

## Research Article

# Zero Controllability Criterion for Discrete Positive Systems with Multiple Delays on Both States and Inputs

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Zero controllability criterion for positive linear discrete systems with multiple delays in both states and inputs is obtained and proved. An example is given to support our main result.

## 1. Introduction and Preliminaries

Throughout this paper, we use the following notation:  $\mathbb{N}$  is the set of nonnegative integers,  $\mathbb{N}^*$  is the set of positive integers,  $\sigma_n^m = \{n, n+1, \dots, m\}$  is the finite subset of  $\mathbb{N}$  with  $n \leq m$ ,  $\mathbb{R}^n$  is the set of real vectors with  $n$  components,  $\mathbb{R}_+^n$  is the set of vectors in  $\mathbb{R}^n$  with nonnegative components,  $\mathbb{R}^{n \times m}$  is the set of real matrices of size  $n \times m$ ,  $I_n$  is the identity matrix in  $\mathbb{R}^{n \times n}$ , and  $\mathbb{R}_+^{n \times m}$  is the set of real matrices with nonnegative entries.

Positive systems are a wide class of systems in which state variables are constrained to be positive or at least nonnegative for all time whenever the initial conditions and inputs are nonnegative [1, 2]. The mathematical theory of positive linear systems is based on the theory of nonnegative matrices developed by Perron and Frobenius, see, e.g., [3, 4]. Since positive systems are not defined on linear spaces but on cones, then many concepts of linear systems cannot be directly generalized to linear positive systems without reformulation. One such property is the notion of controllability of linear positive systems.

Controllability is one of the fundamental concepts in the mathematical control theory. A positive system is controllable if it is possible to transfer it from an arbitrary nonnegative initial state to an arbitrary nonnegative final state using only certain admissible nonnegative controls. Since late 1980s, controllability of discrete positive linear systems

without delays has been a subject of much research [5–9]. In particular, Coxson and Shapiro in [6] showed that the discrete linear positive system is controllable if and only if it is reachable (controllability from zero initial conditions) and zero controllable (controllability to zero final state). The reachability of positive linear discrete systems with multiple delays in both state and control is addressed in [10]. On zero controllability of positive linear discrete systems with delay, the authors of [11] show that the following system with a single state delay

$$x_{i+1} = A_0 x_i + A_1 x_{i-1} + B u_i, \quad i \in \mathbb{N}, \quad (1)$$

is zero controllable if and only if the matrix  $A = \begin{pmatrix} A_0 & A_1 \\ I_n & 0 \end{pmatrix}$

is nilpotent. In this paper, we will extend the result of zero controllability in [11] to the more general case, namely, positive discrete systems with multiple time delays both in state and in input. For this, we consider the general discrete linear time delay systems:

$$x_{i+1} = \sum_{j=0}^p A_j x_{i-j} + \sum_{j=0}^q B_j u_{i-j}, \quad i \in \mathbb{N}, \quad (2)$$

where  $x_i \in \mathbb{R}^n$  is the state,  $u_i \in \mathbb{R}^m$  is the input,  $A_j \in \mathbb{R}^{n \times n}$  ( $j \in \sigma_0^p$ ),  $B_j \in \mathbb{R}^{n \times m}$  ( $j \in \sigma_0^q$ ), and  $p$  and  $q$  are the nonnegative integer maximal values of delays on state and

input, respectively. The initial conditions for (2) are given arbitrarily by  $x_{-j} \in \mathbb{R}^n$  for  $j \in \sigma_0^p$  and  $u_{-j} \in \mathbb{R}^m$  for  $j \in \sigma_1^q$ .

**Definition 1** (positivity). System (2) is said to be positive if the state  $x_i \in \mathbb{R}_+^n$ ,  $i \in \mathbb{N}$ , for any initial states  $x_{-j} \in \mathbb{R}_+^n$  ( $j \in \sigma_0^p$ ), for any initial inputs  $u_{-j} \in \mathbb{R}_+^m$  ( $j \in \sigma_1^q$ ), and all inputs  $u_i \in \mathbb{R}_+^m$ ,  $i \in \mathbb{N}$ .

