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Research Article

Investigation of the Spectral Properties of a Non-Self-Adjoint Elliptic Differential Operator

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Non-self-adjoint operators have many applications, including quantum and heat equations. On the other hand, the study of these types of operators is more difficult than that of self-adjoint operators. In this paper, our aim is to study the resolvent and the spectral properties of a class of non-self-adjoint differential operators. So we consider a special non-self-adjoint elliptic differential operator (Au)(x) acting on Hilbert space and first investigate the spectral properties of space $H_1 = L^2(\Omega)^1$. Then, as the application of this new result, the resolvent of the considered operator in ℓ-dimensional space Hilbert $H_\ell = L^2(\Omega)^\ell$ is obtained utilizing some analytic techniques and diagonalizable way.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ (*i.e.*, $\partial\Omega \in \mathbb{C}^\infty$). We introduce the weighted Sobolev space $H_1 = W^2_{2,\beta}(\Omega)^\ell$ as the space of complex value functions $u(x)$ defined on Ω with the finite norm:

$$|u|_+ = \left(\sum_{i=1}^{n}\int_\Omega \rho^{2\beta}(x)\left|\frac{\partial u}{\partial x_i}\right|^2_{\mathbb{C}^\ell}dx + \int_\Omega |u(x)|^2_{\mathbb{C}^\ell}dx\right)^{1/2}. \quad (1)$$

We denote by $\mathring{H}_1$ the closure of $\mathbb{C}_0^\infty(\Omega)^\ell$ in H with respect to the above norm, i.e., $\mathring{H}$ is the closure of $\mathbb{C}_0^\infty(\Omega)$ in $H_1 = W^2_{2,\beta}(\Omega)^\ell$. The notion $\mathbb{C}_0^\infty(\Omega)$ stands for the space of infinitely differentiable functions with compact support in Ω. In this paper, we investigate the spectral properties. In particular, we estimate the resolvent of a non-self-adjoint elliptic differential operator of type

$$(Au)(x) = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(\rho^{2\beta}(x)\alpha_{ij}(x)q(x)\frac{\partial}{\partial x_i}u(x)\right) \quad (2)$$

acting on Hilbert space $H_\ell = L^2(\Omega)^\ell$ with Dirichlet-type boundary conditions. Here, $\rho(x) \in \mathbb{C}^1(0,1)$ is a positive function that satisfies the following conditions:

$$c_1 x^\alpha (1-x)^\beta \le \left|\rho^2(x)\right| \le M,$$
$$\left|\left(\rho^2\right)'(x)\right| \le M x^{(\alpha/2)-1+\varepsilon_1}(1-x)^{(\beta/2)-1+\varepsilon_2}, \quad (3)$$

where $\alpha, \beta \ge 0, \varepsilon_1 = 0$ if $\alpha \ne 1$and $\varepsilon_1 > 0$ if $\alpha = 1$,and $\varepsilon_2 = 0$if $\beta \ne 1$and $\varepsilon_2 > 0$if $\beta = 1$., $\alpha_{ij}(x) = \overline{\alpha_{ji}(x)}(i, j = 1, \dots, n)$, $\alpha_{ij}(x) \in C^2(\overline{\Omega})(i, j = 1, \dots, n)$ and the functions $\alpha_{ij}(x)$ satisfy the uniformly elliptic condition, i.e., there exists $c > 0$ such that

$$c|s|^2 \le \sum_{i,j=1}^{n}\alpha_{ij}(x), \quad s_i\overline{s_j}\left(s = (s_1, \dots, s_n)\right) \in \mathbb{C}^n, x \in \Omega. \quad (4)$$

Furthermore, suppose that $q(x) \in \mathbb{C}^2(\overline{\Omega}; \mathrm{End}\mathbb{C}^\ell)$ such that for each $x \in (\overline{\Omega})$, the matrix function $q(x)$ has nonzero simple eigenvalues $\mu_j(x) \in \mathbb{C}^2(\overline{\Omega})(1 \le j \le \ell)$ arranged in the complex plane in the following way:

$$\mu_1(x), \dots, \mu_\ell(x) \in \mathbb{C}\backslash\Phi, \quad (5)$$

where $\Phi = \{z \in \mathbb{C}: |\arg z| \le \varphi\}, \varphi \in (0, \pi)$ is a closed angle with zero vertex (i.e., the eigenvalues $\mu_j(x)$ of $q(x)$ lie on the complex plane and outside of the closed angle Φ). For a closed extension of operator A with respect to space $H =$

