q^2}(1/2)$. Moreover, if $t > a > 0$ and $\gamma > 0$ (γ is not a positive integer), then $1 - a/tq^j < 1 - a/tq^{\gamma+j}$, for each $j = 0, 1, \ldots$, and

$$(t-a)_q^\gamma = t^\gamma \prod_{j=0}^{\infty} \frac{1 - a/tq^j}{1 - a/tq^{\gamma+j}} < t^\gamma. \tag{42}$$

Applying the first mean value theorem for definite integrals [33], (41) and (42), and $w_1(t) < M_1$ and $w_2(t) < M_2$, there exists a $\xi \in [a, 1]_q$ such that

$$\begin{aligned}\lim_{n\longrightarrow\infty}(\mathrm{AB})^n u(t) &\le \lim_{n\longrightarrow\infty} u(\xi)\frac{M_1^n M_2^n}{\Gamma_q(n(\alpha+\beta))}\int_a^t (t-\mathrm{q}r)_q^{n(\alpha+\beta)-1}\nabla_q r\\ &= \lim_{n\longrightarrow\infty} u(\xi)\frac{M_1^n M_2^n}{\Gamma_q(n(\alpha+\beta)+1)}(t-a)_q^{n(\alpha+\beta)}\\ &\le \lim_{n\longrightarrow\infty} u(\xi)\frac{M_1^n M_2^n}{\Gamma_q(n(\alpha+\beta)+1)}t^{n(\alpha+\beta)}\\ &= \lim_{n\longrightarrow\infty} u(\xi)\frac{M_1^n M_2^n}{D(1-q)^{1/2-(n(\alpha+\beta)+1)}}t^{n(\alpha+\beta)}\\ &\le \lim_{n\longrightarrow\infty} u(\xi)\frac{M_1^n M_2^n}{D(1-q)^{1/2-(n(\alpha+\beta)+1)}}\\ &= \lim_{n\longrightarrow\infty}\frac{u(\xi)\sqrt{1-q}}{D}\left[M_1M_2(1-q)^{\alpha+\beta}\right]^n.\end{aligned} \tag{43}$$

From (20), for each $t\in[a,1]_q$, one has

$$\left[M_1M_2(1-q)^{\alpha+\beta}\right]^n \longrightarrow 0, \quad \text{as}\, n\longrightarrow\infty. \tag{44}$$

Thus, $(\mathrm{AB})^n u(t)\longrightarrow 0$ as $n\longrightarrow\infty$, Similarly, we can obtain that $(\mathrm{BA})^n u(t)\longrightarrow 0$ as $n\longrightarrow\infty$, for each $t\in\mathbb{T}_a$. Therefore, (30) is proved.

Let $n\longrightarrow\infty$ in (26); with the help of the semigroup property ${}_q\nabla_a^{-\alpha}\,{}_q\nabla_a^{-\mu}={}_q\nabla_a^{-(\alpha+\mu)}$ and the definition of A and B, one obtains

$$\begin{aligned}u(t) &\le g_1(t)+Ag_2(t)+\sum_{k=1}^{\infty}(\mathrm{AB})^k g_1(t)+\sum_{k=1}^{\infty}(\mathrm{AB})^k Ag_2(t)\\ &= g_1(t)+\frac{w_1(t)}{\Gamma_q(\alpha)}\int_a^t (t-\mathrm{q}s)_q^{\alpha-1}g_2(s)\nabla_q s\\ &\quad+\sum_{k=1}^{\infty}w_1(t)^k w_2(t)^k\,{}_q\nabla_a^{-k(\alpha+\beta)}\left(g_1(t)+\frac{w_1(t)}{\Gamma_q(\alpha)}\int_a^t (t-\mathrm{q}s)_q^{\alpha-1}g_2(s)\nabla_q s\right).\end{aligned} \tag{45}$$

Similarly, let $n\longrightarrow\infty$ in (27), and we obtain (22). This completes the proof. □