**Lemma 1** (see [10]). System (2) is positive if and only if  $A_j \in \mathbb{R}_+^{n \times n}$  ( $j \in \sigma_0^p$ ) and  $B_j \in \mathbb{R}_+^{n \times m}$  ( $j \in \sigma_1^q$ ).

In all the sequels in this paper, we assume that system (2) is positive.

**Definition 2** (zero controllability). System (2) is said to be zero controllable if any initial state sequence  $x_{-j} \in \mathbb{R}_+^n$  ( $j \in \sigma_0^p$ ) and any initial input sequence  $u_{-j} \in \mathbb{R}_+^m$  ( $j \in \sigma_1^q$ ), there exist a positive integer  $N$  and an input sequence  $u_i \in \mathbb{R}_+^m$  ( $i \in \sigma_1^{N-1}$ ) such that the state of the system is driven from  $x_{-j}$  to 0, that is,  $x_N = 0$ .

The paper is organized as follows. In Section 2, we give and prove the criterion of the zero controllability of the general system (2) which is the main result of this paper. A numerical example is given in Section 3. Finally, the conclusion is provided in Section 4.

## 2. Main Result

In this section, we give the proof of the main result of this paper, which is accomplished in Theorem 1.

**Lemma 2** (see [12, 13]). The general solution to (2) is given by

$$x_i = G_i x_0 + \sum_{j=1}^p \sum_{k=1}^{p-j+1} G_{i-k} A_{k-1+j} x_{-j} + \sum_{j=1}^q \sum_{k=1}^{q-j+1} G_{i-k} B_{k-1+j} u_{-j} + \sum_{j=0}^{i-1} \sum_{k=0}^q G_{i-1-j-k} B_k u_j, \quad i \in \mathbb{N}, \tag{3}$$

where the transition matrix  $G_i \in \mathbb{R}^{n \times n}$  ( $i \in \mathbb{N}$ ) is determined by the recurrence relation

$$G_i = \begin{cases} I_n, & \text{for } i = 0, \\ \sum_{k=0}^p A_k G_{i-1-k}, & \text{for } i \in \mathbb{N}^*, \end{cases} \tag{4}$$

with the assumption

$$G_i = 0, \quad \text{for } i < 0. \tag{5}$$

**Lemma 3.** The transition matrix  $G_i$  also satisfies the following equation:

$$G_i = \sum_{k=0}^p G_{i-1-k} A_k, \quad \text{for } i \in \mathbb{N}^*. \tag{6}$$

*Proof.* See Appendix A.

Then, for any  $i \in \mathbb{N}$ , we pose  $H_i^0 = G_i$ , and hence, for all  $i \in \mathbb{N}^*$ , we pose

$$\begin{cases} H_i^j = \sum_{k=1}^{p-j+1} H_{i-k}^0 A_{k-1+j}, & j \in \sigma_0^p, \\ L_i^j = \sum_{k=1}^{q-j+1} H_{i-k}^0 B_{k-1+j}, & j \in \sigma_1^q, \end{cases} \tag{7}$$

with

$$\begin{aligned} H_i^j &= 0, & \text{for } j \in \sigma_1^p \text{ and } i \leq 0, \\ L_i^j &= 0, & \text{for } j \in \sigma_1^q \text{ and } i \leq 0. \end{aligned} \tag{8}$$

Moreover, for  $i \in \mathbb{N}$ , we put

$$K_i = \sum_{k=0}^q H_{i-k}^0 B_k, \tag{9}$$

with  $K_i = 0$  for  $i < 0$ .

Clearly by (7) and (9), solution (3) is given by the following new formula:

$$x_i = \sum_{j=0}^p H_i^j x_{-j} + \sum_{j=1}^q L_i^j u_{-j} + \sum_{j=0}^{i-1} K_{i-1-j} u_j, \quad i \in \mathbb{N}. \tag{10}$$

□

**Theorem 1.** System (2) is zero controllable if and only if the matrix

$$A = \begin{pmatrix} A_0 & A_1 & \cdots & \cdots & A_p & B_1 & \cdots & \cdots & \cdots & B_q \\ I_n & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\ 0 & \cdots & 0 & I_n & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & I_m & \ddots & & & \vdots \\ \vdots & & & & \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & I_m & 0 \end{pmatrix} \tag{11}$$

is nilpotent.

