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Research Article

Some Generalizations of Frames in Hilbert Modules

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Frames play significant role in various areas of science and engineering. In this paper, we introduce the concept of frames for the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$ and their generalizations. Moreover, we obtain some new results for generalized frames in Hilbert modules.

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [2] by Daubechies et al., frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [3].

Traditionally, frames have been used in signal processing, image processing, data compression, and sampling theory. A discreet frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements.

Many generalizations of the concept of frame have been defined in Hilbert spaces and Hilbert C^*-modules [4–9].

This paper generalizes the papers' G-frames as special frames [10], Frames and Operator Frames for $B(\mathscr{H})$ [11], and Generalized Frames for $B(\mathscr{H}, \mathscr{K})$ [12], in framework of the Hilbert C^*-modules.

Let I be a finite or countable index subset of $\mathbb{N}$. In this section, we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathscr{A}$-modules, frames in Hilbert $\mathscr{A}$-modules, and their generalizations. For information about frames in Hilbert spaces, we refer to [13]. Our references for C^*-algebras are [14, 15]. For a C^*-algebra $\mathscr{A}$, if $a \in \mathscr{A}$ is positive, we write $a \geq 0$, and $\mathscr{A}^+$ denotes the closed cone of positive elements in $\mathscr{A}$.

Definition 1 (see [14]). If $\mathscr{A}$ is a Banach algebra, an involution is a map $a \longmapsto a^*$ of $\mathscr{A}$ into itself such that, for all a and b in $\mathscr{A}$ and all scalar α, the following conditions hold:

(1) $(a^*)^* = a$
(2) $(ab)^* = b^* a^*$
(3) $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$

Definition 2 (see [14]). A C^*-algebra $\mathscr{A}$ is a Banach algebra with involution such that

$$\|a * a\| = \|a\|^2, \tag{1}$$

for every a in $\mathscr{A}$.

Examples 1

(1) $B(\mathscr{H})$, the algebra of bounded operators on a Hilbert space $\mathscr{H}$, is a C^*-algebra, where, for each operator A, A^* is the adjoint of A
(2) $C(X)$, the algebra of continuous functions on a compact space X, is an abelian C^*-algebra, where $f^*(x) := \overline{f(x)}$
(3) $C_0(X)$, the algebra of continuous functions on a locally compact space X that vanishes at infinity, is an abelian C^*-algebra, where $f^*(x) := \overline{f(x)}$

Definition 3 (see [16]). Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{H}$ be a left $\mathscr{A}$-module, such that the linear structures of $\mathscr{A}$ and $\mathscr{H}$ are compatible. $\mathscr{H}$ is a pre-Hilbert $\mathscr{A}$-module if $\mathscr{H}$ is equipped with an $\mathscr{A}$-valued inner product $\langle .,. \rangle$: $\mathscr{H} \times \mathscr{H} \longrightarrow \mathscr{A}$, such that it is sesquilinear and positive definite and respects the module action. In other words,

(1) $\langle x, x \rangle \geq 0$, for all $x \in \mathscr{H}$ and $\langle x, x \rangle = 0$, if and only if $x = 0$

(2) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$, for all $a \in \mathscr{A}$ and $x, y, z \in \mathscr{H}$

(3) $\langle x, y \rangle = \langle y, x \rangle^*$, for all $x, y \in \mathscr{H}$

For $x \in \mathscr{H}$, we define $\|x\| = \|\langle x, x \rangle\|^{(1/2)}$. If $\mathscr{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathscr{A}$-module or a Hilbert C^*-module over $\mathscr{A}$. For every a in C^*-algebra $\mathscr{A}$, we have $|a| = (a * a)^{(1/2)}$, and the $\mathscr{A}$-valued norm on $\mathscr{H}$ is defined by $|x| = \langle x, x \rangle^{(1/2)}$ for $x \in \mathscr{H}$.

Examples 2. Let X be a locally compact Hausdorff space and $\mathscr{H}$ a Hilbert space, and the Banach space $C_0(X, \mathscr{H})$ of all continuous $\mathscr{H}$-valued functions vanishing at infinity is a Hilbert C^*-module over the C^*-algebra $C_0(X)$ with inner product $\langle f, g \rangle (x) := \langle f(x), g(x) \rangle$ and module operation $(\phi f)(x) = \phi(x) f(x)$, for all $\phi \in C_0(X)$ and $f \in C_0(X, \mathscr{H})$. If $\{\mathscr{H}_k\}_{k\in\mathbb{N}}$ is a countable set of Hilbert $\mathscr{A}$-modules, then one can define their direct sum $\oplus_{k\in\mathbb{N}}\mathscr{H}_k$. On the $\mathscr{A}$-module $\oplus_{k\in\mathbb{N}}\mathscr{H}_k$ of all sequences $x = (x_k)_{k\in\mathbb{N}}$: $x_k \in \mathscr{H}_k$, such that the series $\sum_{k\in\mathbb{N}} \langle x_k, x_k \rangle_{\mathscr{A}}$ is norm-convergent in the $\mathscr{C}^*$-algebra $\mathscr{A}$, we define the inner product by

$$\langle x, y \rangle := \sum_{k\in\mathbb{N}} \langle x_k, y_k \rangle_{\mathscr{A}}, \tag{2}$$

for $x, y \in \oplus_{k\in\mathbb{N}}\mathscr{H}_k$. Then, $\oplus_{k\in\mathbb{N}}\mathscr{H}_k$ is a Hilbert $\mathscr{A}$-module. The direct sum of a countable number of copies of a Hilbert C^*-module $\mathscr{H}$ is denoted by $l^2(\mathscr{H})$.