$W^2_{2,\beta}(\Omega)^\ell$ above, we need to extend its domain to the closed domain

$$D(A)=\left\{u\in H^0_\ell\cap W^2_{2,\mathrm{loc}}(0,1)^\ell:\ \frac{\partial}{\partial x_j}\sum_{i,j=1}^{n}\left(\rho^{2\beta}\alpha_{i,j}q\frac{\partial}{\partial x_i}u\right)\in H_\ell\right\}, \tag{6}$$

(see [1, 2]), where the local space $W^2_{2,\mathrm{loe}}(\Omega)^\ell$ is the functions $u(x)\quad x\in\Omega$ in this form $W^2_{2,\mathrm{loc}}(\Omega)=\{u(x): \sum_{i=0}^{2}\int_J|u^{(i)}(x)|^2dx<\infty, J\text{ is an open subset of }\Omega\}$. Here, and in the sequel, the value of the function $\arg z\in(-\pi,\pi]$ and $\|A\|$ denotes the norm of the bounded operator A: $H\longrightarrow H$.

To get a feeling for the history of the subject under study, refer to our earlier papers [3–5]. Indeed, this paper was written in continuing on our earlier papers. This study is sufficiently more general than our earlier papers; here, we obtain the resolvent estimate of operator A, which satisfies the special and general conditions.

2. The Resolvent Estimate of Degenerate Elliptic Differential Operators on H in Some Special Cases

Theorem 1. *Let A in (2), i.e., assume that operator A is acting on Hilbert space $H=L^2(\Omega)$ with Dirichlet-type boundary conditions, and the sector Ω be defined as in Section 1. Let the complex function q (x) satisfy the following conditions:*

$$\begin{gathered}q(x)\in\mathbb{C}^1(\overline{\Omega}),\\ q(x)\in\mathbb{C}\Phi,\ (\forall x\in\overline{\Omega}),\end{gathered} \tag{7}$$

$$\left|\arg\{q(x_1)q^{-1}(x_2)\}\right|\le\frac{\pi}{8},\quad(\forall x_1,x_2\in\overline{\Omega}). \tag{8}$$

Then, for sufficiently large modulus $\lambda\in\Phi$, the inverse operator $(A-\lambda I)^{-1}$ exists and is continuous in H, and the following estimates are valid:

$$\|(A-\lambda I)^{-1}\|\le M_\Phi|\lambda|^{-1}\ (\lambda\in\Phi,|\lambda|>\mathbb{C}_\Phi), \tag{9}$$

$$\left\|\rho^\beta\frac{\partial}{\partial_{xi}}(A-\lambda I)^{-1}\right\|\le M'_\Phi|\lambda|^{-(1/2)}\ (\lambda\in\Phi,|\lambda|>\mathbb{C}_\Phi),\quad \text{for } i=1,\ldots,n, \tag{10}$$

where $M_\Phi,\mathbb{C}_\Phi>0$ are sufficiently large numbers depending on S (Φ set is defined in the previous sections). The symbol $\|.\|$ stands for the norm of a bounded arbitrary operator T in H.

Proof. Here, to establish Theorem 1, we will first prove the assertion of Theorem 1 together with estimate (9). So, as in Section 1 for a closed extension of operator A (for more explanation, see chapter 6 in [3]), we need to extend its domain to the closed set

$$D(A)=\left\{u\in H^0_\ell\cap W^2_{2,\mathrm{loc}}(0,1)^\ell:\ \frac{\partial}{\partial x_j}\sum_{i,j=1}^{n}\left(\rho^{2\beta}\alpha_{i,j}q\frac{\partial}{\partial x_i}u\right)\in H_\ell\right\}. \tag{11}$$

Let operator A now satisfy (7), (8). Then, there exists a complex number $Z\in\mathbb{C}$ (notice that we can take $Z=e^{iy}$, for a fix real $\Upsilon\in(-\pi,\pi]$) such that $|Z=e^{i\Upsilon}|=1$, and so

$$\begin{gathered}c'\le\mathrm{Re}\{Zq(x)\},\\ c'|\lambda|\le-\mathrm{Re}\{Z\lambda\},\quad c'>0\,(\forall x\in\overline{\Omega},\lambda\in\Phi).\end{gathered} \tag{12}$$

