

## Research Article

# Lie Symmetry Analysis and Explicit Solutions for the Time-Fractional Regularized Long-Wave Equation

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This paper systematically investigates the Lie group analysis method of the time-fractional regularized long-wave (RLW) equation with Riemann–Liouville fractional derivative. The vector fields and similarity reductions of the time-fractional (RLW) equation are obtained. It is shown that the governing equation can be transformed into a fractional ordinary differential equation with a new independent variable, where the fractional derivatives are in Erdélyi–Kober sense. Furthermore, the explicit analytic solutions of the time-fractional (RLW) equation are obtained using the power series expansion method. Finally, some graphical features were presented to give a visual interpretation of the solutions.

## 1. Introduction

In recent decades, fractional differential equations have been used to describe several natural phenomena in applied sciences such as fluid flow, chemistry, physics, biology, and other areas [1–5], for modeling different phenomena which may depend on the previous time as well as the current time. Besides, fractional differential equations can be considered more efficient to investigate the process of scientific phenomena with complex irregular conditions, see, for example, [6–11]. Therefore, a large number of powerful methods have been developed to look for exact and numerical solutions such as the variational iteration method [12], transformation method [13], finite-difference method [14], Exp-function method [15], homotopy analysis method [16], Adomian decomposition method [17], the first integral method [18], and sine-cosine method [19].

Lie symmetry analysis was advocated by the Norwegian mathematician Sophis Lie (1842–1899) in the beginning of the nineteenth century, which is one of the most powerful methods for studying nonlinear partial differential equations and looking for its analytical solutions. It has several applications including linearization of some nonlinear equations, construction of new solutions from trivial ones, construction of integrator factor, reduction of order, and reduction of the independent variables.

Gazizov et al. [20, 21] are the first who started rigorous studies of symmetries admitted by fractional differential equations focusing on Riemann–Liouville and Caputo derivatives. In [22], Wang and Xu investigated the symmetry properties of the time-fractional KDV equation (see also [23]). It is worth to mention that Yusuf [24] analyzed the equation for fluid flow in porous media using the Lie symmetry method and invariant subspace method for constructing exact solutions and conservation laws of the equation. In addition, the authors of [25] have made an attempt to apply the Lie group method to the time-fractional generalized Burgers and Korteweg de Vries equations. In [26], authors have investigated the Lie symmetry analysis of time-fractional Harry-Dym equation with Riemann–Liouville derivative to obtain invariant solutions and symmetry reductions. Moreover, the interested reader can be also referred to the following papers and books [22, 25, 27–32].

In this paper, we investigate the time-fractional regularized long-wave equation given by

$$D_t^\alpha u + u_x - \frac{a}{2} uu_x - bu_{xxt} = 0, \quad (1)$$

where  $D_t^\alpha$  is the fractional derivative of order  $0 < \alpha \leq 1$  and  $a$  and  $b$  are arbitrary constants.

The regularized long-wave (RLW) equation was first introduced by Peregrine [33] to describe the development of an undular bore. It is one of the most important nonlinear evolution equations which have been used to model many physical phenomena such as shallow water waves and ion-acoustic plasma waves. Many researchers have tried in the past to construct solution of the nonlinear regularized long-

wave (RLW) equation [34–36]. Especially, fractional version of this model has been studied (see [37, 38]).

We would like to mention that there is no unique definition of fractional derivatives [5, 11]; in this paper, we use the Riemann–Liouville fractional derivative of order  $\alpha$  ( $\alpha > 0$ ) which is defined by

$$D^\alpha u(t, x) = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{\partial^m u}{\partial t^m}, \alpha = m \in \mathbb{N}, \\ \frac{1}{\Gamma(m - \alpha)} \frac{\partial^m}{\partial t^m} \int_0^t (t - \tau)^{m - \alpha - 1} u(\tau, x) d\tau, m - 1 < \alpha < m, m \in \mathbb{N}, \end{cases} \quad (2)$$

where  $\Gamma(z)$  is the Gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1}. \quad (3)$$

The main purpose of this paper is to investigate the symmetry approach for determining the Lie point symmetries and symmetry reductions of the time-fractional regularized long-wave equation (RLW). Then, we have shown that the time fractional (RLW) can be transformed into an ordinary differential equation of fractional order with Erdélyi–Kober fractional differential derivative. Using the power series method [39], the exact solutions of the time-fractional (RLW) equation are derived.

The remainder of this paper is organized as follows: in Section 2, the general procedure of the Lie symmetry method for fractional partial differential equations (FPDEs) is presented. By employing the proposed method, Lie point symmetries of equation (1) are obtained in Section 3. In Section 4, by using similarity variables, the reduced equations are obtained, solving some of them, and then the similarity solutions of equation (1) are deduced. Section 5 is devoted to constructing the explicit analytical power series solutions. Some graphical features for the obtained solutions are presented in Section 6. Finally, a brief conclusion is given in Section 7.