Corollary 1. *Under the hypothesis of Theorem 3, let $g_1(t)$ and $g_2(t)$ be two nondecreasing functions on $t\in\mathbb{T}_a$. Then,*

$$\begin{aligned}u(t) &\le g_1(t)\,{}_qE_{\alpha+\beta}(w_1(t)w_2(t),t-a)\\ &\quad+w_1(t)g_2(t)\,{}_qe_{\alpha+\beta,\alpha+1}(w_1(t)w_2(t),t-a)\end{aligned} \tag{46}$$

and

$$\begin{aligned}v(t) &\le g_2(t)\,{}_qE_{\alpha+\beta}(w_1(t)w_2(t),t-a)\\ &\quad+w_2(t)g_1(t)\,{}_qe_{\alpha+\beta,\beta+1}(w_1(t)w_2(t),t-a).\end{aligned} \tag{47}$$

Proof. By (16) and the assumption that $g_1(t)$, $g_2(t)$, and $w_1(t)$ are three nondecreasing functions for $t\in\mathbb{T}_a$, we have

$${}_q\nabla_a^{-\alpha}g_2(t)\le g_2(t)\,{}_q\nabla_a^{-\alpha}1=\frac{g_2(t)}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha}, \tag{48}$$

and

$$\begin{aligned}&{}_q\nabla_a^{-k(\alpha+\beta)}\left(g_1(t)+w_1(t)\,{}_q\nabla_a^{-\alpha}g_2(t)\right)\\ &\le\left(g_1(t)+w_1(t)\,{}_q\nabla_a^{-\alpha}g_2(t)\right){}_q\nabla_a^{-k(\alpha+\beta)}1\\ &=\left(g_1(t)+w_1(t)\,{}_q\nabla_a^{-\alpha}g_2(t)\right)\frac{(t-a)_q^{k(\alpha+\beta)}}{\Gamma_q(k(\alpha+\beta)+1)}.\end{aligned} \tag{49}$$

Thus, from (21), (48), and (49), one can conclude that

$$\begin{aligned}
u(t) &\le \left(g_1(t) + w_1(t) g_2(t)_q\nabla_a^{-\alpha}1\right)\left[1 + \sum_{k=1}^{\infty} w_1(t)^k w_2(t)^k{}_q\nabla_a^{-k(\alpha+\beta)}1\right] \\
&= g_1(t)\left[1 + \sum_{k=1}^{\infty} w_1(t)^k w_2(t)^k{}_q\nabla_a^{-k(\alpha+\beta)}1\right] + \frac{w_1(t) g_2(t)}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha} \\
&\quad + w_1(t) g_2(t)\sum_{k=1}^{\infty} w_1(t)^k w_2(t)^k{}_q\nabla_a^{-k(\alpha+\beta)}{}_q\nabla_a^{-\alpha}1 \\
&= g_1(t)\sum_{k=0}^{\infty}\frac{(w_1(t) w_2(t))^k (t-a)_q^{k(\alpha+\beta)}}{\Gamma_q(k(\alpha+\beta)+1)} + \frac{w_1(t) g_2(t)}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha} \\
&\quad + w_1(t) g_2(t)\sum_{k=1}^{\infty} w_1(t)^k w_2(t)^k{}_q\nabla_a^{-k(\alpha+\beta)-\alpha}1 \\
&= g_1(t)_q E_{\alpha+\beta}(w_1(t) w_2(t), t-a) + \frac{w_1(t) g_2(t)}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha} \\
&\quad + w_1(t) g_2(t)\sum_{k=1}^{\infty}\frac{(w_1(t) w_2(t))^k (t-a)_q^{k(\alpha+\beta)+\alpha}}{\Gamma_q(k(\alpha+\beta)+\alpha+1)} \\
&= g_1(t)_q E_{\alpha+\beta}(w_1(t) w_2(t), t-a) + w_1(t) g_2(t)\sum_{k=0}^{\infty}\frac{(w_1(t) w_2(t))^k (t-a)_q^{k(\alpha+\beta)+\alpha}}{\Gamma_q(k(\alpha+\beta)+\alpha+1)} \\
&= g_1(t)_q E_{\alpha+\beta}(w_1(t) w_2(t), t-a) + w_1(t) g_2(t)_q e_{\alpha+\beta,\alpha+1}(w_1(t) w_2(t), t-a).
\end{aligned} \tag{50}$$