We introduce the following useful two lemmas that will aid us in the proof of our main result.

**Lemma 4.** For all  $i \in \mathbb{N}$ , we have

$$\begin{cases} H_{i+1}^j = H_i^{j+1} + H_i^0 A_j, & j \in \sigma_0^{p-1}, \\ H_{i+1}^p = H_i^0 A_p, \end{cases} \tag{12}$$

$$\begin{cases} L_{i+1}^j = L_i^{j+1} + H_i^0 B_j, & j \in \sigma_1^{q-1}, \\ L_{i+1}^q = H_i^0 B_q. \end{cases} \tag{13}$$

*Proof.* See Appendix B.

Without loss of generality, we assume that  $p \geq q$ . Indeed, if  $p < q$ , we can set  $A_j = 0$  for  $j \in \sigma_{p+1}^q$ , and then we come back to  $p = q$  case.  $\square$

**Lemma 5.** For all  $i \geq p$ , we have

$$A^i = \begin{pmatrix} H_i^0 & H_i^1 & \dots & H_i^p & L_i^1 & \dots & L_i^q \\ H_{i-1}^0 & H_{i-1}^1 & \dots & H_{i-1}^p & L_{i-1}^1 & \dots & L_{i-1}^q \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ H_{i-p}^0 & H_{i-p}^1 & \dots & H_{i-p}^p & L_{i-p}^1 & \dots & L_{i-p}^q \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (14)$$

*Proof.* See Appendix C.  $\square$

*Remark 1.* Since  $H_{p-p}^0 = H_0^0 = I_n$ , then  $A^p \neq 0$ . Now, we prove our main result.

*Proof of Theorem 1 (sufficiency).* Since  $A$  is nilpotent, then there exists a positive integer  $N$  such that  $A^{N+p} = 0$ . Hence, by Lemma 5, we have  $H_N^i = 0$  for  $i \in \sigma_0^p$  and  $L_N^j = 0$  for  $j \in \sigma_1^q$ . Thus, system (2) is zero controllable.

*(Necessity).* Since system (2) is zero controllable, there exists a positive integer  $N$  such that  $H_N^j = 0$  for  $j \in \sigma_0^p$  and  $L_N^j = 0$  for  $j \in \sigma_1^q$ . According to Lemma 4, we get that  $H_{N+k}^j = 0$  and  $L_{N+k}^j = 0$  for  $k \in \sigma_0^p$ . Thus, by Lemma 5, we have  $A^{N+p} = 0$ . This implies that  $A$  is nilpotent. The theorem is proved.  $\square$

*Remark 2.* If one diagonal element of the matrix  $A_0$  is nonzero, system (2) is nonzero controllable.

### 3. Example

Consider system (2) with  $p = q = 2$  and matrices

$$\begin{aligned} A_0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ A_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ A_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ B_1 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}. \end{aligned} \quad (15)$$

By calculation, we get that matrix  $A = \begin{pmatrix} A_0 & A_1 & A_2 & B_1 & B_2 \\ I_3 & 0 & 0 & 0 & 0 \\ 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_1 & 0 \end{pmatrix}$  is nilpotent with index  $k = 6$ , that is,  $A^{k-1} \neq 0$  and  $A^k = 0$ . Thus, by Theorem 1, system (2) is zero controllable.

### 4. Conclusion

In this paper, we have investigated the zero controllability of discrete linear positive systems with delays. Necessary and sufficient conditions have been established for the zero controllability discrete linear positive systems with multiple delays in both state variables and input signals. A numerical example is presented to explore the proposed theory.

**Appendix**

For  $i \in \sigma_1^p$ , we have

**A. Proof of Lemma 3**

*Proof.* First, for  $i = 1$ , we have  $G_1 = A_0 = \sum_{k=0}^p G_{-k}A_k$  and (6) holds. Secondly, suppose that (6) holds for  $k \in \sigma_1^i$ . We prove that it holds for  $k = i + 1$ .