## 2. The General Procedure of Lie Symmetry Analysis Method for FNPDE

In this section, we present brief details of Lie symmetry analysis for fractional partial differential equations (FPDEs) of the form

$$D_t^\alpha u = F(x, t, u, u_t, u_x, u_{xt}, u_{xx}, u_{xxt}, \dots), \quad \alpha > 0. \quad (4)$$

We assume that equation (4) is invariant under a one parameter  $\varepsilon$  continuous transformations and the construction of the symmetry group is equivalent to the determination of its infinitesimal transformations:

$$\begin{aligned} \hat{x} &= x + \varepsilon \xi(x, t, u) + 0(\varepsilon^2), \\ \hat{t} &= t + \varepsilon \tau(x, t, u) + 0(\varepsilon^2), \\ \hat{u} &= u + \varepsilon \eta(x, t, u) + 0(\varepsilon^2), \\ D_t^\alpha \hat{u} &= D_t^\alpha u + \varepsilon \eta_\alpha^0(x, t, u) + 0(\varepsilon^2), \\ \frac{\partial^3 \hat{u}}{\partial \hat{x}^3} &= \frac{\partial u}{\partial x} + \varepsilon \eta^x(x, t, u) + 0(\varepsilon^2), \\ \frac{\partial^3 \hat{u}}{\partial \hat{x}^2 \partial \hat{t}} &= \frac{\partial u}{\partial x^2 \partial t} + \varepsilon \eta^{xxt}(x, t, u) + 0(\varepsilon^2), \\ &\vdots, \end{aligned} \quad (5)$$

where  $\varepsilon \ll 1$  is a group parameter and  $\xi$ ,  $\eta$ , and  $\tau$  are the infinitesimals of transformations for the dependent and independent variables, respectively. The explicit expressions of  $\eta^x$  and  $\eta^{xxt}$  are given by

$$\begin{aligned} \eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau), \\ \eta^t &= D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau), \\ \eta^{xt} &= D_x(\eta_t) - u_{xt} D_x(\xi) - u_{tt} D_x(\tau), \\ \eta^{xxt} &= D_x(\eta_{xt}) - u_{xxx} D_x(\xi) - u_{xxt} D_x(\tau), \\ &\vdots, \end{aligned} \quad (6)$$

where  $D_x$  and  $D_t$  are the total derivatives with respect to  $x$  and  $t$ , respectively, which are defined as

$$D_{x^i} = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \dots, \quad i, j = 1, 2, 3, \dots, \quad (7)$$

where  $u_i = \partial u / (\partial x^i)$ ,  $u_{ij} = \partial^2 u / \partial x_i \partial x_j$ , and so on.

The infinitesimal generator  $X$  is represented by the following expression:

$$X = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}, \quad (8)$$

Since the lower limit of integral (2) is fixed, so the structure of the Riemann–Liouville derivative must be

invariant under transformations (5). The invariance condition yields

$$\tau(x, t, u)|_{t=0} = 0. \tag{9}$$

The  $\alpha$ -th extended infinitesimal related to the Riemann–Liouville fractional time derivative (see [20, 21]) can be written as follows:

$$\begin{aligned} \eta_\alpha^0 &= D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) \\ &+ D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u), \end{aligned} \tag{10}$$

where the total fractional derivative operator is denoted as  $D_t^\alpha$ .

Here, for making equation (10) more general, the generalized Leibnitz rule [5] has been presented, which is given as

$$D_t^\alpha((f(t)g(t))) = \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}(f(t))D_t^n g(t), \quad \alpha > 0, \tag{11}$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(n-\alpha)(-1)^{n-1}\alpha}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{12}$$

Now, by using Leibnitz rule as presented below, we get

$$\begin{aligned} \eta_\alpha^0 &= D_t^\alpha(\eta) - \alpha D_t(\tau)D_t^\alpha u - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi)D_t^{\alpha-n}u_x \\ &- \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^n(\tau)D_t^{\alpha-n}(u). \end{aligned} \tag{13}$$

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$$\eta_\alpha^0 = \frac{(\partial^\alpha \eta)}{(\partial t^\alpha)} + (\eta_u - \alpha D_t(\tau)) \frac{(\partial^\alpha \eta u)}{(\partial t^\alpha)} - u \frac{(\partial^\alpha \eta_u)}{(\partial t^\alpha)} + \mu + \sum_{n=1}^{\infty} \left[ \binom{\alpha}{n} \frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_x^{n+1}(\tau) \right] D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x). \tag{17}$$

**Theorem 1** (see [23])

A solution  $u = \theta(x, t)$  is an invariant solution of equation (4) if and only if

