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Research Article

Existence and Regularity of Solutions for Unbounded Elliptic Equations with Singular Nonlinearities

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For $q, \gamma > 0$, we study existence and regularity of solutions for unbounded elliptic problems whose simplest model is $\begin{cases} -\mathrm{div}[(1+|u|^q)\nabla u] = (f/|u|^c) & \text{in } \Omega \\ u = 0 & \text{on } z\Omega \end{cases}$, where $f \in L^m(\Omega)$, $m \geq 1$.

1. Introduction

Consider the Dirichlet problem for some nonlinear elliptic equations:

$$-\mathrm{div}\left(\left[a(x) + |u|^q\right]\nabla u\right) = \frac{f}{|u|^\gamma}, \quad x \in \Omega, u \in H_0^1(\Omega), \tag{1}$$

under the following assumptions. The set Ω is a bounded open subset of $\mathbb{R}^N$, with $N \geq 3$:

$$q, \gamma > 0. \tag{2}$$

$a\colon \Omega \longrightarrow \mathbb{R}$ is a measurable function satisfying the following conditions:

$$\alpha \leq a(x) \leq \beta, \tag{3}$$

for almost every $x \in \Omega$, where α and β are positive constant, and

$$0 \lneqq f \in L^m(\Omega), \quad \text{with } m \geq 1. \tag{4}$$

A possible motivation for studying the existence of these types of problems arises from the calculation of variations and stochastic control. For example, if we consider the functional

$$J(v) = \frac{1}{2}\int_\Omega \left[a(x) + |v|^{1-\theta}\right]|\nabla v|^2 - \int_\Omega f(x)v, \tag{5}$$

the Euler–Lagrange equation associated to the functional J is

$$-\mathrm{div}\left(\left[a(x) + |v|^{1-\theta}\right]\nabla v\right) + \frac{1-\theta}{2}\frac{|\nabla v|^2}{|v|^\theta}\mathrm{sign}(v) = f. \tag{6}$$

Several papers deal with existence of solutions to the singular elliptic problems with lower order terms having a quadratic growth with respect to the gradient (for example, [1–9]), namely, with the model problem

$$\begin{cases} -\mathrm{div}(M(x,u)\nabla u) + \dfrac{|\nabla u|^2}{|u|^\theta}\mathrm{sign}(u) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{7}$$

where θ is a positive constant and $M\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function. More precisely, existence of positive solutions for (7) was shown in [1–3], for $M(x,t) = 1$ and $0 < \theta \leq 1$, and the uniqueness of positive solution, for $M(x,t) = 1$ and $0 < \theta < 1$, in [4]. On the contrary, the existence of positive solutions of (7) is shown in [6] for $0 < \theta \leq 1$, provided M is a bounded uniformly elliptic matrix and $0 \lneqq f \in L^m(\Omega)$ $(m > (2N/N+2))$. Later, in [9], it is proved the existence of solution for (7) with $0<\theta<1$, where $M(x,t)=1$ and the data $f\in L^m(\Omega)$ with $m>(N/2)$, and does not satisfy any sign assumption. Recently, a problem introduced by L. Boccardo (see [7, 10]) has given a strong impulse to the study of quasilinear problems having the unbounded divergence operator. In particular, in [7], the authors have proved the existence of positive solutions to problem (7) under the assumption that $0<\theta<1$, $M(x,t)=1+|t|^q$, and $0\lneqq f\in L^m(\Omega)$. We refer also that, in [5], the author has shown the same result as in [7], in the case $0<\theta<1$ and without any sign restriction over f.

Let us now consider the Dirichlet boundary value problem (7) in the simple case:

$$\begin{cases} -2\Delta u+\dfrac{|\nabla u|^2}{u}=f(x), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega\,. \end{cases}\tag{8}$$

If we define $v=2(u/\sqrt{|u|})$, then the function v is solution of

$$\begin{cases} -\Delta v=\dfrac{f(x)}{|v|}, & x\in\Omega,\\ v(x)=0, & x\in\partial\Omega\,, \end{cases}\tag{9}$$

which is singular on the right-hand side. Let us remark that, in the case of nonnegative f, in [11], the authors considered the elliptic semilinear problems whose model is

$$\begin{cases} -\Delta u=\dfrac{f}{u^\gamma}, & x\in\Omega,\\ u=0, & x\in\partial\Omega\,, \end{cases}\tag{10}$$

where $\gamma>0$. More precisely, they have shown that the term $(f/|u|^\gamma)$ has a regularizing effect on the solutions u. In [12], the author has shown the existence of solutions to the following elliptic problem with degenerate coercivity:

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(1+|u|)^p}\right)=\dfrac{f}{|u|^\gamma}, & x\in\Omega,\\ u=0, & x\in\partial\Omega\,, \end{cases}\tag{11}$$

where $p,\gamma>0$.

The purpose of this paper is to study the same kind of lower order term as in problems (7) and (9) (indeed, $(f/|u|^\gamma)$) in the case of an elliptic operator with unbounded coefficients. The main difficulties posed by this problem were that the principal part of the differential operator $\operatorname{div}((a(x)+|u|^q)\nabla u)$ is not well defined on the whole $H_0^1(\Omega)$; the solutions did not belong, in general, to $H_0^1(\Omega)$ and the lower order term has a singularity at $u=0$. Despite these difficulties, we prove that, in our case too, the lower order term $(f/|u|^\gamma)$ has a regularizing effect.