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-modules, and a map T: $\mathscr{H} \longrightarrow \mathscr{K}$ is said to be adjointable if there exists a map T^*: $\mathscr{K} \longrightarrow \mathscr{H}$ such that $\langle Tx, y \rangle_{\mathscr{A}} = \langle x, T^* y \rangle_{\mathscr{A}}$, for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.

We also reserve the notation $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ for the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$ and $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{H})$ is abbreviated to $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$.

$\mathrm{End}^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ is a Hilbert $\mathrm{End}^*_{\mathscr{A}}(\mathscr{K})$-module with the inner product $\langle T, S \rangle = TS^*$, $\forall T, S \in B(\mathscr{H}, \mathscr{K})$.

Lemma 1 (see [17]). *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-modules and $T \in End^*_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$.*

(i) If T is injective and T has closed range, then the adjointable map T^*T is invertible and

$$\left\|(T * T)^{-1}\right\|^{-1} I_{\mathscr{H}} \leq T * T \leq \|T\|^2 I_{\mathscr{H}}. \tag{3}$$

(ii) If T is surjective, then the adjointable map TT^* is invertible and

$$\left\|(TT^*)^{-1}\right\|^{-1} I_{\mathscr{K}} \leq TT^* \leq \|T\|^2 I_{\mathscr{K}}. \tag{4}$$

Definition 4 (see [18]). We call a sequence $\{\Lambda_i \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H}, V_i): i \in I\}$ a g-frame in Hilbert $\mathscr{A}$-module $\mathscr{H}$ with respect to $\{V_i: i \in I\}$ if there exist two positive constants C, D, such that, for all $x \in \mathscr{H}$,

$$C\langle x, x \rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathscr{A}} \leq D\langle x, x \rangle_{\mathscr{A}}, \tag{5}$$

where the numbers C and D are called lower and upper bounds of the g-frame, respectively. If $C = D = \lambda$, the g-frame is λ-tight. If $C = D = 1$, it is called a g-Parseval frame. If the sum in the middle of (5) is convergent in norm, the g-frame is called standard.

Definition 5 (see [17]). Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module over a unital C^*-algebra. A family $\{x_i\}_{i\in I}$ of elements of $\mathscr{H}$ is a $*$-frame for $\mathscr{H}$, if there exist strictly nonzero elements A, B in $\mathscr{A}$, such that, for all $x \in \mathscr{H}$,

$$A\langle x, x \rangle_{\mathscr{A}} A^* \leq \sum_{i\in I} \langle x, x_i \rangle_{\mathscr{A}} \langle x_i, x \rangle_{\mathscr{A}} \leq B\langle x, x \rangle_{\mathscr{A}} B^*, \tag{6}$$

where the elements A and B are called lower and upper bounds of the $*$-frame, respectively. If $A = B = \lambda_1$, the $*$-frame is λ_1-tight. If $A = B = 1$, it is called a normalized tight $*$-frame or a Parseval $*$-frame. If the sum in the middle of (6) is convergent in norm, the $*$-frame is called standard.

Definition 6 (see [19]). We call a sequence $\{\Lambda_i \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H}, V_i): i \in I\}$ a $*$-g-frame in Hilbert $\mathscr{A}$-module $\mathscr{H}$ over a unital C^*-algebra with respect to $\{V_i: i \in I\}$ if there exist strictly nonzero elements A, B in $\mathscr{A}$, such that, for all $x \in \mathscr{H}$,

$$A\langle x, x \rangle_{\mathscr{A}} A^* \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathscr{A}} \leq B\langle x, x \rangle_{\mathscr{A}} B^*, \tag{7}$$

where the elements A and B are called lower and upper bounds of the $*$-g-frame, respectively. If $A = B = \lambda_1$, the $*$-g-frame is λ_1-tight. If $A = B = 1$, it is called a $*$-g-Parseval frame. If the sum in the middle of (7) is convergent in norm, the $*$-g-frame is called standard.

Definition 7 (see [20]). Let $K \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ and $\Lambda_i \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H}, V_i)$, for all $i \in I$; then, $\{\Lambda_i\}_{i\in I}$ is said to be a K-g-frame for $\mathscr{H}$ with respect to $\{V_i\}_{i\in I}$ if there exist two constants $C, D > 0$ such that

$$C\langle K * x, K * x \rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathscr{A}} \leq D\langle x, x \rangle_{\mathscr{A}}, \quad \forall x \in \mathscr{H}. \tag{8}$$

The numbers C and D are called K-g-frame bounds. Particularly, if

$$C\langle K * x, K * x \rangle = \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x \rangle, \quad \forall x \in \mathscr{H}. \tag{9}$$

The K-g-frame is C-tight.

Definition 8. (see [21]) Let $K \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$. A family $\{x_i\}_{i\in I}$ of elements of $\mathscr{H}$ is a $*$-K-frame for $\mathscr{H}$ if there exist strictly nonzero elements A and B in $\mathscr{A}$, such that, for all $x \in \mathscr{H}$,

$$A\langle K^* x, K^* x\rangle_{\mathscr{A}} A^* \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle x_i, x\rangle_{\mathscr{A}} \leq B\langle x, x\rangle_{\mathscr{A}} B^*, \tag{10}$$

where the elements A and B are called lower and upper bound of the $*$-K-frame, respectively.