Similarly,we can obtain (47) holds. □

4. Main Result

Throughout this paper, we make the following assumptions:

(H1) $f, g \in D(\mathbb{T}_a \times \mathbb{R}^n, \mathbb{R}^n)$ are two Lipschitz-type functions. That is, for any x, y, u, v: $\mathbb{T}_{\tau a} \longrightarrow \mathbb{R}^n$, there exist two positive constants $L_1, L_2 > 0$ such that

$$\begin{aligned}
&\|f(t, y(t), y(\tau t)) - g(t, v(t), tvn(\tau t))\| \le L_1(\|y(t) - v(t)\| + \|y(\tau t) - v(\tau t)\|), \\
&\|g(t, x(t), x(\tau t)) - g(t, u(t), tun(\tau t))\| \le L_2(\|x(t) - u(t)\| + \|x(\tau t) - u(\tau t)\|),
\end{aligned} \tag{51}$$

For $t \in [a, 1]_q$,

$$\begin{aligned}
&(\text{H2}) f(t,0,0) = \underbrace{[0,0,\ldots,0]}_{n}^{T}, \quad g(t,0,0) = \underbrace{[0,0,\ldots,0]}_{n}^{T}, \\
&(\text{H3})\left(\|A_0\| + \|A_1\| + 2L_1\right)\left(\|B_0\| + \|B_1\| + 2L_2\right)(1-q)^{\alpha+\beta} < 1.
\end{aligned} \tag{52}$$

Let us denote $\|(\phi, \psi)\| = \max\left\{\max_{t\in\mathbb{I}_\tau}\|\phi(t)\|, \max_{t\in\mathbb{I}_\tau}\|\psi(t)\|\right\}$.

Definition 3. System (6) is finite-time stable with respect to $\{\delta, \varepsilon, T\}$, with $\delta < \epsilon$, if and only if $\|(\phi, \psi)\| < \delta$ implies $\|(x(t), y(t))\| = \max\{\|x(t)\|, \|y(t)\|\} < \varepsilon$, $\forall t \in \mathbb{T}_{\tau a}$.

Theorem 4. *Assume that conditions (H1)–(H3) hold. Then, system (6) is finite-time stable if the following conditions are satisfied:*

$$\begin{aligned}&\left(1+\frac{\|A_1\|+L_1}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha}\right)_q E_{\alpha+\beta}(c,t-a)\\&+\left(\|A_0\|+\|A_1\|+2L_1\right)\left(1+\frac{\|B_1\|+L_2}{\Gamma_q(\beta+1)}(t-a)_q^{\beta}\right)_q e_{\alpha+\beta,\alpha+1}(c,t-a)<\frac{\varepsilon}{\delta},\end{aligned}\tag{53}$$

and

$$\begin{aligned}&\left(1+\frac{\|B_1\|+L_2}{\Gamma_q(\beta+1)}(t-a)_q^{\beta}\right)_q E_{\alpha+\beta}(c,t-a)\\&+\left(\|B_0\|+\|B_1\|+2L_2\right)\left(1+\frac{\|A_1\|+L_1}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha}\right)_q e_{\alpha+\beta,\beta+1}(c,t-a)<\frac{\varepsilon}{\delta},\end{aligned}\tag{54}$$

where $c=(\|A_0\|+\|A_1\|+2L_1)(\|B_0\|+\|B_1\|+2L_2)$.