In view of the uniformly elliptic condition, we have

$$\begin{gathered}c|s|^2=c\sum_{i=1}^{n}|s_i|^2\le\sum_{i,j=1}^{n}\alpha_{ij}(x)s_i\overline{s_j},\\ (c>0,s=(s_1,\ldots,s_n)\in\mathbb{C}^n,x\in\Omega),\end{gathered} \tag{13}$$

and take $s_i=(\partial y/\partial x_i)$ which implies that $c\sum_{i=1}^n|(\partial y/\partial x_i)(x)|^2\le\sum_{i,j=1}^n\alpha_{ij}(x)(\partial y/\partial x_i)\quad(x)(\partial y/\partial x_i)(x)$. From this, and according to $c'\le\mathrm{Re}\{Zq(x)\}$ in (10), we then multiply these two positive relations with each other, implying that

$$c_1\sum_{i=1}^{n}\left|\frac{\partial y}{\partial x_i}(x)\right|^2\le\mathrm{Re}Zq(x)\sum_{i,j=1}^{n}\alpha_{ij}(x)\frac{\partial y}{\partial x_i}(x)\overline{\frac{\partial y}{\partial x\ j}(x)}\quad\text{for } y\in D(A). \tag{14}$$

Multiplying both sides of the latter relation by the positive term $\rho^{2\alpha}(x)$ and then integrating both sides, we will have

$$\begin{aligned}&c_1\sum_{i=1}^{n}\int_\Omega\rho^{2\beta}\left|\frac{\partial y}{\partial x_i}(x)\right|^2dx\\ &\le\mathrm{Re}Zq(x)\sum_{i,j=1}^{n}\int_\Omega\rho^{2\beta}(x)\alpha_{ij}(x)q(x)\frac{\partial y}{\partial x_i}(x)\overline{\frac{\partial y}{\partial x\ j}(x)}.\end{aligned} \tag{15}$$

Now by applying the integration by parts and using Dirichlet-type condition, then the right sides of the latter relation without multiple ReZ become

$$\begin{aligned}&\sum_{i,j=1}^{n}\int_\Omega\rho^{2\beta}(x)\alpha_{ij}(x)q(x)\frac{\partial y}{\partial x_i}(x)\overline{\frac{\partial y}{\partial x\ j}(x)}dx\\ &=-\sum_{i,j=1}^{n}\int_\Omega\rho^{2\beta}(x)\alpha_{ij}(x)q(x)\frac{\partial y}{\partial x_i}(x)y(x)dx\\ &=-\frac{\partial y}{\partial x\ j}\left(\sum_{i,j=1}^{n}\rho^{2\beta}(x)\alpha_{ij}(x)q(x)\frac{\partial y}{\partial x_i}(x)y(x)\right)=(Ay,y).\end{aligned} \tag{16}$$

Hence, $(Ay)(x)=-(\partial y/\partial x_j)\sum_{i,j=1}^n\rho^{2\beta}(x)\alpha_{ij}(x)q(x)(\partial y/\partial x_i)(x)$

Here, the symbol (,) denotes the inner product in H.

Notice that the above equality in (16) is obtained by the well-known theorem of the m-sectorial operators which are closed by extending its domain to the closed domain in H. These operators are associated with the closed sectorial bilinear forms that are densely defined in H (for more explanation of the well-known Theorem 1, see chapter 6 in [2]). This is why we extend the domain of operator A to the closed domain in space H above. Therefore,

$$c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\beta}(x) \left| \frac{\partial y}{\partial x_i}(x) \right|^2 dx \le \operatorname{Re} Z(Ay, y). \tag{17}$$

From (10), we have $c'|\lambda| \le -\operatorname{Re}\{Z\lambda\}, c' > 0, \forall \lambda \in \Phi$. Multiply this inequality by $\int_\Omega |y(x)|^2 dx = (y, y) = \|y\|^2 > 0$. It follows that

$$c'|\lambda| \int_\Omega |y(x)|^2 dt \le -\operatorname{Re}\{Z\lambda\}(y, y). \tag{18}$$