Definition 9 (see [22]). Let $K \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$. We call a sequence $\{\Lambda_i \in \mathrm{End}_{A}^*(\mathscr{H}, \mathscr{H}_i)\colon i \in I\}$ a $*$-K-g-frame in Hilbert $\mathscr{A}$-module $\mathscr{H}$ with respect to $\{\mathscr{H}_i\colon i \in I\}$ if there exist strictly nonzero elements A and B in $\mathscr{A}$ such that

$$A\langle K^* x, K^* x\rangle_{\mathscr{A}} A^* \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathscr{A}} \leq B\langle x, x\rangle_{\mathscr{A}} B^*, \quad \forall x \in \mathscr{H}. \tag{11}$$

The numbers A and B are called lower and upper bounds of the $*$-K-g-frame, respectively. If

$$A\langle K^* x, K^* x\rangle A^* = \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle, \quad \forall x \in \mathscr{H}, \tag{12}$$

The $*$-K-g-frame is A-tight.

2. Some Results for Generalized Frames in Hilbert $\mathscr{A}$-Modules

We begin this section with the following hTeorem.

Theorem 1. *Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module over a commutative C^*-algebra. A family $\{x_i\}_{i\in I}$ of elements of $\mathscr{H}$ is a $*$-frame for $\mathscr{H}$ if and only if there exist strictly positive elements A and B in $\mathscr{A}$, such that, for all $x \in \mathscr{H}$,*

$$A\langle x, x\rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle x_i, x\rangle_{\mathscr{A}} \leq B\langle x, x\rangle_{\mathscr{A}}. \tag{13}$$

Proof. Let $\{x_i\}_{i\in I}$ be a $*$-frame for $\mathscr{H}$; then, there exist strictly nonzero elements a and b in $\mathscr{A}$, such that, for all $x \in \mathscr{H}$,

$$a\langle x, x\rangle_{\mathscr{A}} a^* \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle x_i, x\rangle_{\mathscr{A}} \leq b\langle x, x\rangle_{\mathscr{A}} b^*. \tag{14}$$

By commutativity, we have

$$aa^* \langle x, x\rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle x_i, x\rangle_{\mathscr{A}} \leq bb^* \langle x, x\rangle_{\mathscr{A}}. \tag{15}$$

We pose $A = aa^*$ and $B = bb^*$; then, A and B are strictly positive elements in $\mathscr{A}$. Hence,

$$A\langle x, x\rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle x_i, x\rangle_{\mathscr{A}} \leq B\langle x, x\rangle_{\mathscr{A}}. \tag{16}$$

Conversely, let A and B be strictly positive elements in $\mathscr{A}$, such that, for all $x \in \mathscr{H}$,

$$A\langle x, x\rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle x_i, x\rangle_{\mathscr{A}} \leq B\langle x, x\rangle_{\mathscr{A}}, \tag{17}$$

where A and B are strictly positive elements in $\mathscr{A}$; then, there exist a and b in $\mathscr{A}$ such that $A = aa^*$ and $B = bb^*$.

So,

$$aa^* \langle x, x\rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle x_i, x\rangle_{\mathscr{A}} \leq bb^* \langle x, x\rangle_{\mathscr{A}}. \tag{18}$$

By commutativity, we have

$$a\langle x, x\rangle_{\mathscr{A}} a^* \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle x_i, x\rangle_{\mathscr{A}} \leq b\langle x, x\rangle_{\mathscr{A}} b^*. \tag{19}$$

Therefore, $\{x_i\}_{i\in I}$ is an $*$-frame for $\mathscr{H}$. □

Corollary 1. *A sequence $\{\Lambda_i \in End_{\mathscr{A}}^*(\mathscr{H}, V_i)\colon i \in I\}$ is a $*$-g-frame in Hilbert $\mathscr{A}$-module $\mathscr{H}$ over a commutative C^*-algebra with respect to $\{V_i\colon i \in I\}$ if and only if there exist strictly positive elements A, B in $\mathscr{A}$, such that for all $x \in \mathscr{H}$,*

$$A\langle x, x\rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathscr{A}} \leq B\langle x, x\rangle_{\mathscr{A}}. \tag{20}$$

Corollary 2. *Let $K \in End_{\mathscr{A}}^*(\mathscr{H})$. A family $\{x_i\}_{i\in I}$ of elements of $\mathscr{H}$ is a $*$-K-frame for $\mathscr{H}$ if and only if there exist strictly positive elements A and B in $\mathscr{A}$, such that, for all $x \in \mathscr{H}$,*

$$A\langle K^* x, K^* x\rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle x_i, x\rangle_{\mathscr{A}} \leq B\langle x, x\rangle_{\mathscr{A}}. \tag{21}$$

Corollary 3. *Let $K \in End_{\mathscr{A}}^*(\mathscr{H})$. A sequence $\{\Lambda_i \in End_{A}^*(\mathscr{H}, \mathscr{H}_i)\colon i \in I\}$ is a $*$-K-g-frame in Hilbert $\mathscr{A}$-module $\mathscr{H}$ over a commutative C^*-algebra, with respect to $\{\mathscr{H}_i\colon i \in I\}$ if and only if there exist strictly positive elements A and B in $\mathscr{A}$, such that, for all $x \in \mathscr{H}$,*

$$A\langle K^* x, K^* x\rangle_{\mathscr{A}} \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle_{\mathscr{A}} \leq B\langle x, x\rangle_{\mathscr{A}}. \tag{22}$$