(i)  $u = \theta(x, t)$  is an invariant surface; in other words,

$$\begin{aligned} X\theta &= 0 \Leftrightarrow \left( \xi(x, t, u) \frac{\partial}{\partial x} + \zeta(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u} \right) \\ \theta &= 0. \end{aligned} \tag{18}$$

Then, the chain rule for composite function [5] can be written as

$$\frac{d^n \phi(h(t))}{dt^n} = \sum_{k=0}^n \sum_{r=0}^k \binom{\alpha}{n} \frac{1}{k!} [-h(t)]^r \frac{d^r}{dt^r} [(h(t))^{k-r}] \frac{d^k \phi(h)}{dh^k}. \tag{14}$$

By applying this rule and the generalized Leibnitz rule with  $f(t) = 1$ , we have

$$D_t^\alpha(\eta) = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_u \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty} \binom{\alpha}{n} \frac{\partial^\alpha \eta_u}{\partial t^n} D_t^{\alpha-n}(u) + \mu, \tag{15}$$

where

$$\begin{aligned} \mu &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{(t^{n-\alpha})}{(\tau(n+1-\alpha))} \\ &\cdot (-u)^r \frac{\partial^m}{\partial t^m}(u^{k-r}) \frac{\partial^{n-m+k} \eta}{\partial x^{n-m} \partial u^k}. \end{aligned} \tag{16}$$

Therefore, the  $\alpha$ -th extended infinitesimal given in (13) becomes

(ii)  $u = \theta(x, t)$  is the solution of FPDE (4).

**3. Symmetry Analysis of Time-Fractional Regularized Long-Wave Equation**

We complete this section in the light of references [20, 21, 25, 29]. We employ the Lie symmetry analysis to derive the similarity solution for nonlinear time-fractional equation (1) and to reduce it to be a FODE as shown in the next sections.

Let us assume that equation (1) is invariant under one-parameter transformations (5), and we get the following transformed equation:

$$D_t^\alpha \widehat{u} + \widehat{u}_x - \frac{a}{2} \widehat{u} \widehat{u}_x - b \widehat{u}_{xxt} = 0. \tag{19}$$

Using point transformation equations (6) in equation (19), we obtain the following symmetry determining equation:

$$\eta_\alpha^0 - \left(1 - \frac{a}{2} u\right) \eta^x - \frac{a}{2} \eta u_x - b \eta^{xxt} = 0, \tag{20}$$

such that  $u = u(x, t)$  satisfies equation (1).

By substituting the expressions  $\eta_\alpha^0$  given in equations (6) and (17) into equation (20), we get

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$$\begin{aligned}
 & D_t^\alpha (\eta) + (\eta_u - \alpha D_t(\tau)) D_t^\alpha u - u D_t^\alpha \eta_u + \\
 & \mu + \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n (\xi) D_t^{\alpha-n} u_x \\
 & + \sum_{n=1}^{\infty} \left[ \binom{\alpha}{n} D_t^n (\eta_u) - \binom{\alpha}{n+1} D_t^{n+1} (\tau) \right] D_t^{\alpha-n} (u) \\
 & + \left(1 - \frac{a}{2} u\right) (\eta_x + u_x (\eta_u - \xi_x) - u_x^2 \xi_u - u_t \tau_x - u_t u_x \tau_u) - \frac{a}{2} \eta u_x \\
 & - b (\eta_{xxt} + u_x (2\eta_{xtu} - \xi_{txx}) + u_{xx} (\eta_{tu} - 2\xi_{tx}) + u_x^2 (\eta_{tuu} - 2\xi_{txu})) \\
 & + u_{txx} (\eta_u - \tau_t - \xi_x - \tau_x) + u_{tx} (2\eta_{ux} - 2\tau_{tx} - \xi_{xx}) + u_x u_{tx} (2\eta_{uu} - 4\xi_{ux} - 2\tau_{tu}) \\
 & + u_t (\eta_{uxx} - \tau_{txx}) + u_x u_t (2\eta_{uux} - 2\tau_{txu} - \xi_{uux}) + u_{xx} u_t (\eta_{uu} - 2\xi_{ux} - \tau_{tu}) \\
 & + u_x^2 u_t (\eta_{uuu} - 2\xi_{uux} - \tau_{tuu}) - 3u_x u_{xx} \xi_{tu} - u_x^3 \xi_{tuu} - \xi_u u_{xxx} u_t - 3u_{xx} u_{tx} \xi_u \\
 & - 3u_x u_{xx} u_t \xi_{uu} - u_x u_{txx} (2\xi_u + \tau_u) - 3u_x^2 u_{tx} \xi_{uu} - u_x^3 u_t \xi_{uuu} - u_{tx}^2 \tau_u - 2u_t u_{txx} \tau_u \\
 & - 4u_t u_{tx} \tau_{ux} - 4u_x u_t u_{tx} \tau_{uu} - u_t^2 \tau_{uux} - u_x u_t^2 (2\tau_{uux} + \tau_{uuu}) - \tau_{uu} u_{xx} u_t^2 \\
 & - \tau_x u_{xtt} - 2\tau_{ux} u_x u_{tt} - u_{ttx} u_x \tau_u - \\
 & u_{tt} u_{xx} \tau_u - u_x^2 u_{tt} \tau_{uu} - u_{xxx} u_x \xi_u + u_{tt} \tau_{xx} \\
 & - u_{xxx} (\xi_x + u_x \xi_u) - u_{xxt} (\tau_x + u_x \tau_u) = 0,
 \end{aligned} \tag{21}$$