Proof. From Theorem 4 in [19], it is easy to see that $(x,y)\colon \mathbb{T}_{\tau a}\times\mathbb{T}_{\tau a}\longrightarrow\mathbb{R}^n\times\mathbb{R}^n$ is a solution of system (6) if and only if

$$\begin{cases}x(t)=\phi(\alpha)+\dfrac{1}{\Gamma_q(\alpha)}\displaystyle\int_a^t(t-\mathrm{qs})_q^{\alpha-1}\left[A_0y(s)+A_1y(\tau s)+f(s,y(s),y(\tau s))\right]\nabla_q s,\\[2ex] y(t)=\psi(\alpha)+\dfrac{1}{\Gamma_q(\beta)}\displaystyle\int_a^t(t-\mathrm{qs})_q^{\beta-1}\left[B_0x(s)+B_1x(\tau s)+g(s,x(s),x(\tau s))\right]\nabla_q s,\\[2ex] x(t)=\phi(t),\ y(t)=\psi(t),\ t\in\mathbb{I}_{\tau}.\end{cases}\tag{55}$$

For $t\in\mathbb{T}_a$, we have, by (55), that

$$\begin{aligned}\|x(t)\|&\le\|\phi(a)\|+\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-\mathrm{qs})_q^{\alpha-1}\left\|A_0y(s)+A_1y(\tau s)+f(s,y(s)y(\tau s))\right\|\nabla_q s\\&\le\|\phi\|+\frac{\|A_0\|}{\Gamma_q(\alpha)}\int_a^t(t-\mathrm{qs})_q^{\alpha-1}\|y(s)\|\nabla_q s+\frac{\|A_1\|}{\Gamma_q(\alpha)}\int_a^t(t-\mathrm{qs})_q^{\alpha-1}\|y(\tau s)\|\nabla_q s\\&\quad+\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-\mathrm{qs})_q^{\alpha-1}\|f(s,y(s),y(s))krqs\\&\le\|\phi\|+\frac{\|A_0\|+L_1}{\Gamma_q(\alpha)}\int_a^t(t-\mathrm{qs})_q^{\alpha-1}\|y(s)\|\nabla_q s+\frac{\|A_1\|+L_1}{\Gamma_q(\alpha)}\int_a^t(t-\mathrm{qs})_q^{\alpha-1}\|y(\tau s)\|\nabla_q s\end{aligned}$$

$$\begin{aligned}
&\le \|\phi\| + \frac{\|A_0\| + L_1}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} \|y(s)\| \nabla_q s \\
&\quad + \frac{\|A_1\| + L_1}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} \left[\sup_{\theta\in\mathbb{J}_\tau} \|y(\theta s)\| + \|\psi\|\right] \nabla_q s \\
&= \|\phi\| + \frac{\|\psi\|\left(\|A_1\| + L_1\right)}{\Gamma_q(\alpha+1)} (t-a)_q^{\alpha} + \frac{\|A_0\| + L_1}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} \|y(s)\| \nabla_q s \\
&\quad + \frac{\|A_1\| + L_1}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} \sup_{\theta\in\mathbb{J}_\tau} \|y(\theta s)\| \nabla_q s,
\end{aligned} \tag{56}$$

where $\mathbb{J}_\tau = \{\{\tau, \tau q^{-1}, \ldots, 1\}$. Similarly, we can obtain

$$\begin{aligned}
\|y(t)\| &\le \|\psi\| + \frac{\|\phi\|\left(\|B_1\| + L_2\right)}{\Gamma_q(\beta+1)} (t-a)_q^{\beta} + \frac{\|B_0\| + L_2}{\Gamma_q(\beta)} \int_a^t (t-\mathrm{qs})_q^{\beta-1} \|x(s)\| \nabla_q s \\
&\quad + \frac{\|B_1\| + L_2}{\Gamma_q(\beta)} \int_a^t (t-\mathrm{qs})_q^{\beta-1} \sup_{\theta\in\mathbb{J}_\tau} \|x(\theta s)\| \nabla_q s.
\end{aligned} \tag{57}$$

Let $g_1(t) = \|\phi\| + \|\psi\|(\|A_1\| + L_1)/\Gamma_q(\alpha+1)(t-a)_q^{\alpha}$ and $g_2(t) = \|\psi\| + \|\phi\|(\|B_1\| + L_2)/\Gamma_q(\beta+1)(t-a)_q^{\beta}$; then, g_1 and g_2 are two nondecreasing functions.