For a sequence of Hilbert $\mathscr{A}$-module $\{\mathscr{K}_i\}_{i\in I}$, the space $\oplus_{i\in I}\mathscr{K}_i$ is a Hilbert $\mathscr{A}$-module with the inner product

$$\langle \{x_i\}_{i\in I}, \{y_i\}_{i\in I}\rangle = \sum_{i\in I} \langle x_i, y_i\rangle_{\mathscr{A}}. \tag{23}$$

Proposition 1. *Let $\mathscr{K} = \oplus_{i\in I}\mathscr{K}_i$.*

(1) The sequence $\{\Lambda_i \in End_{\mathscr{A}}^*(\mathscr{H}, \mathscr{K}_i)\colon i \in I\}$ is a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_i\}_{i\in I}$ if and only if the sequence $\{\widetilde{\Lambda}_i \in End_{\mathscr{A}}^*(\mathscr{H}, \mathscr{K})\colon i \in I\}$ is a g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$, with $\widetilde{\Lambda}_i x = (\ldots, 0, 0, \Lambda_i x, 0, 0, \ldots)$, $\forall x \in \mathscr{H}$

(2) The sequence $\{\Lambda_i \in End_{\mathscr{A}}^*(\mathscr{H}, \mathscr{K}_i)\colon i \in I\}$ is a *-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_i\}_{i\in I}$ if and only if the sequence $\{\widetilde{\Lambda}_i \in End_{\mathscr{A}}^*(\mathscr{H}, \mathscr{K})\colon i \in I\}$ is a $*$-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$, with $\widetilde{\Lambda}_i x = (\ldots, 0, 0, \Lambda_i x, 0, 0, \ldots)$, $\forall x \in \mathscr{H}$

(3) For $K \in End_{\mathscr{A}}^*(\mathscr{H})$, the sequence $\{\Lambda_i \in End_{\mathscr{A}}^*(\mathscr{H}, \mathscr{K}_i)\colon i \in I\}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_i\}_{i\in I}$ if and only if the sequence $\{\widetilde{\Lambda}_i \in End_{\mathscr{A}}^*(\mathscr{H}, \mathscr{K})\colon i \in I\}$ is a K-g-frame for $\mathscr{H}$ with

respect to $\mathcal{K}$, *with* $\widetilde{\Lambda}_i x = (\ldots, 0, 0, \Lambda_i x, 0, 0, \ldots)$, $\forall x \in \mathcal{H}$

(4) *For* $K \in End^*_{\mathcal{A}}(\mathcal{H})$, *the sequence* $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_i): i \in I\}$ *is a* $*$-*K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{K}_i\}_{i\in I}$ *if and only if the sequence* $\{\widetilde{\Lambda}_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}): i \in I\}$ *is a* $*$-*K-g-frame for* $\mathcal{H}$ *with respect to* $\mathcal{K}$, *with* $\widetilde{\Lambda}_i x = (\ldots, 0, 0, \Lambda_i x, 0, 0, \ldots)$, $\forall x \in \mathcal{H}$

3. Frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$

We begin this section with the following definition.

Definition 10. A sequence $\{T_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}): i \in I\}$ is said to be a frame for $\mathrm{End}^*_{A}(\mathcal{H}, \mathcal{K})$ if there exist $0 < A, B < \infty$ such that

$$A\langle T, T\rangle \le \sum_{i\in I}\langle T, T_i\rangle\langle T_i, T\rangle \le \langle T, T\rangle, \quad \forall T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}), \tag{24}$$

where the series converges in the strong operator topology.

Example 3. For $n \in \mathbb{N}^*$, let $\{T_i\}_{i=1}^n \subset \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ such that, for all $i = 1, \ldots, n$, T_i is injective and has a closed range. Then, for all $i = 1, \ldots, n$, we have

$$\left\|(T_i^* T_i)^{-1}\right\|^{-1} I_{\mathcal{H}} \le T_i^* T_i \le \|T_i\|^2 I_{\mathcal{H}}. \tag{25}$$

So,

$$\left(\sum_{i=1}^n \left\|(T_i^* T_i)^{-1}\right\|^{-1}\right) I_{\mathcal{H}} \le \sum_{i=1}^n T_i^* T_i \le \left(\sum_{i=1}^n \|T_i\|^2\right) I_{\mathcal{H}}. \tag{26}$$

Hence,

$$\left(\sum_{i=1}^n \left\|(T_i^* T_i)^{-1}\right\|^{-1}\right) TT^* \le \sum_{i=1}^n TT_i^* T_i T^* \le \left(\sum_{i=1}^n \|T_i\|^2\right) TT^*, \quad \forall T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}). \tag{27}$$

Thus,

$$\left(\sum_{i=1}^n \left\|(T_i^* T_i)^{-1}\right\|^{-1}\right)\langle T, T\rangle \le \sum_{i=1}^n \langle T, T_i\rangle\langle T_i, T\rangle \le \left(\sum_{i=1}^n \|T_i\|^2\right)\langle T, T\rangle, \tag{28}$$

for all $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$.

This shows that $\{T_i\}_{i=1}^n$ is a frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$.