Our main existence results are as follows.

Theorem 1. *Assume that (2) and (3) hold true. If $0\lneqq f\in L^m(\Omega)$ with $m>(N/2)$, then there is a positive solution $u\in L^\infty(\Omega)$ of (1), in the sense of distributions, that is,*

$$\int_\Omega [a(x)+u^q]\nabla u\nabla\varphi=\int_\Omega\frac{f\varphi}{u^\gamma},\tag{12}$$

for any test function φ in $C_0^1(\Omega)$. Moreover, we have the following summability results for u:

(1) Let $0<q<1$:

(i) If $0<\gamma\le 1-q$, then $u\in H_0^1(\Omega)$.
(ii) If $\gamma>1-q$, then $u\in H_{loc}^1(\Omega)$.

(2) Let $q=1$:

(i) If $0<\gamma\le 1$, then $u\in H_0^1(\Omega)$.
(ii) If $\gamma>1$, then $u\in H_{loc}^1(\Omega)$.

(3) Let $q>1$, then $u\in H_{loc}^1(\Omega)$.

When $f\in L^m(\Omega)$, $1<m<(N/2)$, we will prove the following regularizing effects.

Theorem 2. *We suppose that $0\lneqq f\in L^m(\Omega)$, $1<m<(N/2)$ and that (2) and (3) are satisfied. If $0<q<1$, then, there exists a solution u of (1) in the sense (19), such that*

(1) If $\gamma<1-q$ and $(2^*/2^*+q-1+\gamma)\le m<(N/2)$, then $u\in H_0^1(\Omega)\cap L^{m^{**}(1+q+\gamma)}(\Omega)$, where

$$m^{**}=(m^*)^*=\frac{Nm}{N-2m}.\tag{13}$$

(2) If $\gamma=1-q$, then $u\in H_0^1(\Omega)$.
(3) If $\gamma>1-q$, then $u\in L^{(1+q+\gamma/2)2^*}(\Omega)\cap H_{loc}^1(\Omega)$.

Notation: throughout this paper, we fix an integer $N\ge 3$. For any $p>1$, $p'=(p/p-1)$ will be the Hölder conjugate exponent of p, and if $1\le p<N$, we will denote by $p^*=(Np/N-p)$ the Sobolev conjugate exponent of p. As usual, let us denote by $\mathcal{S}$ the Sobolev constant, i.e.,

$$\mathcal{S}=\inf_{u\in H_0^1(\Omega)-\{0\}}\frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.\tag{14}$$

We denote by $\mathcal{P}$ the Poincaré constant given by

$$\mathcal{P}=\inf_{u\in H_0^1(\Omega)-\{0\}}\frac{\|\nabla u\|_2^2}{\|u\|_2^2}.\tag{15}$$

For all $k>0$, we recall the definition of a truncated function $T_k(s)$ defined by

$$T_k(s)=\max\{\min\{k,s\}-k\}.\tag{16}$$

We also consider

$$G_k(s)=s-T_k(s).\tag{17}$$

As usual, we consider the positive and negative part of a measurable function $u(x)$

$$u(x)=u^{+}(x)-u^{-}(x),\quad \text{where } u^{+}(x) = u(x)\chi_{\{u\ge 0\}} \text{ and } u^{-}(x)=-u(x)\chi_{\{u<0\}}. \tag{18}$$

2. The Approximated Problem

To prove our existence results, we will use the following approximating problems:

$$-\operatorname{div}\left(\left[a(x)+\left|u_n\right|^q\right]\nabla u_n\right)=\frac{f_n}{\left(\left|u_n\right|+(1/n)\right)^{\gamma}},\quad x\in\Omega, \tag{19}$$

where $n\in\mathbb{N}^*$, and

$$f_n(x)=\frac{f(x)}{1+(1/n)|f(x)|}. \tag{20}$$

As in [11], we prove existence of positive solution of the approximated problem.

Lemma 1. *Let g be positive function belonging to $L^{\infty}(\Omega)$. Suppose that (2) and (3) are satisfied. Then, there exists a positive solution $u_n\in H_0^1(\Omega)\cap L^{\infty}(\Omega)$ of the problem*

$$-\operatorname{div}\left(\left[a(x)+\left|u_n\right|^q\right]\nabla u_n\right)=\frac{g}{\left(\left|u_n\right|+(1/n)\right)^{\gamma}},\quad x\in\Omega, u_n\in H_0^1(\Omega). \tag{21}$$

Proof. To prove it, we define the following operator S_n: $L^2(\Omega)\longrightarrow L^2(\Omega)$ which associates to every $v\in L^2(\Omega)$ the solution $w_n\in H_0^1(\Omega)$ to

$$\begin{cases}-\operatorname{div}\left(\left[a(x)+\left|T_n(w_n)\right|^q\right]\nabla w_n\right)=\dfrac{g}{(|v|+(1/n))^{\gamma}}, & \text{in } \Omega,\\ w_n=0, & \text{on } \partial\Omega.\end{cases} \tag{22}$$