and equating various powers of derivatives of  $u$  to zero, we obtain the following overdetermined system of linear equations

$$\begin{aligned} \xi_u &= \tau_u = \tau_x \\ \xi_{uu} &= \tau_{uu} = \tau_{xx} = \eta_{uu} \\ \eta_{ut} - 2\xi_{xt} &= 0, \\ 2\xi_x + (1 - \alpha)\tau_t &= 0, \\ D_t^n(\xi) &= 0, \quad n = 1, 2, 3, \dots, \\ \frac{\partial^\alpha \eta}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \left(1 - \frac{au}{2}\right) \eta_x - b\eta_{xxt} &= 0, \\ \binom{\alpha}{n} \frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) &= 0, \quad n \in \mathbb{N}, \\ \left(\frac{a}{2}u - 1\right) \xi_x - \frac{a}{2} \eta + \alpha \left(1 - \frac{au}{2}\right) \tau_t &= 0. \end{aligned} \tag{22}$$

By solving the previous system, we get the following explicit form of infinitesimals:

$$\begin{aligned} \xi &= -\frac{C_1 a (\alpha - 1)}{\alpha + 1} x + C_3, \\ \eta &= C_1 (au - 2), \\ \tau &= -\frac{2aC_1}{\alpha + 1} t + C_2, \end{aligned} \tag{23}$$

where  $C_1, C_2$ , and  $C_3$  are arbitrary constants. Then, the Lie algebra of infinitesimal symmetries of equation (1) is given by

$$X_1 = \frac{\partial}{\partial x}, \tag{24}$$

$$X_2 = \frac{\partial}{\partial t}, \tag{25}$$

$$X_3 = -\frac{a(\alpha - 1)}{\alpha + 1} x \frac{\partial}{\partial x} + (au - 2) \frac{\partial}{\partial u} - \frac{2a}{\alpha + 1} t \frac{\partial}{\partial t}. \tag{26}$$

Then, the infinitesimal generator of equation (1) can be written as follows:

$$X = C_1 X_1 + C_2 X_2 + C_3 X_3. \tag{27}$$

### 4. The Similarity Reduction of Fractional Regularized Long-Wave Equation

In this section, we will use the characteristic equations of vector fields obtained in equations (24)–(26) for obtaining the reduction equations.

4.1. Case 1. The characteristic equation for infinitesimal generator (24) can be expressed symbolically as follows:

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}. \tag{28}$$

By solving the above characteristic equation, we obtain the trivial solution.

4.2. Case 2. The characteristic equation for infinitesimal generator (25) can be expressed symbolically as follows:

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}. \tag{29}$$

By solving the above characteristic equation, we obtain the trivial solution.

4.3. Case 3. The characteristic equation for infinitesimal generator (26) can be expressed as follows:

$$\frac{dx}{a((1 - \alpha)/(\alpha + 1))x} = \frac{dt}{((-2at)/(\alpha + 1))} = \frac{du}{au - 2}. \tag{30}$$

By solving the above characteristic equation, we obtain

$$z = xt^{((1 - \alpha)/2)}, \tag{31}$$

with the group invariant solution

$$t^{(\alpha+1)/2} (au - 2) = f(z). \tag{32}$$

Hence, we get

$$\begin{aligned} u(t, x) &= \frac{1}{a} t^{-(\alpha+1)/2} f(z) + \frac{2}{a}, \\ &= h(z) + \frac{2}{a}, \end{aligned} \tag{33}$$

where  $h(z) = (1/a)t^{-(\alpha+1)/2} f(z)$ .

By means of this similarity transformation, time-fractional regularized long-wave equation (1) can be reduced to a nonlinear FODE with a new independent variable  $z$ . Thus, one can get the following theorem.