Set $\overline{x}(t) = \sup_{\theta\in\mathbb{J}_\tau}\|x(\theta t)\|$ and $\overline{y}(t) = \sup_{\theta\in\mathbb{J}_\tau}\|y(\theta t)\|$; then, by (56), we obtain

$$\begin{aligned}
\overline{x}(t) &\le g_1(t) + \frac{\|A_0\| + L_1}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} \overline{y}(s) \nabla_q s \\
&\quad + \frac{\|A_1\| + L_1}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} \overline{y}(s) \nabla_q s \\
&= g_1(t) + \frac{\|A_0\| + \|A_1\| + 2L_1}{\Gamma_q(\alpha)} \int_a^t (t-\mathrm{qs})_q^{\alpha-1} \overline{y}(s) \nabla_q s \\
&= g_1(t) + \left(\|A_0\| + \|A_1\| + 2L_1\right) {}_q\nabla_a^{-\alpha} \overline{y}(t).
\end{aligned} \tag{58}$$

Similarly, we obtain

$$\overline{y}(t) \le g_2(t) + \left(\|B_0\| + \|B_1\| + 2L_2\right) {}_q\nabla_a^{-\beta} \overline{x}(t). \tag{59}$$

Hence, by (53) and (54) and applying the result of Corollary 1, we have

$$\begin{aligned}
\|x(t)\| &\le \overline{x}(t) \le g_1(t)\, {}_qE_{\alpha+\beta}(c, t-a) \\
&\quad + \left(\|A_0\| + \|A_1\| + 2L_1\right) g_2(t)\, {}_qe_{\alpha+\beta,\alpha+1}(c, t-a) \\
&= \left(1 + \frac{\|A_1\| + L_1}{\Gamma_q(\alpha+1)} (t-a)_q^{\alpha}\right) \|(\phi,\psi)\|\, {}_qE_{\alpha+\beta}(c, t-a)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\left\|A_0\right\|+\left\|A_1\right\|+2L_1\right)\left(1+\frac{\left\|B_1\right\|+L_2}{\Gamma_q(\beta+1)}(t-a)_q^{\beta}\right)\|(\phi,\psi)\|_q e_{\alpha+\beta,\alpha+1}(c,t-a)\\
&\leq \delta\left(1+\frac{\left\|A_1\right\|+L_1}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha}\right)_q E_{\alpha+\beta}(c,t-a)\\
&+\delta\left(\left\|A_0\right\|+\left\|A_1\right\|+2L_1\right)\left(1+\frac{\left\|B_1\right\|+L_2}{\Gamma_q(\beta+1)}(t-a)_q^{\beta}\right)_q e_{\alpha+\beta,\alpha+1}(c,t-a)<\varepsilon,
\end{aligned}
\tag{60}
$$

and

$$
\begin{aligned}
\|y(t)\| &\leq \overline{y}(t)\leq g_2(t)_q E_{\alpha+\beta}(c,t-a)\\
&+\left(\left\|B_1\right\|+\left\|B_0\right\|+2L_2\right)g_1(t)_q e_{\alpha+\beta,\beta+1}(c,t-a)\\
&=\left(\|\psi\|+\frac{\|\phi\|\left(\left\|B_1\right\|+L_2\right)}{\Gamma_q(\beta+1)}(t-a)_q^{\beta}\right)_q E_{\alpha+\beta}(c,t-a)\\
&+\left(\left\|B_1\right\|+\left\|B_0\right\|+2L_2\right)\left(\|\phi\|+\frac{\|\psi\|\left(\left\|A_1\right\|+L_1\right)}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha}\right)_q e_{\alpha+\beta,\beta+1}(c,t-a).\\
&\leq \delta\left(1+\frac{\left\|B_1\right\|+L_2}{\Gamma_q(\beta+1)}(t-a)_q^{\beta}\right)_q E_{\alpha+\beta}(c,t-a)\\
&+\delta\left(\left\|B_1\right\|+\left\|B_0\right\|+2L_2\right)\left(1+\frac{\left\|A_1\right\|+L_1}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha}\right)_q e_{\alpha+\beta,\beta+1}(c,t-a)<\varepsilon.
\end{aligned}
\tag{61}
$$

□

Remark 1. In [34], Lyu and Vong proposed and analyzed a scheme on nonuniform mesh for solving the q-Volterra equation which provides the same solution to the q-fractional initial value problem. We will apply the proposed scheme as in (3.12) in [32] to solve Example 1.