Example 4. Let $\{T_i\}_{i\in\mathbb{N}^*} \subset \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ such that, for all $i \in \mathbb{N}^*$, T_i is injective and has a closed range and $\|T_i\| \le (1/i)$.

Then, for all $i \in \mathbb{N}^*$, we have

$$\left\|(T_i^* T_i)^{-1}\right\|^{-1} I_{\mathcal{H}} \le T_i^* T_i \le \|T_i\|^2 I_{\mathcal{H}}. \tag{29}$$

So,

$$\left(\sum_{i\in\mathbb{N}^*} \left\|(T_i^* T)_i^{-1}\right\|^{-1}\right) I_{\mathcal{H}} \le \sum_{i\in\mathbb{N}^*} T_i^* T_i \le \left(\sum_{i\in\mathbb{N}^*} \|T_i\|^2\right) I_{\mathcal{H}}. \tag{30}$$

Hence,

$$\left(\sum_{i\in\mathbb{N}^*} \left\|(T_i^* T_i)^{-1}\right\|^{-1}\right) TT^* \le \sum_{i\in\mathbb{N}^*} TT_i^* T_i T^* \le \left(\sum_{i\in\mathbb{N}^*} \|T_i\|^2\right) TT^*, \quad \forall T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}). \tag{31}$$

Thus,

$$\left(\sum_{i\in\mathbb{N}^*} \left\|(T_i^* T_i)^{-1}\right\|^{-1}\right)\langle T, T\rangle \le \sum_{i\in\mathbb{N}^*} \langle T, T_i\rangle\langle T_i, T\rangle \le \left(\sum_{i\in\mathbb{N}^*} \|T_i\|^2\right)\langle T, T\rangle, \tag{32}$$

for all $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$.

This shows that $\{T_i\}_{i\in\mathbb{N}^*}$ is a frame for $\text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$.

Theorem 2. *A sequence $\{T_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}): i \in I\}$ is a frame for $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ if and only if it is a g-frame for $\mathcal{H}$ with respect to $\mathcal{K}$.*

Proof. Let $\{T_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\mathcal{K}$.

Then, there exist two positive constants A and B, such that

$$\begin{aligned} &A\langle x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle T_i x, T_i x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}, \quad \forall x \in \mathcal{H}, \\ \Longleftrightarrow &A\langle x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle T_i^* T_i x, x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}, \quad \forall x \in \mathcal{H}. \end{aligned} \tag{33}$$

So,

$$AI_{\mathcal{H}} \leq \sum_{i\in I} T_i^* T_i \leq BI_{\mathcal{H}}. \tag{34}$$

Hence,

$$ATT^* \leq \sum_{i\in I} TT_i^* T_i T^* \leq BTT^*, \quad \forall T \in \text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}). \tag{35}$$

Thus,

$$A\langle T, T\rangle \leq \sum_{i\in I} \langle T, T_i\rangle \langle T_i, T\rangle \leq B\langle T, T\rangle, \quad \forall T \in \text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}), \tag{36}$$

i.e., $\{T_i\}_{i\in I}$ is a frame for $\text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$.

Conversely, assume that $\{T_i\}_{i\in I}$ is a frame for $\text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$.

Then, there exist two positive constants A and B, such that

$$\begin{aligned} &A\langle T, T\rangle \leq \sum_{i\in I} \langle T, T_i\rangle \langle T_i, T\rangle \leq B\langle T, T\rangle, \quad \forall T \in \text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}), \\ \Longleftrightarrow &ATT^* \leq \sum_{i\in I} TT_i^* T_i T^* \leq BTT^*, \quad \forall T \in \text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}). \end{aligned} \tag{37}$$

So,

$$A\langle TT^* x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle TT_i^* T_i T^* x, x\rangle_{\mathcal{A}} \leq B\langle TT^* x, x\rangle_{\mathcal{A}}, \quad \forall T \in \text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}), \forall x \in \mathcal{K}. \tag{38}$$

Let $y \in \mathcal{H}$ and $T \in \text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ such that $T^* x = y$; then,

$$A\langle y, y\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle T_i^* T_i y, y\rangle_{\mathcal{A}} \leq B\langle y, y\rangle_{\mathcal{A}}, \quad \forall y \in \mathcal{H}, \tag{39}$$

i.e.,

$$A\langle y, y\rangle_{\mathcal{A}} \leq \sum_{i\in I} \langle T_i y, T_i y\rangle_{\mathcal{A}} \leq B\langle y, y\rangle_{\mathcal{A}}, \quad \forall y \in \mathcal{H}, \tag{40}$$

and thus, $\{T_i\}_{i\in I}$ is a g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$. □

Corollary 4. *A sequence $\{T_i \in End_A^*(\mathscr{H},\mathscr{K})\colon i \in I\}$ is a tight frame for $End_A^*(\mathscr{H},\mathscr{K})$ if and only if it is a tight g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$.*

Corollary 5. *A sequence $\{T_i \in End_A^*(\mathscr{H},\mathscr{K})\colon i \in I\}$ is a Bessel sequence for $End_A^*(\mathscr{H},\mathscr{K})$ if and only if it is a g-Bessel sequence for $\mathscr{H}$ with respect to $\mathscr{K}$.*

4. Generalized Frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$

We begin this section with the following definition.