From this and the above inequality, we will have

$$\begin{aligned} c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\beta}(x) \left| \frac{\partial y}{\partial x_i}(x) \right|^2 dx + c'|\lambda| \int_\Omega |y(x)|^2 dx \\ \le \operatorname{Re} Z(Ay, y) - Z\lambda(y, y) \\ = \operatorname{Re}\{Z((A - \lambda I)y, y)\} \\ \le \|Z\| \|y\| \|(A - \lambda I)y\| \\ = \|y\| \|(A - \lambda I)y\|', \end{aligned} \tag{19}$$

i.e.,

$$c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\beta}(x) \left| \frac{\partial y}{\partial x_i}(x) \right|^2 dx + c'|\lambda| \int_\Omega |y(x)|^2 dx \le \|y\| \|(A - \lambda I)y\|. \tag{20}$$

Since $c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\beta}(x) |(\partial y/\partial x_i)(x)|^2 dx$ is positive, we will have either

$$c'|\lambda| \|y(x)\|^2 = |\lambda| \int_\Omega \left| \frac{\partial y}{\partial x_i}(x) \right|^2 dx \le \|y\| \|(A - \lambda I)y\|, \tag{21}$$

or

$$|\lambda| \|y(x)\| \le M_\Phi \|(A - \lambda I)y\|. \tag{22}$$

This inequality ensures that the operator $(A - \lambda I)$ is one to one, which implies that ker $(A - \lambda I) = 0$. Therefore, the inverse operator $(A - \lambda I)^{-1}$ exists, and its continuity follows from the proof of estimate (9) of Theorem 1. To prove (9), we set $v = (A - \lambda I)^{-1} f, f \in H$ in (19), implying that

$$\begin{aligned} |\lambda| \int_\Omega \left| (A - \lambda I)^{-1} f \right|^2 dx \\ \le M_\Phi \left\| (A - \lambda I)^{-1} f \right\| \left\| (A - \lambda I)(A - \lambda I)^{-1} f \right\|. \end{aligned} \tag{23}$$

Since $(A - \lambda I)(A - \lambda I)^{-1} f = I(f) = f$, then

$$|\lambda| \int_\Omega \left| (A - \lambda I)^{-1} f \right|^2 dx \le M_\Phi \left\| (A - \lambda I)^{-1} f \right\| |f|, \tag{24}$$

so

$$|\lambda| \left\| (A - \lambda I)^{-1} f \right\| \le M_\Phi \left\| (A - \lambda I)^{-1} f \right\| |f|, \tag{25}$$

which implies that $|\lambda| \| (A - \lambda I)^{-1} (f) \| \le M_\Phi |f|$. Since $\lambda \ne 0$, then $\| (A - \lambda I)^{-1} f \| \le M_\Phi |\lambda|^{-1} |f|$; i.e., $\| (A - \lambda I)^{-1} \| \le M_\Phi |\lambda|^{-1}$. This estimate completes the proof of the assertion of Theorem 1 together with estimate (9). Now, we start to prove estimate (10) of Theorem 1. As in the above argument, we drop the positive term $c'|\lambda| \int_\Omega |y(x)|^2$ dx from

$$c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\beta}(x) \left| \frac{\partial y}{\partial x_i}(x) \right|^2 dx + c'|\lambda| \int_\Omega |y(x)|^2 dx \le \|y\| \|(A - \lambda I)y\|. \tag{26}$$

It follows that

$$c_1 \sum_{i=1}^{n} \int_\Omega \rho^{2\beta}(x) \left| \frac{\partial y}{\partial x_i}(x) \right|^2 dx \le \|y\| \|(A - \lambda I)y\|. \tag{27}$$

Equivalently

$$c_1 \left\| \rho^\beta \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \|y\| \|(A - \lambda I)y\|. \tag{28}$$

Set $(A - \lambda I)^{-1} f, f \in H$in the latter relation, and proceeding by similar calculation as in the proof of estimate (9), we then obtain

$$c_1 \left\| \rho^\beta \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \le \left\| (A - \lambda I)^{-1} f \right\| \left\| (A - \lambda I)(A - \lambda I)^{-1} f \right\|. \tag{29}$$

Since $(A - \lambda I)(A - \lambda I)^{-1} f = I(f) = f$, then

$$c_1 \left\| \rho^\beta \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \le \left\| (A - \lambda I)^{-1} \right\| \|f\|^2. \tag{30}$$

Consequently, by (9), this implies that

$$c_1 \left\| \rho^\beta \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \le M'_\Phi \|\lambda\|^{-1} \|\lambda\|^{-1} \|f\|^2. \tag{31}$$

To this end, we will have

$$\left\| \rho^\beta \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} \right\| \le M'_\Phi \|\lambda\|^{-(1/2)}. \tag{32}$$

Thus, here, the proof of estimate (10) is finished; i.e., this completes the proof of Theorem 1.