**Theorem 2.** Transformation (27) reduces equation (1) to the following nonlinear ordinary differential equation of fractional order:

$$\left(P_{(2/(\alpha-1))}^{((1-3\alpha/2), \alpha)} f\right)(z) + 2 \frac{t^{(\alpha+1/2)}}{\Gamma(1-\alpha)} - \frac{1}{2} f f_z + b \frac{(3\alpha - 1)}{2} f_{zz} = 0, \tag{34}$$

with the Erdélyi–Kober fractional operator [11],

$$\left(P_\xi^{\tau, \alpha} f\right)(z) = \prod_{j=0}^{n-1} \left(\tau + j - \frac{1}{\xi} z \frac{d}{dz}\right) \left(K_\xi^{\tau+\alpha, n-\alpha} f\right)(z), \quad z > 0, \xi > 0, \alpha > 0, \tag{35}$$

$$n = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases} \tag{36}$$

where

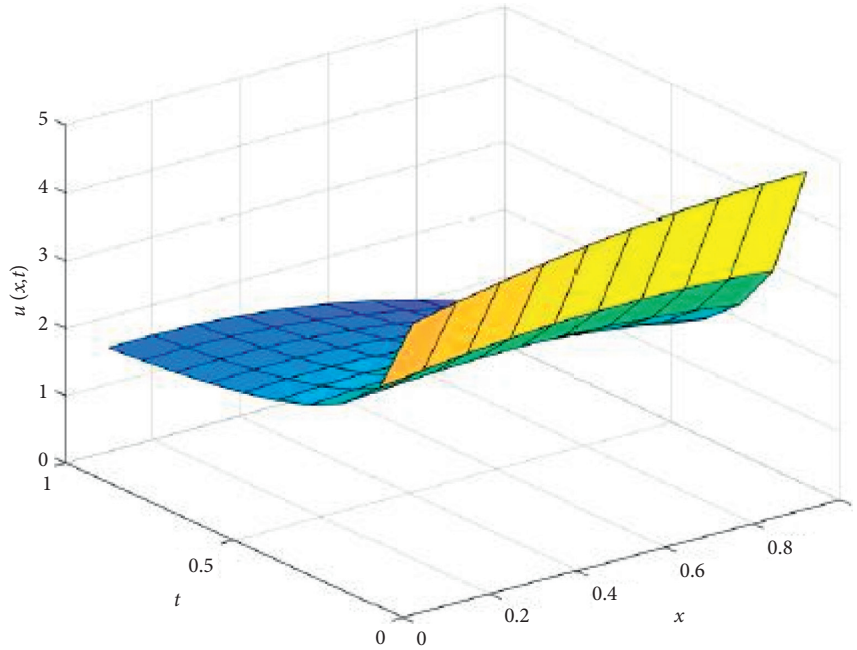


FIGURE 1: 3D plot of equation (52) with the parameter values  $c_0 = c_1 = 1$ ,  $\alpha = 0.5$ ,  $a = 2$ , and  $b = 1$ .

$$(K_{\xi}^{\tau, \alpha} f)(z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (v-1)^{\alpha-1} v^{-(\tau+\alpha)} f(zv^{1/\xi}) dv, & \alpha > 0; \\ f(z), & \alpha = 0, \end{cases} \tag{37}$$

which is the Erdélyi–Kober fractional integral operator.

*Proof.* Let  $n - 1 < \alpha < n$ ,  $n = 1, 2, 3, \dots$ . The Riemann–Liouville fractional derivative becomes

$$D_t^\alpha h(z) = \frac{1}{a} \frac{\partial^n}{\partial t^n} \left[ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} s^{-(\alpha+1)/2} \cdot f(xs^{(1-\alpha/2)}) ds \right] \tag{38}$$

Let  $ds = -1/s^2 dv$ . So, equation (38) can be written as

$$\frac{\partial^n h(z)}{\partial t^n} = \frac{1}{a} \frac{\partial^n}{\partial t^n} \left[ t^{n-((3\alpha+1)/2)} \frac{1}{\Gamma(n-\alpha)} \int_1^{\infty} v^{-(n-\alpha+(1-\alpha/2))} \cdot (v-1)^{n-\alpha-1} f(zv^{(2/1-\alpha)}) dv \right] \tag{39}$$

According to the definition of Erdélyi–Kober fractional integral operator (29), we have