Definition 11. A sequence $\{T_i \in \mathrm{End}_A^*(\mathscr{H},\mathscr{K})\colon i \in I\}$ is said to be a generalized frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$ if there exist strictly positive elements A and B in $\mathscr{A}$ such that

$$A\langle T,T\rangle \le \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \le B\langle T,T\rangle, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}), \tag{41}$$

where the series converges in the strong operator topology.

In the following, $\mathscr{A}$ is a commutative C^*-algebra.

Theorem 3. *A sequence $\{T_i \in End_A^*(\mathscr{H},\mathscr{K})\colon i \in I\}$ is a generalized frame for $End_A^*(\mathscr{H},\mathscr{K})$ if and only if it is a $*$-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$.*

Proof. Let $\{T_i\}_{i\in I}$ be a $*$-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$. Then, there exist strictly positive elements A and B in $\mathscr{A}$, such that

$$\begin{aligned} &A\langle x,x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_i x,T_i x\rangle_{\mathscr{A}} \le B\langle x,x\rangle_{\mathscr{A}}, \quad \forall x \in \mathscr{H},\\ &\Longleftrightarrow A\langle x,x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_i^* T_i x,x\rangle_{\mathscr{A}} \le B\langle x,x\rangle_{\mathscr{A}}, \quad \forall x \in \mathscr{H}. \end{aligned} \tag{42}$$

So,

$$AI_{\mathscr{H}} \le \sum_{i\in I} T_i^* T_i \le BI_{\mathscr{H}}. \tag{43}$$

Hence,

$$ATT^* \le \sum_{i\in I} TT_i^* T_i T^* \le BTT^*, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}). \tag{44}$$

Thus,

$$A\langle T,T\rangle \le \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \le B\langle T,T\rangle, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}), \tag{45}$$

i.e., $\{T_i\}_{i\in I}$ is a generalized frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$.

Conversely, assume that $\{T_i\}_{i\in I}$ is a generalized frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$.

Then, there exist strictly positive elements A and B in $\mathscr{A}$, such that

$$\begin{aligned} &A\langle T,T\rangle \le \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \le B\langle T,T\rangle, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}),\\ &\Longleftrightarrow ATT^* \le \sum_{i\in I} TT_i^* T_i T^* \le BTT^*, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}). \end{aligned} \tag{46}$$

So,

$$A\langle TT^* x,x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle TT_i^* T_i T^* x,x\rangle_{\mathscr{A}} \le B\langle TT^* x,x\rangle_{\mathscr{A}}, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}), \forall x \in \mathscr{K}. \tag{47}$$

Let $y \in \mathscr{H}$ and $T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K})$ such that $T^* x = y$; then,

$$A\langle y,y\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_i^* T_i y,y\rangle_{\mathscr{A}} \le B\langle y,y\rangle_{\mathscr{A}}, \quad \forall y \in \mathscr{H}, \tag{48}$$

i.e.,

$$A\langle y,y\rangle_{\mathscr{A}} \leq \sum_{i\in I}\langle T_i y, T_i y\rangle_{\mathscr{A}} \leq B\langle y,y\rangle_{\mathscr{A}}, \quad \forall y\in\mathscr{H}, \tag{49}$$

and thus, $\{T_i\}_{i\in I}$ is a $*$-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$. □

Corollary 6. *A sequence $\{T_i \in End_A^*(\mathscr{H},\mathscr{K})\colon i\in I\}$ is a generalized tight frame for $End_A^*(\mathscr{H},\mathscr{K})$ if and only if it is a tight $*$-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$.*

Corollary 7. *A sequence $\{T_i \in End_A^*(\mathscr{H},\mathscr{K})\colon i\in I\}$ is a generalized Bessel sequence for $End_A^*(\mathscr{H},\mathscr{K})$ if and only if it is a $*$-g-Bessel sequence for $\mathscr{H}$ with respect to $\mathscr{K}$.*

5. K-Frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$

We begin this section with the following definition.

Definition 12. Let $K\in \mathrm{End}_A^*(\mathscr{H})$. A sequence $\{T_i\in \mathrm{End}_A^*(\mathscr{H},\mathscr{K})\colon i\in I\}$ is said to be a K-frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$ if there exist $0<A,B<\infty$ such that

$$A\langle TK,TK\rangle \leq \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \leq B\langle T,T\rangle, \quad \forall T\in \mathrm{End}_A^*(\mathscr{H},\mathscr{K}), \tag{50}$$

where the series converges in the strong operator topology.

Example 5. For $n\in\mathbb{N}^*$, let $\{T_i\}_{i=1}^n \subset \mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$.

Then, for $K=(\sum_{i=1}^n T_i^* T_i)^{(1/2)}$, we have

$$K^*K=\sum_{i=1}^n T_i^*T_i. \tag{51}$$

So,

$$TKK^*T^* = \sum_{i=1}^n TT_i^*T_iT^*. \tag{52}$$

Hence,

$$\langle TK,TK\rangle = \sum_{i=1}^n \langle T,T_i\rangle\langle T_i,T\rangle, \quad \forall T\in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H}). \tag{53}$$

This shows that $\{T_i\}_{i=1}^n$ is a Parseval K-frame for $\mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$.

Similar to Remark 1 in [22], we have the following.