From the results of [13], the operator S_n is well defined and w_n is bounded by the results of [14]. We take w_n as a test function in (19), and we use Hölder's inequality and (3) to deduce that

$$\begin{aligned}\alpha\int_{\Omega}\left|\nabla w_n\right|^2 &\le \int_{\Omega}\left[a(x)+\left|T_n(w_n)\right|^q\right]\left|\nabla w_n\right|^2=\int_{\Omega}\frac{g w_n}{(|v|+(1/n))^{\gamma}},\\ &\le n^{\gamma}\|g\|_{L^{\infty}(\Omega)}\int_{\Omega}\left|w_n\right|\\ &\le n^{\gamma}\|g\|_{L^{\infty}(\Omega)}\sqrt{|\Omega|}\left\|w_n\right\|_{L^2(\Omega)}.\end{aligned} \tag{23}$$

Thanks to Poincaré's inequality, we deduce

$$\alpha\mathscr{P}\left\|w_n\right\|_{L^2(\Omega)}^2\le n^{\gamma}\|g\|_{L^{\infty}(\Omega)}\sqrt{|\Omega|}\left\|w_n\right\|_{L^2(\Omega)}. \tag{24}$$

Hence, there exists an invariant ball for S_n. On the contrary, from the $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ embedding, it is easily seen that S_n is continuous and compact. The Schauder theorem shows that S_n has a fixed point or equivalently, and there exists a solution $u_n\in H_0^1(\Omega)$ to problems

$$\begin{cases}-\operatorname{div}\left(\left[a(x)+\left|T_n(u_n)\right|^q\right]\nabla u_n\right)=\dfrac{g}{\left(\left|u_n\right|+(1/n)\right)^{\gamma}}, & \text{in } \Omega,\\ u_n=0, & \text{on } \partial\Omega.\end{cases} \tag{25}$$

Moreover, by the maximum principle, it is clear that the sequence u_n is nonnegative since g is nonnegative, and we choose $G_k(u_n)$ as test function in (25) and use (3) to obtain

$$\alpha\int_{A_k}\left|G_k(u_n)\right|^2\le\frac{1}{k^{\gamma}}\int_{A_k} g G_k(u_n), \tag{26}$$

where $A_k=\{x\in\Omega\colon |u_n|>k\}$. By the method of Stampacchia (see [14]), the sequence u_n is bounded in $L^{\infty}(\Omega)$. Supposing that u_n is bounded by d_n in $L^{\infty}(\Omega)$, we have that $u_n:=u_{n+[d_n]+1}\in L^{\infty}(\Omega)\cap H_0^1(\Omega)$ is a solution of (13).

By Lemma 1, it follows the existence of a solution $u_n\in L^{\infty}(\Omega)\cap H_0^1(\Omega)$ of (19).

Now, we are going to prove that the sequence u_n is not 0 in Ω. For this, we are going to prove that it is uniformly away from zero in every compact set in Ω. We will follow a similar technique to that one in [12]. □

Lemma 2. *Assume that (2) and (3) hold true. If $0\lneqq f\in L^1(\Omega)$ and u_n is the solution of problem (19), then for every $n\in\mathbb{N}^*$: $u_n\le u_{n+1}$ a.e. in Ω. Furthermore, if $\omega\subset\subset\Omega$, then, for every $n\in\mathbb{N}^*$, there exists $c_{\omega}>0$ such that $u_n\ge c_{\omega}>0$ a.e. in ω.*

Proof. Let us consider $T_k[(u_n-u_{n+1})^{+}]$ as a test function in problems (19). Then,

$$\int_{\Omega}\left[a(x)+u_n^q\right]\nabla u_n\nabla T_k\left[(u_n-u_{n+1})^{+}\right]=\int_{\Omega}\frac{f_n}{\left(\left|u_n\right|+(1/n)\right)^{\gamma}}T_k\left[(u_n-u_{n+1})^{+}\right]. \tag{27}$$

Observing that $f_n \le f_{n+1}$, we have

$$\begin{aligned}\int_\Omega \frac{f_n}{(u_n+(1/n))^\gamma} T_k[(u_n-u_{n+1})^+] &\le \int_\Omega \frac{f_{n+1}}{(u_{n+1}+(1/n+1))^\gamma} T_k[(u_n-u_{n+1})^+] \\ &= \int_\Omega [a(x)+u_{n+1}^q]\nabla u_{n+1}\nabla T_k[(u_n-u_{n+1})^+] \\ &\le \int_\Omega [a(x)+u_n^q]\nabla u_{n+1}\nabla T_k[(u_n-u_{n+1})^+].\end{aligned} \tag{28}$$

Therefore, by (3), we deduce that

$$\alpha\int_\Omega \left|\nabla T_k[(u_n-u_{n+1})^+]\right|^2 \le \int_\Omega [a(x)+u_n^q]\left|\nabla T_k[(u_n-u_{n+1})^+]\right|^2 \le 0. \tag{29}$$

Consequently, we obtain $\int_\Omega |\nabla T_k[(u_n-u_{n+1})^+]|^2 = 0$, so by Poincaré's inequality, we have $T_k[(u_n-u_{n+1})^+] = 0$ for every $k>0$. Thus, $u_n \le u_{n+1}$ a.e. $x\in\Omega$.