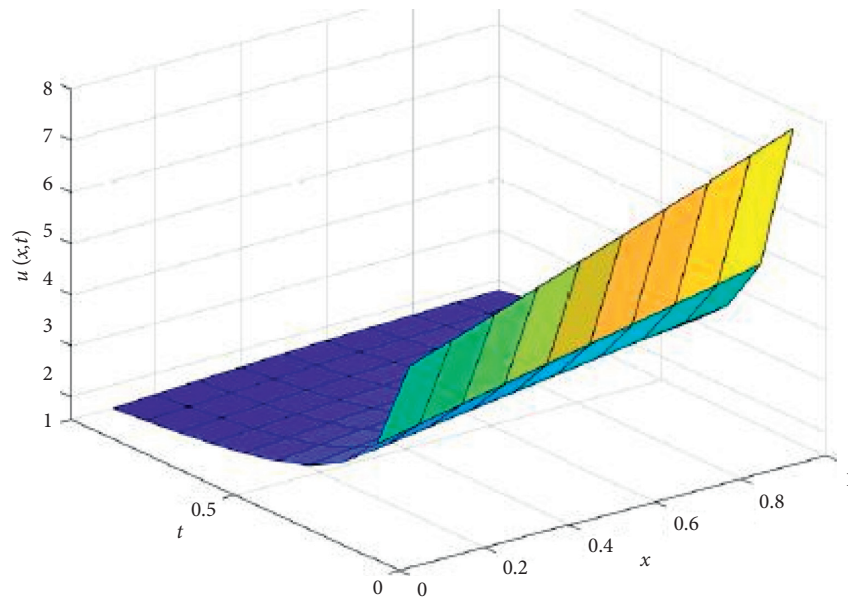


FIGURE 2: 3D plot of equation (52) with the parameter values  $c_0 = c_1 = 1$ ,  $\alpha = 0.75$ ,  $a = 2$ , and  $b = 1$ .

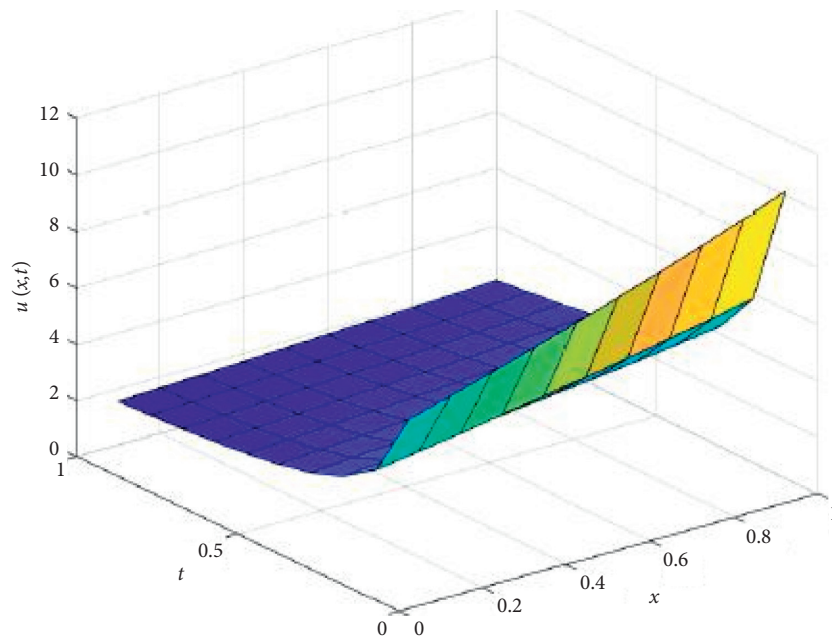


FIGURE 3: 3D plot of equation (52) with the parameter values  $c_0 = c_1 = 1$ ,  $\alpha = 0.90$ ,  $a = 2$ , and  $b = 1$ .



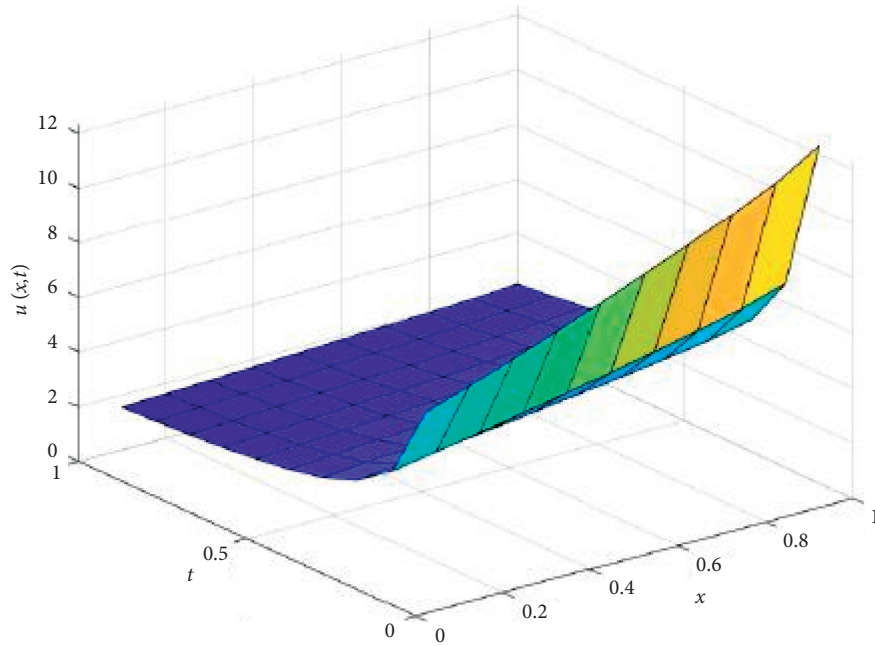


FIGURE 4: 3D plot of equation (52) with the parameter values  $c_0 = c_1 = 1$ ,  $\alpha = 0.99$ ,  $a = 2$ , and  $b = 1$ .