Now let condition (8) not hold. Then we will have the following statement. □

3. The Resolvent Estimate of Some Classes of Degenerate Elliptic Differential Operators on H

In this section, we will derive a new general theorem by dropping the assumption (8) from Theorem 1 in Section 2.

Theorem 2. *As in Section 1, let Φ be some closed sector with vertex at 0 in the complex plane (for more explanation, see*

[3]), and let the complex function q (x) satisfy the following equations:

$$q(x) \in \mathbb{C}^1(\overline{\Omega}),\quad q(x) \in \mathbb{C}\Phi; \quad (\forall x \in \overline{\Omega}). \tag{33}$$

Then, for sufficiently large modulus $\lambda \in \Phi$, the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in H, and the following estimates hold:

$$\left\| (A - \lambda I)^{-1} f \right\| \le M'_\Phi \|\lambda\|^{-1}, \quad (\lambda \in \Phi, |\lambda| > \mathbb{C}_\Phi), \tag{34}$$

where $M'_\Phi, \mathbb{C}_\Phi > 0$ are sufficiently large numbers depending on Φ.

Proof. Let us (9) not satisfy. To prove the assertion of Theorem 2 together with (34), we construct the functions $\varphi_1(x), \ldots, \varphi_m(x), q_1(x), \ldots, q_m(x)$ so that each one of the functions$q_1(x), \ldots, q_m(x) (x \in \overline{\Omega})$ as the function $q\ (x)$ in Theorem 1 satisfies (8).

Therefore, let

$$\varphi_1(x), \ldots, \varphi_m(x), q_1(x), \ldots, q_m(x) \in C_0^\infty(\Omega) \tag{35}$$

satisfy

$$\begin{gathered}
0 \le \varphi_r(x), \quad r = 1, \ldots, m,\\
\varphi_1^2(x) + \cdots + \varphi_m^2(x) \equiv 1, \quad (x \in \overline{\Omega})\\
\frac{\mathrm{d}}{\mathrm{d}t}\varphi_r(x) \in \mathbb{C}_0^\infty(\Omega), q_r(x) = q(x), \quad \forall x \in \operatorname{supp}\varphi_r,\\
q_r(x) \in \mathbb{C}\backslash\Phi, \quad (\forall x \in \overline{\Omega}), r = 1, \ldots, m,\\
\left|\arg\{q_r(x_1) q_r^{-1}(x_2)\}\right| \le \frac{\pi}{8}, \quad (\forall x_1, x_2 \in \operatorname{supp}\varphi_r), r = 1, \ldots, m.
\end{gathered} \tag{36}$$

In view of Theorem 1 and by (9) and (10), set $A_r = A$ in the definition of the differential operator, which implies that

$$A_r u(x) = -\sum_{i,j=1}^{n} \rho^{2\beta}(x) \alpha_{ij}(x) q(x) u'_{xi}(x)'_{xj} \tag{37}$$

is acting on H where

$$D(A_r) = \left\{ u \in \overset{o}{H} \cap W^2_{2,\mathrm{loc}}(\Omega); \frac{\partial u}{\partial x_j} \sum_{i,j=1}^{n} \left(\rho^{2\beta} \alpha_{ij} q_r \frac{\partial u}{\partial x_i} \right) \in H \right\}. \tag{38}$$

Due to the assertion of Theorem 1, for $0 \ne \lambda \in \Phi$, the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in space $H = L^2(\Omega)$ and satisfies

$$\begin{gathered}
\left\| (A - \lambda I)^{-1} \right\| \le M_\Phi \|\lambda\|^{-1},\\
\left\| \rho^\beta \frac{\partial}{\partial_{xi}} (A - \lambda I)^{-1} \right\| \le M'_\Phi |\lambda|^{-(1/2)}, \quad (\lambda \in \Phi, |\lambda| > \mathbb{C}_\Phi), (0 \ne \lambda \in \Phi).
\end{gathered} \tag{39}$$