We remark that u_1 is bounded; indeed, $|u_1|\le c$, for some positive constant c. Then, it follows that

$$-\operatorname{div}\left(\left[a(x)+|u_1|^q\right]\nabla u_1\right) \ge \frac{f_1}{(c+1)^\gamma}, \quad x\in\Omega. \tag{30}$$

Thanks to (3), we have $\alpha \le a(x)+|u_1|^q \le \beta + c^q$. Thus, we infer that u_1 is a supersolution of a linear Dirichlet problem with a strictly positive and bounded, measurable coefficient. The strong maximum principle implies that $u_1>0$. In addition, Harnack's inequality gives the stronger conclusion: for every $\omega \subset\subset \Omega$, there exists c_ω such that $u_1 \ge c_\omega$ a.e. in ω. Finally, using that the sequence u_n is increasing, one deduces that $u_n \ge c_\omega$ a.e. in ω for every $n\in\mathbb{N}^*$. □

2.1. Existence of Bounded Solutions. In this section, we will prove existence of bounded weak solutions for (1).

Lemma 3. *Let $0 \lneqq f \in L^m(\Omega)$ with $m>(N/2)$. Suppose that (2) and (3) hold true. Let $\{u_n\}$ be a sequence solutions of (19) with $f_n = f$ for every $n\in\mathbb{N}^*$. Then, the norm of the sequence $\{u_n\}$ in $L^\infty(\Omega)$ is bounded by a constant which depends on q, m, N, α, γ, meas(Ω) and on the norm of f in $L^m(\Omega)$.*

Proof. The use of $G_k(u_n)$ as test function in (19) and (3), implies that

$$\alpha\int_{A_k} |\nabla G_k(u_n)|^2 \le \frac{1}{k^\gamma}\int_{A_k} f G_k(u_n), \tag{31}$$

where $A_k = \{x\in\Omega: |u_n|>k\}$. Hence, we can use Theorem 4.1 in [14] and obtain a positive constant, say M, that only depends on the parameters: q, N, α, γ, meas(Ω) and $\|f\|_{L^m(\Omega)}$ such that: $\|u_n\|_{L^\infty(\Omega)} \le M$ for all $n\in\mathbb{N}^*$. □

Lemma 4. *We assume that $0 \lneqq f \in L^m(\Omega)$ with $m>(N/2)$, and (2) and (3) are satisfied. Let $\{u_n\}$ be a sequence solutions of (19) with $f_n = f$ for every $n\in\mathbb{N}^*$. If $q<1$ and $\gamma \le 1-q$, then the sequence $\{u_n\}$ is uniformly bounded in $H_0^1(\Omega)$.*

Proof. We denote by C a positive constant which may only depend on the parameters of our problem, and its value may vary from line to line.

We use $(1+u_n)^{1-q}-1$ as test function in (19) to obtain

$$(1-q)\int_\Omega \frac{a(x)+u_n^q}{(1+u_n)^q}|\nabla u_n|^2 \le C\int_\Omega f|u_n|^{1-q-\gamma}, \tag{32}$$

and thus (since $q\le 1$),

$$(1-q)\min(\alpha,1)\int_\Omega |\nabla u_n|^2 \le C\int_\Omega |f||u_n|^{1-q-\gamma} \le C\|u\|_n L^{1-q-\gamma}_{\infty(\Omega)}\int_\Omega f \le C. \tag{33}$$

from which the sequence u_n is bounded in $H_0^1(\Omega)$. □

Lemma 5. *Let $0 \lneqq f \in L^m(\Omega)$ with $m > (N/2)$, and we suppose that (2) and (3) are satisfied. If $q < 1$ and $\gamma > 1 - q$ and u_n is a solution to problem (19), then u_n is uniformly bounded in $H^1_{loc}(\Omega)$.*

Proof. Let $\varphi \in C^1_0(\Omega)$ and $\omega = \text{Supp}\varphi$ be the support of φ; then, from Lemma 2, there exists $c_\omega > 0$ such that $u_n \geq c_\omega$ for a.e. $x \in \omega$.

Choosing $[(u_n + 1)^{1-q} - 1]\varphi^2$ as test function in (19) and using (3), we obtain

$$\alpha(1-q)\int_\Omega |\nabla u_n|^2 \varphi^2 + 2\int_\Omega [a(x) + u_n^q][(u_n+1)^{1-q} - 1]\nabla u_n \nabla\varphi\varphi \leq \int_\Omega \frac{f_n}{(u_n + (1/n))^\gamma}[(u_n+1)^{1-q} - 1]\varphi^2 \leq \frac{\|\varphi\|^2_{L^\infty(\Omega)}}{c_\omega^\gamma}\int_\Omega f, \tag{34}$$

which then implies

$$\alpha(1-q)\int_\Omega |\nabla u_n|^2 \varphi^2 \leq \frac{\|\varphi\|^2_{L^\infty(\Omega)}}{c_\omega^\gamma}\int_\Omega f - 2\int_\Omega [a(x) + u_n^q][(u_n+1)^{1-q} - 1]\nabla u_n \nabla\varphi\varphi. \tag{35}$$

We can use Young's inequality with ϵ, and we obtain

$$2\left|\int_\Omega [a(x) + u_n^q][(u_n+1)^{1-q} - 1]\nabla u_n \nabla\varphi\varphi\right| \leq \varepsilon\int_\Omega |\nabla u_n|^2\varphi^2 + C(\varepsilon)\int_\Omega [a(x) + u_n^q]^2[(u_n+1)^{1-q} - 1]^2|\nabla\varphi|^2. \tag{36}$$

Using (3), we have

$$a(x) + t^q \leq c_0(1+t)^q, \tag{37}$$

for every $q > 0$ and $t \geq 0$ (and for a suitable c_0 independent on n).