Remark 1

(1) Every frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$ is a K-frame, for any $K\in \mathrm{End}_A^*(\mathscr{H})$: $K\neq 0$

(2) If $K\in \mathrm{End}_A^*(\mathscr{H})$ is a surjective operator, then every K-frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$ is a frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$

Theorem 4. *Let $K\in End_A^*(\mathscr{H})$ and $\{T_i\in End_{\mathscr{A}}^*(\mathscr{H},\mathscr{K})\colon i\in I\}$ be a frame for $End_A^*(\mathscr{H},\mathscr{K})$. Then, $\{T_iK^*\}_{i\in I}$ is a K-frame for $End_A^*(\mathscr{H},\mathscr{K})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$.

Then, there exist two positive constants A and B, such that

$$A\langle T,T\rangle \leq \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \leq B\langle T,T\rangle, \quad \forall T\in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}). \tag{54}$$

Hence,

$$A\langle TK,TK\rangle \leq \sum_{i\in I}\langle TK,T_i\rangle\langle T_i,TK\rangle \leq B\langle TK,TK\rangle, \quad \forall T\in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}). \tag{55}$$

Thus,

$$A\langle TK, TK\rangle \leq \sum_{i\in I}\langle T, T_iK^*\rangle\langle T_iK^*, T\rangle \leq B\|K\|^2\langle T, T\rangle, \quad \forall T \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}). \tag{56}$$

Then, $\{T_iK^*\}_{i\in I}$ is a K-frame for $\mathrm{End}_A^*(\mathcal{H}, \mathcal{K})$. □

Theorem 5. *Let $K_1, K_2 \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H})$ and $\{T_i \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}): i \in I\}$ be a K_1-frame for $\mathrm{End}_A^*(\mathcal{H}, \mathcal{K})$. Then, $\{T_iK^*\}_{i\in I}$ is a K_2K_1-frame for $\mathrm{End}_A^*(\mathcal{H}, \mathcal{K})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a K_1-frame for $\mathrm{End}_A^*(\mathcal{H}, \mathcal{K})$.

Then, there exist two positive constants A and B, such that

$$A\langle TK_1, TK_1\rangle \leq \sum_{i\in I}\langle T, T_i\rangle\langle T_i, T\rangle \leq B\langle T, T\rangle, \quad \forall T \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}). \tag{57}$$

Hence,

$$A\langle TK_2K_1, TK_2K_1\rangle \leq \sum_{i\in I}\langle TK_2, T_i\rangle\langle T_i, TK_2\rangle \leq B\langle TK_2, TK_2\rangle, \forall T \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}). \tag{58}$$

Thus,

$$A\langle TK_2K_1, TK_2K_1\rangle \leq \sum_{i\in I}\langle T, T_iK_2^*\rangle\langle T_iK_2^*, T\rangle \leq B\left\|K_2\right\|^2\langle T, T\rangle, \quad \forall T \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}). \tag{59}$$

Then, $\{T_iK^*\}_{i\in I}$ is a K_2K_1-frame for $\mathrm{End}_A^*(\mathcal{H}, \mathcal{K})$. □

Corollary 8. *Let $K \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H})$ and $\{T_i \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}): i \in I\}$ be a K-frame for $\mathrm{End}_A^*(\mathcal{H}, \mathcal{K})$. Then, $\left\{T_i(K^*)^N\right\}_{i\in I}$ is a K^{N+1}-frame for $\mathrm{End}_A^*(\mathcal{H}, \mathcal{K})$.*

Proof. It follows from the previous theorem. □

Theorem 6. *A sequence $\{T_i \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}): i \in I\}$ is a K-frame for $\mathrm{End}_A^*(\mathcal{H}, \mathcal{K})$ if and only if it is a K-g-frame for $\mathcal{H}$ with respect to $\mathcal{K}$.*

Proof. Let $\{T_i\}_{i\in I}$ be a K-g-frame for $\mathcal{H}$ with respect to $\mathcal{K}$.

Then, there exist two positive constants A and B, such that

$$\begin{aligned} &A\langle K^*x, K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_ix, T_ix\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}, \quad \forall x \in \mathcal{H}, \\ &\Longleftrightarrow A\langle KK^*x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_i^*T_ix, x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}, \quad \forall x \in \mathcal{H}. \end{aligned} \tag{60}$$

So,

$$AKK^* \leq \sum_{i\in I}T_i^*T_i \leq BI_{\mathcal{H}}. \tag{61}$$

Hence,

$$ATKK^*T^* \leq \sum_{i\in I}TT_i^*T_iT^* \leq BTT^*, \quad \forall T \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}). \tag{62}$$

Thus,

$$A\langle TK, TK\rangle \leq \sum_{i\in I}\langle T, T_i\rangle\langle T_i, T\rangle \leq B\langle T, T\rangle, \quad \forall T \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}), \tag{63}$$

i.e., $\{T_i\}_{i\in I}$ is a K-frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$.

Conversely, assume that $\{T_i\}_{i\in I}$ is a K-frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$.