Let us introduce

$$G(\lambda) = \sum_{r=1}^{m} \varphi_r (A_r - \lambda I)^{-1} \varphi_r. \tag{40}$$

Here φ_r is the multiplication operator in H by the function $\varphi_r(x)$. Consequently, it is easily verified that

$$\begin{aligned}
(A_r - \lambda I) G(\lambda) = I &+ \rho^{2\beta-1}(x) \sum_{r=1}^{m} \eta_r(x) (A_r - \lambda I)^{-1} \varphi_r\\
&+ \rho^\beta(x) \sum_{i=1}^{n} \sum_{r=1}^{m} \Upsilon i_r(x) \frac{\partial}{\partial_{xi}} (A_r - \lambda I)^{-1} \varphi_r,
\end{aligned} \tag{41}$$

where $\eta_r, \Upsilon i_r \in L_\infty(\Omega)$; supp β_r and supp Υi_r are contained in supp φ_r. Let us take the right side of (41) equal to $\mathrm{I} + T(\lambda)$. Thus, we will have

$$(A - \lambda I) G(\lambda) = I + T(\lambda). \tag{42}$$

Now according to Section 2, if we put $A = \mathrm{Ar}$ for $r = 1, \ldots, m$ in (8), we will have

$$\begin{gathered}
\left\| (A_r - \lambda I)^{-1} \right\| \le M1_s \|\lambda\|^{-1},\\
\left\| \rho^\beta \frac{\partial}{\partial_{xi}} (A_r - \lambda I)^{-1} \right\| \le M'_\Phi |\lambda|^{-(1/2)}.
\end{gathered} \tag{43}$$

Owing to the definition of T (λ) in (41) easily, it follows that

$$\|T(\lambda)\| \le M'_\Phi |\lambda|^{-(1/2)}, \quad (\lambda \in \Phi, |\lambda| > 1). \tag{44}$$

Since $|\lambda|$ is a sufficiently large number, it easily implies that $\|T(\lambda)\| < (1/2) < 1$. From this and using the well-known theorem in the operator theory, we conclude that $\mathrm{I} + T(\lambda)$ and so $(\mathrm{A} - \lambda \mathrm{I})\, G(\lambda)$ are invertible. Hence, $((\mathrm{A} - \lambda \mathrm{I})\, \mathrm{G}(\lambda))^{-1}$ exists and is equal to

$$\left(G(\lambda)^{-1} (A - \lambda I)^{-1} \right) = (I + T(\lambda))^{-1}. \tag{45}$$

By adding +I and –I to the right side of (44), it follows that

$$\left(G(\lambda)^{-1} (A - \lambda I)^{-1} \right) = (I + T(\lambda))^{-1} - I + I. \tag{46}$$

We now set

$$F(\lambda) = (I + T(\lambda))^{-1} - I. \tag{47}$$

Then

$$\left(G(\lambda)^{-1} (A - \lambda I)^{-1} \right) = I + F(\lambda). \tag{48}$$

In view of kT $(\lambda) < 1$ and (44), we now estimate $F(\lambda)$ by the following geometric series:

$$\begin{aligned}
\|F(\lambda)\| &\le \sum_{i=2}^{+\infty} \left\| T^k(\lambda) \right\| \le \|T(\lambda)\|^2 \left(1 + \|T(\lambda)\| + \|T(\lambda)\|^2 + \cdots \right)\\
&\le \|T(\lambda)\|^2 M_\Phi \left(1 + \frac{1}{2} + \cdots \right) \le 2 M_\Phi \left(M'_\Phi |\lambda|^{-(1/2)} \right)^2,
\end{aligned} \tag{49}$$

i.e., $\|F(\lambda)\| \le 2M1_{\Phi}|\lambda|^{-1}$.By $\|(A_r - \lambda I)^{-1}\| \le M1_{\Phi}|\lambda|^{-1}$, for we will have

$$\|G(\lambda)\| = \left\| \sum_{r=1}^{m} \varphi_r (A_r - \lambda I)^{-1} \varphi_r \right\| \le M_{\Phi}'' \left\| (A_r - \lambda I)^{-1} \right\| \le M_{\Phi}'' M_{1\Phi} |\lambda|^{-1}, \tag{50}$$

i.e., $\|G(\lambda)\| \le M_{2\Phi}|\lambda|^{-1}$. Now from (45), we have

$$(A - \lambda I)^{-1} = G(\lambda)(I + T(\lambda))^{-1} = G(\lambda)(I + F(\lambda)). \tag{51}$$