We then have

$$2\left|\int_\Omega [a(x) + u_n^q][(u_n+1)^{1-q} - 1]\nabla u_n \nabla\varphi\varphi\right| \leq \varepsilon\int_\Omega |\nabla u_n|^2\varphi^2 + C(\varepsilon)c_0^2\int_\Omega u_n^2|\nabla\varphi|^2. \tag{38}$$

Applying (38) to (35) and letting $\varepsilon = (\alpha(1-q)/2)$, we obtain

$$\int_\Omega |\nabla u_n|^2\varphi^2 \leq C + C\int_\Omega u_n^2|\nabla\varphi|^2 \leq C + C\|u_n\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla\varphi|^2 \leq C, \tag{39}$$

and this gives that u_n is bounded in $H^1_{\text{loc}}(\Omega)$. □

Lemma 6. *Let $q = 1$. Suppose that (2) and (3) hold. If $0 \lneqq f \in L^m(\Omega)$ with $m > (N/2)$, then the sequence $\{u_n\}$ defined by (19) satisfies the following summability:*

(1) If $0 < \gamma \leq 1$, then u_n is uniformly bounded in $H^1_0(\Omega)$

(2) If $\gamma > 1$, then u_n is uniformly bounded in $H^1_{loc}(\Omega)$

Proof. (1) Let us take $\log(1 + u_n)$ as test function in (19) and use (3) to obtain that

$$\min(1,\alpha)\int_\Omega |\nabla u_n|^2 \leq \int_\Omega f\frac{\log(1+u_n)}{(u_n + (1/n))^\gamma} \leq \int_\Omega f u_n^{1-\gamma} \leq \|u_n\|^{1-\gamma}_{L^\infty(\Omega)}\int_\Omega f \leq C. \tag{40}$$

(2) Let $\varphi \in C_0^1(\Omega)$ and choose $\log(1+u_n)\varphi^2$, as a test function in problem (19). From assumption (19), one has

$$\min(1,\alpha)\int_\Omega |\nabla u_n|^2\varphi^2 + 2\int_\Omega [a(x)+u_n^q]\log(1+u_n)\nabla u_n \nabla\varphi\varphi$$
$$\le \int_\Omega f\frac{\log(1+u_n)}{(u_n+(1/n))^\gamma}\varphi^2 \le \int_\Omega f\frac{\varphi^2}{u_n^{\gamma-1}} \le \frac{\|\varphi\|^2_{L^\infty(\Omega)}}{c_\omega^{\gamma-1}}\int_\Omega f, \tag{41}$$

where $\omega = \mathrm{Supp}\varphi$. By Young's inequalities, it is easy to prove

$$2\left|\int_\Omega [a(x)+u_n^q]\log(1+u_n)\nabla u_n\nabla\varphi\varphi\right| \le \varepsilon\int_\Omega |\nabla u_n|^2\varphi^2 + C(\varepsilon). \tag{42}$$

Hence, equality (41) implies that

$$\min(1,\alpha)\int_\Omega |\nabla u_n|^2\varphi^2 \le \frac{\|\varphi\|^2_{L^\infty(\Omega)}}{c_\omega^{\gamma-1}}\int_\Omega f + \varepsilon\int_\Omega |\nabla u_n|^2\varphi^2 + C(\varepsilon). \tag{43}$$

Letting $\varepsilon = (\min(1,\alpha)/2)$, we get that u_n is bounded in $H^1_{\mathrm{loc}}(\Omega)$. □

Lemma 7. *Let $q>1$. Assume that (2) and (3) hold true. If $0 \lneqq f \in L^m(\Omega)$ with $m > (N/2)$, then the solution u_n of (19) is uniformly bounded in $H^1_{loc}(\Omega)$.*

Proof. Let φ be a function in $C_0^1(\Omega)$ and $\omega = \mathrm{Supp}\varphi$. Take $[1-(u_n+1)^{1-q}]\varphi^2$ as test function in (19) and use (3) to obtain

$$\frac{\min(1,\alpha)}{2^{q-1}}\int_\Omega |\nabla u_n|^2\varphi^2 \le (q-1)\min(1,\alpha)\int_\Omega \frac{1+u_n^q}{(1+u_n)^q}|\nabla u_n|^2\varphi^2$$
$$\le \int_\Omega \frac{f}{(u_n+(1/n))^\gamma}\varphi^2 - 2\int_\Omega [a(x)+u_n^q]\left[1-(u_n+1)^{1-q}\right]\nabla u_n\nabla\varphi\varphi. \tag{44}$$

Using Young's inequality with ϵ, we have by (3) and Lemma 3 that

$$2\left|\int_\Omega [a(x)+u_n^q]\left[1-(u_n+1)^{1-q}\right]\nabla u_n\nabla\varphi\varphi\right|$$
$$\le \varepsilon\int_\Omega |\nabla u_n|^2\varphi^2 + C(\varepsilon)\int_\Omega |\nabla\varphi|^2. \tag{45}$$

Taking the above estimate in (44) and letting $\varepsilon = (\min(1,\alpha)/2^q)$, we obtain

$$\frac{\min(1,\alpha)}{2^q}\int_\Omega |\nabla u_n|^2\varphi^2 \le \frac{\|\varphi\|^2_{L^\infty(\Omega)}}{c_\omega^\gamma}\int_\Omega f + C, \tag{46}$$

and thus, Lemma 7 is proved. □

Proof. of Theorem 1.