Then, there exist two positive constants A and B, such that

$$
\begin{aligned}
&A\langle TK,TK\rangle \le \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \le B\langle T,T\rangle, \quad \forall T\in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}),\\
&\Longleftrightarrow ATKK^*T^* \le \sum_{i\in I} TT_i^*T_iT^* \le BTT^*, \quad \forall T\in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}).
\end{aligned} \tag{64}
$$

So,

$$
A\langle TKK^*T^*x,x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle TT_i^*T_iT^*x,x\rangle_{\mathscr{A}} \le B\langle TT^*x,x\rangle_{\mathscr{A}}, \quad \forall T\in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}), \forall x\in\mathscr{K}. \tag{65}
$$

Let $y\in\mathscr{H}$ and $T\in End_{\mathscr{A}}^*(\mathscr{H},\mathscr{K})$ such that $T^*x=y$; then,

$$
A\langle KK^*y,y\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_i^*T_iy,y\rangle_{\mathscr{A}} \le B\langle y,y\rangle_{\mathscr{A}}, \quad \forall y\in\mathscr{H}, \tag{66}
$$

i.e.,

$$
A\langle K^*y,K^*y\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_iy,T_iy\rangle_{\mathscr{A}} \le B\langle y,y\rangle_{\mathscr{A}}, \quad \forall y\in\mathscr{H}, \tag{67}
$$

and thus, $\{T_i\}_{i\in I}$ is a K-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$. □

Corollary 9. *A sequence $\{T_i\in End_A^*(\mathscr{H},\mathscr{K})\colon i\in I\}$ is a tight K-frame for $End_A^*(\mathscr{H},\mathscr{K})$ if and only if it is a tight K-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$.*

6. Generalized K-Frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$

We begin this section with the following definition.

Definition 13. Let $K\in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$. A sequence $\{T_i\in \mathrm{End}_A^*(\mathscr{H},\mathscr{K})\colon i\in I\}$ is said to be a generalized K-frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$ if there exist strictly positive elements A and B in $\mathscr{A}$ such that

$$
A\langle TK,TK\rangle \le \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \le B\langle T,T\rangle, \quad \forall T\in \mathrm{End}_A^*(\mathscr{H},\mathscr{K}), \tag{68}
$$

where the series converges in the strong operator topology.

In the following, $\mathscr{A}$ is a commutative C^*-algebra.

Theorem 7. *A sequence $\{T_i \in End_A^*(\mathscr{H},\mathscr{K}) \colon i \in I\}$ is a generalized K-frame for $End_A^*(\mathscr{H},\mathscr{K})$ if and only if it is a $*$-K-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$.*

Proof. *Let $\{T_i\}_{i\in I}$ be an $*$-K-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$.* □

Then, there exist strictly positive elements A and B in $\mathscr{A}$, such that

$$\begin{aligned} &A\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_ix, T_ix\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}, \quad \forall x \in \mathscr{H},\\ &\Longleftrightarrow A\langle KK^*x, x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_i^*T_ix, x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}, \quad \forall x \in \mathscr{H}. \end{aligned} \tag{69}$$

So,

$$AKK^* \le \sum_{i\in I} T_i^*T_i \le BI_{\mathscr{H}}. \tag{70}$$

Hence,

$$ATKK^*T^* \le \sum_{i\in I} TT_i^*T_iT^* \le BTT^*, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}). \tag{71}$$

Thus,

$$A\langle TK, TK\rangle \le \sum_{i\in I}\langle T, T_i\rangle\langle T_i, T\rangle \le B\langle T, T\rangle, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}), \tag{72}$$

i.e., $\{T_i\}_{i\in I}$ is a generalized K-frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$.

Conversely, assume that $\{T_i\}_{i\in I}$ is a generalized K-frame for $\mathrm{End}_A^*(\mathscr{H},\mathscr{K})$.

Then, there exist strictly positive elements A and B in $\mathscr{A}$, such that

$$\begin{aligned} &A\langle TK, TK\rangle \le \sum_{i\in I}\langle T, T_i\rangle\langle T_i, T\rangle \le B\langle T, T\rangle, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}),\\ &\Longleftrightarrow ATKK^*T^* \le \sum_{i\in I} TT_i^*T_iT^* \le BTT^*, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}). \end{aligned} \tag{73}$$

So,

$$A\langle TKK^*T^*x, x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle TT_i^*T_iT^*x, x\rangle_{\mathscr{A}} \le B\langle TT^*x, x\rangle_{\mathscr{A}}, \quad \forall T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K}), \forall x \in \mathscr{K}. \tag{74}$$

Let $y \in \mathscr{H}$ and $T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H},\mathscr{K})$ such that $T^*x = y$; then,

$$A\langle KK^*y, y\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_i^*T_iy, y\rangle_{\mathscr{A}} \le B\langle y, y\rangle_{\mathscr{A}}, \quad \forall y \in \mathscr{H}, \tag{75}$$

i.e.,

$$A\langle K^*y, K^*y\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_iy, T_iy\rangle_{\mathscr{A}} \le B\langle y, y\rangle_{\mathscr{A}}, \quad \forall y \in \mathscr{H}, \tag{76}$$

and thus, $\{T_i\}_{i\in I}$ is a *-K-g-frame for $\mathscr{H}$ with respect to $\mathscr{K}$.

Corollary 10. *A sequence* $\{T_i \in End_A^*(\mathscr{H}, \mathscr{K}): i \in I\}$ *is a generalized tight K-frame for* $End_A^*(\mathscr{H}, \mathscr{K})$ *if and only if it is a tight* **-K-g-frame for* $\mathscr{H}$ *with respect to* $\mathscr{K}$.

Data Availability

No data were used to support this study.

Disclosure

This manuscript is presented as preprint in arXiv (https://arxiv.org/pdf/1805.11655v2.pdf). Also, the manuscript has been submitted as a preprint in the following link: https://www.researchgate.net/publication/325464080.

Conflicts of Interest

There are no conflicts of interest.

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