$$\begin{aligned} G_{i+1} &= \sum_{k=0}^i A_k G_{i-k} = \sum_{k=0}^{i-1} A_k G_{i-k} + A_i = \sum_{k=0}^{i-1} A_k \left( \sum_{j=0}^p G_{i-k-1-j} A_j \right) + A_i \\ &= \sum_{k=0}^{i-1} A_k \left( \sum_{j=0}^{i-k-1} G_{i-k-1-j} A_j \right) + A_i = \sum_{j=0}^{i-1} \left( \sum_{k=0}^{i-j-1} A_k G_{i-j-1-k} \right) A_j + A_i \\ &= \sum_{j=0}^{i-1} G_{i-j} A_j + A_i = \sum_{j=0}^i G_{i-j} A_j = \sum_{j=0}^p G_{i-j} A_j. \end{aligned} \tag{A.1}$$

For  $i \geq p + 1$ , we have

$$\begin{aligned} G_{i+1} &= \sum_{k=0}^p A_k G_{i-k} = \sum_{k=0}^p A_k \left( \sum_{j=0}^p G_{i-k-1-j} A_j \right) \\ &= \sum_{j=0}^p \left( \sum_{k=0}^p A_k G_{i-j-1-k} \right) A_j = \sum_{j=0}^p G_{i-j} A_j. \end{aligned} \tag{A.2}$$

Thus, (6) is satisfied in step  $i + 1$ . Hence, (6) holds for any  $i \in \mathbb{N}^*$ .  $\square$

**B. Proof of Lemma 4**

*Proof.* Let  $i \in \mathbb{N}$ . For  $j \in \sigma_0^{p-1}$ , we have

$$\begin{aligned} H_{i+1}^j - H_i^{j+1} &= \sum_{k=1}^{p-j+1} H_{i+1-k}^0 A_{k-1+j} - \sum_{k=1}^{p-j} H_{i-k}^0 A_{k+j} \\ &= \sum_{k=0}^{p-j} H_{i-k}^0 A_{k+j} - \sum_{k=1}^{p-j} H_{i-k}^0 A_{k+j} \\ &= H_i^0 A_j, \end{aligned} \tag{B.1}$$

and, for  $j = p$ , we have

$$H_{i+1}^p = \sum_{k=1}^1 H_{i+1-k}^0 A_{k-1+p} = H_i^0 A_p. \tag{B.2}$$

Similarly, we prove that (13) holds.  $\square$

**C. Proof of Lemma 5**

*Proof.* We introduce a new state variable  $\tilde{x}_i \in \mathbb{R}^{n(p+1)+mq}$  for  $i \in \mathbb{N}$  by

$$\tilde{x}_i = \begin{pmatrix} x_i \\ x_{i-1} \\ \vdots \\ x_{i-p} \\ u_{i-1} \\ \vdots \\ u_{i-q} \end{pmatrix}. \tag{C.1}$$

It is easy to verify that

$$\begin{cases} \tilde{x}_{i+1} = A\tilde{x}_i + Bu_i, & i \in \mathbb{N}, \\ \tilde{x}_0 \in \mathbb{R}^{n(p+1)+mq}, \end{cases} \tag{C.2}$$

where  $A$  is defined in (11) and

$$B = \begin{pmatrix} B_0 \\ 0 \\ \vdots \\ 0 \\ I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{C.3}$$

Let  $u_i = 0$  for  $i \in \mathbb{N}$ . Then, the solution of system (C.2) is given by

$$\tilde{x}_i = A^i \tilde{x}_0, \quad i \in \mathbb{N}. \tag{C.4}$$

On the other hand, from (10), for all  $i \geq p$ , we have

$$\tilde{x}_i = \begin{pmatrix} x_i \\ x_{i-1} \\ \vdots \\ x_{i-p} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} H_i^0 & H_i^1 & \cdots & H_i^p & L_i^1 & \cdots & L_i^q \\ H_{i-1}^0 & H_{i-1}^1 & \cdots & H_{i-1}^p & L_{i-1}^1 & \cdots & L_{i-1}^q \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ H_{i-p}^0 & H_{i-p}^1 & \cdots & H_{i-p}^p & L_{i-p}^1 & \cdots & L_{i-p}^q \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_{-1} \\ \vdots \\ x_{-p} \\ u_{-1} \\ \vdots \\ u_{-q} \end{pmatrix}. \quad (\text{C.5})$$

Hence, by identification between (C.4) and (C.5), we get that (14) holds.  $\square$

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The author declares no conflicts of interest.

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