We start by proving point (1.i), the rest of the proof of the theorem can be proven similarly. According to Lemmas 3 and 4, there exists a subsequence u_n and a function $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ such that u_n weakly converges to u in $H_0^1(\Omega)$. Now, we can pass to the limit in the equation satisfied by the approximated solutions u_n:

$$\int_\Omega [a(x)+u_n^q]\nabla u_n\nabla\varphi = \int_\Omega \frac{f_n\varphi}{(u_n+(1/n))^\gamma}, \quad \forall\varphi\in C_0^1(\Omega), \tag{47}$$

where $f_n(x) = (f(x)/1+(1/n)f(x))$.

For the term of the left-hand side, it is sufficient to observe that ∇u_n converge to ∇u weakly in $L^2_{\mathrm{loc}}(\Omega)$ and $[a(x)+u_n^q]$ a.e. (and weakly $-^*$ in $L^\infty(\Omega)$ converges towards $[a(x)+u^q]$. On the contrary, for the limit of the right-hand side of (47), let $\omega = \mathrm{Supp}\varphi$, and one can use Lebesgue's dominated convergence theorem, since

$$\left|\frac{f_n\varphi}{(u_n+(1/n))^\gamma}\right| \le \frac{f|\varphi|}{c_\omega^\gamma}. \tag{48}$$

Finally, passing to the limit as n goes to infinity in equation (47), we conclude that

$$\int_\Omega [a(x)+u^q]\nabla u\nabla\varphi = \int_\Omega \frac{f\varphi}{u^\gamma}, \quad \forall\varphi\in C_0^1(\Omega). \tag{49}$$

□

2.2. Further Existence Result. In this section, we suppose (2) and (3) and we assume that

$$0<q<1 \tag{50}$$

holds true.

Lemma 8. *We suppose that (2), (3), and (50) hold true. Let $\gamma < 1-q$ and $0\lneqq f \in L^m(\Omega)$, with*

$$\frac{2^*}{2^*+q-1+\gamma} \le m < \frac{N}{2}. \tag{51}$$

Then, the solutions u_n to problem (19) are uniformly bounded in $H_0^1(\Omega)\cap L^{m^{**}(1+q+\gamma)}(\Omega)$.

Proof. Let us take $(1+u_n)^{1-q}-1$ as a test function in (19) and use assumption (3) to obtain

$$(1-q)\min(1,\alpha)\int_\Omega |\nabla u_n|^2 \le (1-q)\int_\Omega \frac{a(x)+u_n^q}{(1+u_n)^q}|\nabla u_n|^2 \le C\int_\Omega f|u_n|^{1-q-\gamma}. \tag{52}$$

We can use Hölder's inequality on the right-hand side with exponent $p = (2^*/2^* + q - 1 + \gamma) = (2N/N(\gamma+1+q) + 2(1-q-\gamma)) > 1$, and Sobolev inequality on the left-hand side to deduce

$$\mathcal{S}\min(1,\alpha)(1-q)\left(\int_\Omega u_n^{2^*}\right)^{2/2^*} \le C\left(\int_\Omega u_n^{p'(1-q-\gamma)}\right)^{1/p'}. \tag{53}$$

We note that $2^* = p'(1-q-\gamma)$; moreover, $(2/2^*) \ge (1/p')$ (thanks to the fact that $\gamma < 1-q$). This last estimate imply that u_n is uniformly bounded in $L^{2^*}(\Omega)$ and in $H_0^1(\Omega)$.

We are going to prove now that the sequence u_n is bounded in $L^{m^{**}(1+q+\gamma)}(\Omega)$. Let $\lambda = (N(1+q)(m-1) + \gamma m(N-2)/N-2m)$; using $(1+u_n)^\lambda - 1$ as a test function for problem (19), we can deduce

$$\lambda\min(1,\alpha)\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{1-\lambda-q}} \le \int_\Omega \frac{f_n}{((1/n)+u_n)^\gamma}\left[(1+u_n)^\lambda - 1\right], \le C + C\int_\Omega \frac{f}{(1+u_n)^{\gamma-\lambda}}. \tag{54}$$

Now, we rewrite

$$\frac{4\lambda\min(1,\alpha)}{(1+q+\lambda)^2}\int_\Omega \left|\nabla\left[(1+u_n)^{1+q+\lambda/2} - 1\right]\right|^2 = \lambda\min(1,\alpha)\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{1-\lambda-q}} \tag{55}$$

and use the Sobolev inequality and the Hölder inequality in (54) to obtain

$$\left(\int_\Omega \left|(1+u_n)^{1+q+\lambda/2} - 1\right|^{2^*}\right)^{2/2^*} \le \left(\int_\Omega |u_n+1|^{m'(\lambda-\gamma)}\right)^{1/m'}. \tag{56}$$

We note that the choice of λ is equivalent to require $(2/2^*)(1+q+\lambda) = m'(\lambda-\gamma)$; furthermore, $(2/2^*) \ge (1/m')$ and $(2/2^*)(1+q+\lambda) = m^{**}(1+q+\gamma)$. Thus, the sequence $\{u_n\}$ is uniformly bounded in $L^{m^{**}(1+q+\gamma)}(\Omega)$. □