$$\begin{aligned} \frac{\partial^n h(z)}{\partial t^n} &= \frac{1}{a} \frac{\partial^n}{\partial t^n} \left[ t^{n-((3\alpha+1)/2)} \left( K_{(2/1-\alpha)}^{((1-\alpha/2), n-\alpha)} f \right) (z) \right] \\ &= \frac{1}{a} \frac{\partial^{n-1}}{\partial t^{n-1}} \left[ \frac{\partial}{\partial t} \left( t^{n-((3\alpha+1)/2)} \left( K_{(2/1-\alpha)}^{((1-\alpha/2), n-\alpha)} f \right) (z) \right) \right]. \end{aligned} \tag{40}$$

Repeating the similar procedure for  $n - 1$  times, we obtain

$$\begin{aligned} \frac{\partial^n h(z)}{\partial t^n} &= \frac{1}{a} \frac{\partial^{n-1}}{\partial t^{n-1}} \left[ t^{n-((3\alpha+1)/2)-1} \right. \\ &\quad \cdot \left. \left( n - \frac{(3\alpha+1)}{2} - \frac{\alpha-1}{2} z \frac{d}{dz} \right) \left( K_{(2/1-\alpha)}^{(1-\alpha/2)+1, n-\alpha} f \right) (z) \right], \\ &\vdots, \\ &= \frac{1}{a} t^{(-3\alpha+1)/2} \prod_{j=0}^{n-1} \left( j - \frac{(3\alpha+1)}{2} + 1 - \frac{\alpha-1}{2} z \frac{d}{dz} \right) \left( K_{(2/1-\alpha)}^{(1-\alpha/2)+1, n-\alpha} f \right) (z). \end{aligned} \tag{41}$$

By using equation (29), we have

$$D_t^\alpha h(z) = \frac{1}{a} t^{-3\alpha-1/2} \left( P_{(2/\alpha-1)}^{(1-3\alpha/2), \alpha} f \right) (z). \tag{42}$$

According to equation (40), we have

$$\begin{aligned} D_t^\alpha u(x, t) &= D_t^\alpha h(z) + D_t^\alpha \left( \frac{2}{a} \right), \\ &= \frac{1}{a} t^{-3\alpha-1/2} \left( P_{(2/\alpha-1)}^{(1-3\alpha/2), \alpha} f \right) (z) + \frac{2}{a} \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned} \tag{43}$$



Consequently, time-fractional regularized long-wave equation (1) reduces into an ordinary differential equation for fractional order with new independent variable

$$(P_{(2/\alpha-1)}^{(1-3\alpha/2),\alpha} f)(z) + 2 \frac{t^{(\alpha+1/2)}}{\Gamma(1-\alpha)} - \frac{1}{2} f f_z + b \frac{(3\alpha-1)}{2} f_{zz} = 0. \tag{44}$$

The proof of the theorem is complete.  $\square$

### 5. Explicit Power Series Solution of Time-Fractional Regularized Long-Wave Equation

In this section, we investigate the exact analytic solutions of equation (1) via the power series method [39]. Let us assume that

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n, \tag{45}$$

$$f'(\xi) = \sum_{n=0}^{\infty} (n+1)c_{n+1} \xi^n, \tag{46}$$

$$f''(\xi) = \sum_{n=0}^{\infty} (n+1)(n+2)c_{n+2} \xi^n.$$

Substituting (45) and (46) into (44), it yields

$$\left( \sum_{n=0}^{\infty} \left( \frac{\Gamma((3-\alpha/2) + n(\alpha-1)/2)}{\Gamma((3(1-\alpha)/2) + (n(n-1)/2))} c_n \xi^n \right) - \frac{1}{2} \left( \sum_{n=0}^{\infty} c_n \xi^n \right) \left( \sum_{n=0}^{\infty} (n+1)c_{n+1} \xi^n \right) + \frac{b(3\alpha-1)}{2} \sum_{n=0}^{\infty} (n+1)(n+2)c_{n+2} \xi^n + 2 \frac{t^{((\alpha+1)/2)}}{\Gamma(1-\alpha)} \right) = 0. \tag{47}$$