5. Example

Example 1. Consider the nonlinear delay q-fractional differential difference coupled system:

$$
\begin{cases}
{}_qC_a^{0.7}x(t)=0.7y(t)+\sin\left(\dfrac{1}{8}y(\tau t)\right), & t\in[a,1]_q,\\
{}_qC_a^{0.5}y(t)=0.8x(t)+\arctan\left(\dfrac{1}{9}x(\tau t)\right), & t\in[a,1]_q,
\end{cases}
\tag{62}
$$

where $\alpha=0.7$, $\beta=0.5$, $q=0.6$, $a=q^9=0.6^9$, $\tau=q^9=0.6^9$, $f(t,y(t),y(\tau t))=\sin((1/8)y(\tau t))$, $g(t,x(t),x(\tau t))=\arctan((1/9)x(\tau t))$, $\phi(t)=0.07$, and $\psi(t)=0.09$, and $t\in\mathbb{I}_\tau=\{0.6^{18},0.6^{17},\ldots,0.6^9\}$.

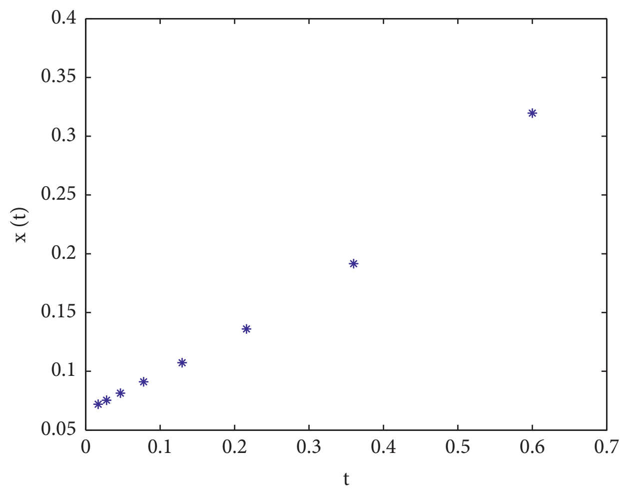

Figure 1: Behavior of the solution of system (62) in the (t, x) plane within $T = 0.6$s.

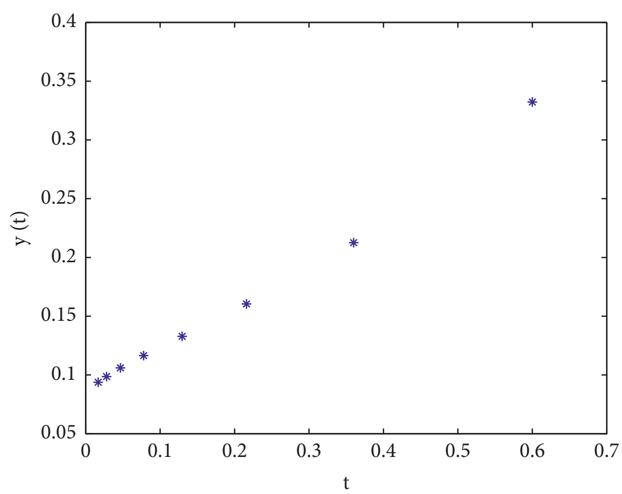

Figure 2: Behavior of the solution of system (62) in the (t, y) plane within $T = 0.6$s.