Lemma 9. *Under the hypotheses $0 \lneqq f \in L^1(\Omega)$, (2), (3), and (50), if $\gamma = 1-q$, then the solutions u_n are uniformly bounded in $H_0^1(\Omega)$.*

Proof. We choose $(1+u_n)^{1-q} - 1$ as test function in (19) to obtain, by hypothesis (3), that

$$(1-q)\min(\alpha,1)\int_\Omega |\nabla u_n|^2 \le (1-q)\int_\Omega \frac{a(x)+u_n^q}{(1+u_n)^q}|\nabla u_n|^2 \le C\int_\Omega f. \tag{57}$$

Therefore, u_n is bounded in $H_0^1(\Omega)$. □

Lemma 10. *Let $0 \lneqq f \in L^1(\Omega)$. Under hypotheses (2), (3), and (50), if $\gamma > 1-q$, then the solutions u_n are uniformly bounded in $L^{(1+q+\gamma/2)2^*}(\Omega) \cap H_{loc}^1(\Omega)$.*

Proof. Choosing u_n^γ as test function in (19) and using Hölder and Sobolev inequalities, thanks to (3), we obtain that

$$\frac{4\gamma\mathcal{S}}{(1+q+\gamma)^2}\left(\int_\Omega u_n^{(1+q+\gamma/2)2^*}\right)^{2/2^*} \le \gamma\int_\Omega |\nabla u_n|^2 u_n^{\gamma+q-1}, \le \gamma\int_\Omega \left[a(x)+u_n^q\right]|\nabla u_n|^2 u_n^{\gamma-1} \le \int_\Omega f. \tag{58}$$

The above inequality implies that

$$\int_\Omega u_n^{(1+q+\gamma/2)2^*} \le C. \tag{59}$$

Now, we prove that the sequence u_n is bounded in $H_{\mathrm{loc}}^1(\Omega)$. Let $\varphi \in C_0^1(\Omega)$ and choose $[(1+u_n)^{1-q} - 1]\varphi^2$, as a test function in problems (19). From assumption (19), one has

$$\min(1,\alpha)\int_\Omega |\nabla u_n|^2\varphi^2 + +2\int_\Omega \left[a(x)+u_n^q\right]\left[(u_n+1)^{1-q} - 1\right]\nabla u_n \nabla\varphi\varphi \int_\Omega \frac{f_n}{(u_n+(1/n))^\gamma}\left[(u_n+1)^{1-q} - 1\right]\varphi^2 \le \frac{\|\varphi\|_{L^\infty(\Omega)}^2}{c_\omega^{\gamma-q+1}}\int_\Omega f, \tag{60}$$

where $\omega = \mathrm{Supp}\varphi$. We can use Young's inequality with ϵ and both (37) and (59) to obtain

$$\begin{aligned}
&2\left|\int_\Omega [a(x)+u_n^q]\left[(u_n+1)^{1-q}-1\right]\nabla u_n \nabla\varphi\varphi\right| \\
&\le \varepsilon\int_\Omega |\nabla u_n|^2\varphi^2 + C(\varepsilon)\int_\Omega [a(x)+u_n^q]\left[(u_n+1)^{1-q}-1\right]|\nabla\varphi|^2 \\
&\le \varepsilon\int_\Omega |\nabla u_n|^2\varphi^2 + C(\varepsilon)c_0\int_\Omega u_n^2|\nabla\varphi|^2 \\
&\le \varepsilon\int_\Omega |\nabla u_n|^2\varphi^2 + C(\varepsilon).
\end{aligned} \tag{61}$$

Hence, equality (60) implies that

$$\min(1,\alpha)\int_\Omega |\nabla u_n|^2\varphi^2 \le \frac{\|\varphi\|^2_{L^\infty(\Omega)}}{c_\omega^{\gamma+q-1}}\int_\Omega f + \varepsilon\int_\Omega |\nabla u_n|^2\varphi^2 + C(\varepsilon). \tag{62}$$

Letting $\varepsilon = (\min(1,\alpha)/2)$, we get that u_n is bounded in $H^1_{\mathrm{loc}}(\Omega)$. □

Lemma 11. *Under the assumptions of Theorem 2, let u_n be a solution to problem (19). Then, the sequence $u_n^q|\nabla u_n|$ is uniformly bounded in $L^\sigma_{loc}(\Omega)$, for every $\sigma < (N/N-1)$.*

Proof. We will prove our proof in two steps:

Step 1: we want to prove that, for every $\lambda > 1$, $(1+u_n)^{q-\lambda}|\nabla u_n|^2 \in L^1_{\mathrm{loc}}(\Omega)$. Indeed, let $\lambda > 1$, $\varphi \in C_0^1(\Omega)$ and $\omega = \mathrm{Supp}\varphi$ is the support of φ. Thanks to (3), we have from (19) with test function $[1-(1/(1+u_n)^{\lambda-1})]\varphi^2$

$$(\lambda-1)\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{\lambda-q}}\varphi^2 + 2\int_\Omega [a(x)+u_n^q]\left[1-(u_n+1)^{1-\lambda}\right]\nabla u_n\nabla\varphi\varphi \le C(\omega). \tag{63}$$