Therefore

$$\left\| (A - \lambda I)^{-1} \right\| = \|G(\lambda)\| \|I + F(\lambda)\| \le M_{2\Phi}|\lambda|^{-1} \left\| \left(1 + 2M_{1\Phi}|\lambda|^{-1}\right) \right\|, \tag{52}$$

i.e., here the assertion of Theorem 2 is proved. Therefore, to complete the proof Theorem 2, we must prove the estimate (34). To the end, according to the latter inequality, we have

$$\left\| (A - \lambda I)^{-1} \right\| \le M_{2\Phi}|\lambda|^{-1} + 2M_{2\Phi}M_{1\Phi}|\lambda|^{-1}|\lambda|^{-1}, \tag{53}$$

and since $|\lambda|^{-1}|\lambda|^{-1} = |\lambda|^{-2} \le |\lambda|^{-1}$, it follows that

$$\left\| (A - \lambda I)^{-1} \right\| \le M_{\Phi}|\lambda|^{-1}, \quad (|\lambda| \ge \mathbb{C}, \lambda \in \Phi). \tag{54}$$

This completes the proof of Theorem 2. □

4. On the Resolvent Estimate of the Differential Operator in H_ℓ

As in Section 1, let the differential operator

$$(Au)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\rho^{2\beta}(x) \alpha_{ij}(x) q(x) \frac{\partial u}{\partial x_i}(x) \right), \tag{55}$$

act on Hilbert space $H_\ell = L^2_\ell(\Omega)$ with Dirichlet-type boundary conditions, and suppose that $q(x) \in \mathbb{C}^2(\overline{\Omega}, \mathrm{End}\mathbb{C}^\ell)$ such that for each $x \in \overline{\Omega}$, the matrix function q (x) has nonzero simple eigenvalues $\mu_j(x) \in \mathbb{C}^2(\overline{\Omega})$, $(1 \le j \le \ell)$ arranged in the complex plane in the following way:

$$\mu_1(x), \ldots, \mu_\ell(x) \in \mathbb{C} \backslash \Phi, \tag{56}$$

where

$$\Phi = \{z \in \mathbb{C}: |\arg z| \le \varphi\}, \quad \varphi \in (0, \pi). \tag{57}$$

Furthermore, suppose that for $j = 1, \ldots, \ell$, we have

$$\mu_j(x) \in \mathbb{C}^1(\overline{\Omega}), \qquad \mu_j(x) \in \mathbb{C} \backslash \Phi, \quad (\forall x \in \overline{\Omega}), \tag{58}$$

$$\left| \arg\{\mu_j(x_1)\mu_j^{-1}(x_2)\} \right| \le \frac{\pi}{8}, \quad (\forall x_1, x_2 \in \overline{\Omega}). \tag{59}$$

Now, according to Theorem 1, but here instead of operator A which acts on the space $H = L^2(\Omega)$, let operator A act on the space $H_\ell = L^2(\Omega)^\ell$. Now by the assumption of Section 1, we will have the following theorem in the general case.

Theorem 3. *Let (58) and (59) and the assumptions of Section 1 hold for operator A as in (2), then for sufficiently large modulus $\lambda \in \Phi$, the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in the space $H_\ell = L^2(\Omega)^\ell$ and the following estimate holds:*

$$\left\| (A - \lambda I)^{-1} \right\| \le M_{\Phi}|\lambda|^{-1}, \qquad \left\| \rho \frac{\mathrm{d}}{\mathrm{d}x} (A - \lambda I)^{-1} \right\| \le M_2' |\lambda|^{-(1/2)}, \quad (\lambda \in \Phi, |\lambda| \ge \mathbb{C}), \tag{60}$$

where $M_\Phi, C_\Phi > 0$ are sufficiently large numbers depending on Φ and $|\lambda| > C_\Phi$.