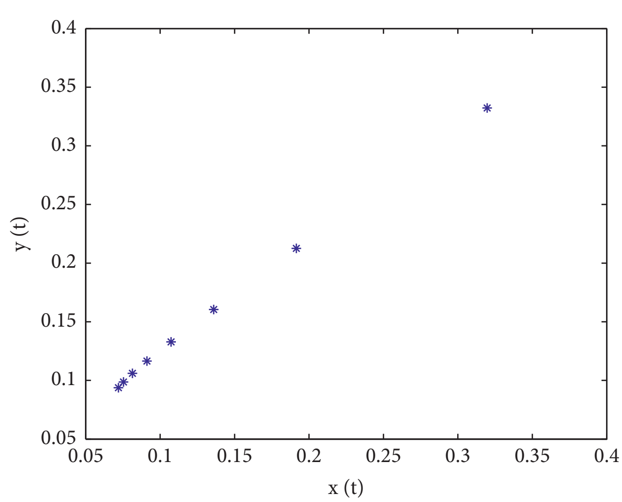

Figure 3: Phase plot of (62).

Let $\delta = 0.1$ and $\epsilon = 0.5$. Obviously, $\|(\phi,\psi)\| = \max\{0.07, 0.09\} < 0.1 = \delta$. We can see that f, g satisfy conditions (H2) and (H1) with $L_1 = 1/8$ and $L_2 = 1/9$. Moreover, we have $\|A_0\| = 0.7$, $\|A_1\| = 0$, $\|B_0\| = 0.8$, and $\|B_1\| = 0$, and (H3) holds. In the following, the aim is to validate the FTS conditions (53) and (54), w.r.t. $\{\delta, \epsilon, T\}$. By using Matlab (the pseudocode to compute different values of $\Gamma_q(\alpha)$, see [35]), when $t = 0.6 \in [a, 1]_q$,

$$\begin{aligned}
&\left(1 + \frac{\|A_1\| + L_1}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha}\right){}_qE_{\alpha+\beta}(c, t-a) \\
&+\left(\|A_0\| + \|A_1\| + 2L_1\right)\left(1 + \frac{\|B_1\| + L_2}{\Gamma_q(\beta+1)}(t-a)_q^{\beta}\right){}_qe_{\alpha+\beta,\alpha+1}(c, t-a) \\
&\approx 3.4304 < 5 = \frac{\varepsilon}{\delta},
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
&\left(1 + \frac{\|B_1\| + L_2}{\Gamma_q(\beta+1)}(t-a)_q^{\beta}\right){}_qE_{\alpha+\beta}(c, t-a) \\
&+\left(\|B_0\| + \|B_1\| + 2L_2\right)\left(1 + \frac{\|A_1\| + L_1}{\Gamma_q(\alpha+1)}(t-a)_q^{\alpha}\right){}_qe_{\alpha+\beta,\beta+1}(c, t-a) \\
&\approx 4.8461 < 5 = \frac{\varepsilon}{\delta}.
\end{aligned} \tag{64}$$

Thus, we obtain that the estimated time of FTS is $T = 0.6$.

Within given parameters, we can observe the finite-time behavior. In Figures 1 and 2, we can see that, within the finite-time $T = 0.6s$, $\|(\phi,\psi)\| = 0.09 < \delta = 0.1$, the norm $\|(x, y)\|$ of solution $(x(t), y(t))$ does not exceed $\varepsilon = 0.5$ which supposes the theorem numerically. The phase plot of (62) is shown in Figure 3.

6. Conclusion

The problem of finite-time stability of coupled q-fractional difference delay systems is emphasized in this work. For this, we obtained a generalized coupled q-fractional Gronwall inequality, and by applying this inequality, a novel and easy to verify sufficient conditions have been provided in this paper to determine the finite-time stability of the solutions for the considered system. Finally, an example is given to illustrate the effectiveness and feasibility of our criterion.

In the future, we will consider more q-fractional difference systems, such as q-fractional difference singular systems or q-fractional difference uncertain systems, and we will study the problems of finite-time stability, and the Ulam–Hyers stability for these systems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

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