We use Young's inequality, and since $q<1$, we deduce from (37) that

$$\begin{aligned}
&(\lambda-1)\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{\lambda-q}}\varphi^2 \\
&\le C(\omega) + \frac{1}{2}\int_\Omega |\nabla u_n|^2\varphi^2 + \frac{1}{2}\int_\Omega [a(x)+u_n^q]^2 \\
&\quad \left[1-(u_n+1)^{1-\lambda}\right]^2|\nabla\varphi|^2 \\
&\le C(\omega) + \frac{1}{2}\int_\Omega |\nabla u_n|^2\varphi^2 + \frac{c_0}{2}\int_\Omega (1+u_n)^{2q}|\nabla\varphi|^2 \\
&\le C(\omega) + \frac{1}{2}\int_\Omega |\nabla u_n|^2\varphi^2 + C\int_\Omega u_n^2|\nabla\varphi|^2 + C.
\end{aligned} \tag{64}$$

Thus, by the above estimate and since u_n is uniformly bounded in $H^1_{\mathrm{loc}}(\Omega)$, this proves Step 1.

Step 2: here, we show that $u_n^q|\nabla u_n|$ is uniformly bounded in $L^r_{\mathrm{loc}}(\Omega)$ for every $r < (N/Nt-n1)$. For this, let $\sigma < 2$, $0 < \varphi \in C_0^1(\Omega)$, and $\omega = \mathrm{Supp}\varphi$. We use Hölder inequality with exponent $2/\sigma$ and by step 1, and1 we obtain

$$\begin{aligned}
&\int_\Omega u_n^{q\sigma}|\nabla u_n|^\sigma\varphi^\sigma \\
&\le \int_\Omega \frac{|\nabla u_n|^\sigma}{(1+u_n)^{\sigma(\lambda-q)/2}}\varphi^{\sigma^2(N-2)/2(N-\sigma)}(1+u_n)^{\sigma(\lambda+q)/2}\varphi^{N\sigma(2-\sigma)/2(N-\sigma)} \\
&\le \left(\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{\lambda-q}}\varphi^{\sigma(N-2)/N-\sigma}\right)^{\sigma/2}\left(\int_\Omega (1+u_n)^{\sigma(\lambda+q)/2-\sigma}\varphi^{\sigma^*}\right)^{2-\sigma/2} \\
&\le C(\omega)\left(\int_\Omega (1+u_n)^{\sigma(\lambda+q)/2-\sigma}\varphi^{\sigma^*}\right)^{2-\sigma/2}.
\end{aligned} \tag{65}$$

Using the Sobolev inequality, we obtain

$$\left(\int_{\Omega} u_n^{(q+1)\sigma^*}\varphi^{\sigma^*}\right)^{\sigma/\sigma^*} \le C(\omega)\left(\int_{\Omega}(1+u_n)^{\sigma(\lambda+q)/2-\sigma}\varphi^{\sigma^*}\right)^{2-\sigma/2} + C(\omega). \tag{66}$$

Noticing that $(\sigma/\sigma^*) > (2-\sigma/2)$ and choosing σ such that $(q+1)\sigma^* = (\sigma(\lambda+q)/2-\sigma)$ yields $\sigma = (N(2+q-\lambda)/N(q+1)-(\lambda+q))$. Using Young's inequality with ϵ, we obtain

$$\left(\int_{\Omega} u_n^{(q+1)\sigma^*}\varphi^{\sigma^*}\right)^{\sigma/\sigma^*} \le \varepsilon\left(\int_{\Omega}(1+u_n)^{(q+1)\sigma^*}\varphi^{\sigma^*}\right)^{\sigma/\sigma^*} + C(\omega,\varepsilon). \tag{67}$$

It is easy to check that the hypotheses $\lambda > 1$ imply $\sigma < (N/N-1) < 2$. □

Proof. of Theorem 2.

The proof of the theorem is similar to the proof of the previous theorem with just a small change for the convergence of the term on the left side of equation (47). Indeed, using Lemma 11, we have that $[a(x)+u_n^q]\nabla u_n \longrightarrow [a(x)+u^q]\nabla u$ is weak in $(L^{\sigma}_{\text{loc}}(\Omega))^N$ for every $\sigma < (N/N-1)$. Hence, for every $\varphi \in C_0^1(\Omega)$, we can pass to the limit with respect to n in the integral in the left-hand side of (47). □

Remark 1. Assume that (2) and (3) are satisfied. We can choose u_n^{γ}, as test function in (19), using (3), and we obtain that

$$\begin{aligned} \frac{4\gamma}{(\gamma+q+1)^2}\int_{\Omega}\left|\nabla\left(u_n^{\gamma+q+1/2}\right)\right|^2 &= \gamma\int_{\Omega}\left|\nabla u_n\right|^2 u_n^{\gamma+q-1}, \\ &\le \gamma\int_{\Omega}\left[a(x)+u_n^q\right]\left|\nabla u_n\right|^2 u_n^{\gamma-1} \le \int_{\Omega} f. \end{aligned} \tag{68}$$

We deduce from (68) that the sequence $u_n^{\gamma+q+1/2}$ is bounded in $H_0^1(\Omega)$. Therefore, $u^{\gamma+q+1/2}$ belongs to $H_0^1(\Omega)$.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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