Thus,

$$\left( \sum_{n=0}^{\infty} \left( \frac{\Gamma(3-\alpha/3 + n(\alpha-1)/2)}{\Gamma(3(1-\alpha)/2 + n(n-1)/2)} c_n \xi^n \right) - \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^n (n-k+1)c_k c_{n-k+1} \xi^n + \frac{b(3\alpha-1)}{2} \sum_{n=0}^{\infty} (n+1)(n+2)c_{n+2} \xi^n + 2 \frac{t^{((\alpha+1)/2)}}{\Gamma(1-\alpha)} \right) = 0. \tag{48}$$

Comparing coefficients in equation (47), when  $n = 0$ , we have

$$c_2 = \frac{1}{b(3\alpha-1)} \left( \frac{1}{2} c_0 c_1 - \frac{\Gamma((3-\alpha/2))}{\Gamma(3(1-\alpha)/2)} c_0 - 2 \frac{t^{((\alpha+1)/2)}}{\Gamma(1-\alpha)} \right). \tag{49}$$

When  $n \geq 1$ , we get

$$c_{n+2} = \frac{-2}{b(3\alpha-1)(n+1)(n+2)} \cdot \frac{(\Gamma((3-\alpha)/2 + (n(\alpha-1))/2))}{(2 + (n(\alpha-1))/2)} c_n - \frac{1}{2} \sum_{k=0}^n (n-k+1)c_k c_{n-k+1} + 2 \frac{t^{((\alpha+1)/2)}}{\Gamma(1-\alpha)}. \tag{50}$$

Then, explicit solution for equation (44) can be written in the form:

$$f(\xi) = c_0 + c_1 \xi + \frac{1}{b(3\alpha-1)} \left( \frac{1}{2} c_0 c_1 - \frac{\Gamma((3-\alpha)/2)}{\Gamma(3(1-\alpha)/2)} c_0 - 2 \frac{t^{((\alpha+1)/2)}}{\Gamma(1-\alpha)} \right) \xi^2 - \frac{2}{b(3\alpha-1)} \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \times \frac{(\Gamma((3-\alpha)/2 + (n(\alpha-1))/2))}{\Gamma((3(1-\alpha)/2) + (n(\alpha-1)/2))} c_n - \frac{1}{2} \sum_{k=0}^n (n-k+1)c_k c_{n-k+1} + 2 \frac{t^{((\alpha+1)/2)}}{\Gamma(1-\alpha)} \xi^{n+2}. \tag{51}$$

Consequently, the exact power series solution for equation (44) has the following form:

$$\begin{aligned}
 u(x, t) = & \frac{1}{a} t^{-(\alpha+1)/2} c_0 + \frac{1}{a} x t^{-\alpha} c_1 + \frac{1}{ab(3\alpha-1)} \left[ \frac{1}{2} c_0 c_1 - \frac{\Gamma((3-\alpha)/2)}{\Gamma(3(1-\alpha)/2)} c_0 - 2 \frac{t^{((\alpha+1)/2)}}{\Gamma(1-\alpha)} \right] x^2 t^{(1-3\alpha/2)} - \\
 & \frac{2}{ab(3\alpha-1)} \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \left[ \frac{\Gamma((3-\alpha/2) + (n(\alpha-1)/2))}{\Gamma(3(1-\alpha)/2 + (n(\alpha-1)/2))} c_n - \frac{1}{2} \sum_{k=0}^n (n-k+1) c_k c_{n-k+1} \right. \\
 & \left. + 2 \frac{t^{((\alpha+1)/2)}}{\Gamma(1-\alpha)} \right] t^{((n(1-\alpha)-3\alpha+1)/2)} x^{n+2}.
 \end{aligned} \tag{52}$$

## 6. Physical Interpretation of the Power Series Solution for Equation (52)

In order to have a clear vision of the physical properties of the power series solution and to help us analyse it, the 3-dimensional plots for solution equation (52) are plotted in Figures 1–4 by using suitable parameter values.

## 7. Conclusion

In this paper, the Lie symmetry method has been successfully applied to the time-fractional regularized long-wave equation with the Riemann–Liouville fractional derivative. Generally, to find Lie point symmetries for FDEs is not an obvious task but still an interesting and efficient tool. In our context, we can obtain Lie point symmetries of the time-fractional regularized long-wave equation basing on the systematic method presented by Gazizov et al. [20, 21]. The obtained nontrivial Lie point symmetries have been used to derive similarity reductions and to transform the initial equation into a nonlinear fractional ordinary differential equation with the well-known Erdélyi–Kober fractional derivative. With the help of the power series method, the exact power series solution of the reduced FODE has been constructed. Some interesting 3-D figures for the obtained solutions were presented. Furthermore, the exact solutions obtained in this paper might be of great importance in the fields of physics and different other branches of physical sciences present the ability of our technique appear to be applicable on many various forms of nonlinear partial differential equations. There are some possible extensions of this study, e.g., symmetry analysis and conservation laws of time-space fractional (RLW) equation, which are in progress and will be discussed in the future work.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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