Proof. Now by applying the eigenvalues $\mu_1(x), \ldots, \mu_\ell(x)$ of the matrix function $q(x)$, we define the operators $A_1, \ldots, A_\ell$ such that

$$\left(A_j u\right)(x) = -\frac{\partial u}{\partial x\ j} \sum_{i,j=1}^{n} \left(\rho^{2\beta}(x) \alpha_{ij}(x) \mu_j(x) \frac{\partial u}{\partial x_i}(x) \right), \quad (j = 1, \ldots, \ell), \tag{61}$$

where its extension domains are

$$D\left(A_j\right) = \left\{ y \in \overset{o}{H} \cap W^2_{2,\mathrm{loc}}(\Omega); \frac{\partial y}{\partial x\ j} \sum_{i,j=1}^{n} \left(\rho^{2\beta} \alpha_{ij} \mu_j \frac{\partial y}{\partial x_i} \right) \in H \right\}, \tag{62}$$

which, as operator A in Theorem 1, the operators A_j, $j = 1, \ldots, \ell$, acts on space $H = L^2(\Omega)$ (notice that here the operators A_j are the same operator A in Section 2, i.e., to define the operators A_j, we just change the function $q(x)$ in operator A by the eigenvalues functions $\mu_j(x)$, $j = 1, \ldots, \ell$of matrix $q(x)$). The conditions which we consider on the eigenvalues μ_j (x) of the matrix function $q(x)$ in Section 1 guarantee that one can convert the matrix $q(x)$ to the diagonal form $q(x) = U(x)\Lambda(x)U^{-1}(x)$, where $U(x)$, $U^{-1}(x) \in \mathbb{C}^2([0,1]; \mathrm{End}\mathbb{C}^\ell)$ and $\Lambda(x) = \mathrm{diag}\{\mu_1(x), \ldots, \mu_\ell(x)\}$. Consider space $H_\ell = H \oplus \cdots H \oplus$ (ℓ times). Put $\Gamma(\lambda) = UB(\lambda)U^{-1}$ where the operator

$$B(\lambda) = \mathrm{diag}\left\{ (A_1 - \lambda I)^{-1}, \ldots, (A_\ell - \lambda I)^{-1} \right\} \tag{63}$$

acts on the direct sum $H_\ell = H \oplus \cdots H \oplus$ (ℓ times) in which $\lambda \in \overline{\Phi} \backslash R_+, |\lambda| \ge C_0$ and $(Uu)(x) = U(x)\ u(x)$; $(u \in H_\ell)$. Consequently, it follows that

$$(A - \lambda I)\Gamma(\lambda)u = -\frac{\mathrm{d}}{\mathrm{d}x} \left(\rho^2 A(x) \frac{\mathrm{d}}{\mathrm{d}x} U(x) B(\lambda) U^{-1}(x) u(x) \right) = T_1 + T_2 + T_3, \tag{64}$$

where

$$\begin{aligned} T_1 &= -\frac{d}{dx}\left(\rho^2 A(x)U(x)\frac{d}{dx}B(\lambda)U^{-1}(x)u(x)\right)\\ &= -\frac{d}{dx}\left(\rho^2 U(t)\Lambda(x)\frac{d}{dx}B(\lambda)U^{-1}(x)u(x)\right)\\ &= -U\frac{d}{dx}\left(\rho^2\Lambda(x)\frac{d}{dx}B(\lambda)U^{-1}(x)u(x)\right) - U'(x)\rho^2\Lambda\frac{d}{dx}B(\lambda)U^{-1}u\\ &= \lambda UB(\lambda)U^{-1}u - U'(x)\rho^2\Lambda\frac{d}{dx}B(\lambda)U^{-1}u + UU^{-1}u,\\ T_2 &= -\frac{d}{dx}\left(\rho^2 qU'B(\lambda)U^{-1}u\right),\\ T_3 &= -\lambda U(x)B(\lambda)U^{-1}u. \end{aligned} \tag{65}$$

Using (9) and (10), we have $(A-\lambda I)T(\lambda) = I + T_1^0 + T_2^0$ where $T_2^0 = (\rho^2)' qU'B(\lambda)U^{-1}$ and $\|T_1^0\| \le M|\lambda|^{-(1/2)}$. Now by the Hardy-type inequality, we estimate the operator T_2^0 as follows:

$$\begin{aligned} &\int_0^1 t^{-1+\varepsilon_1'}(1-t)^{-1+\varepsilon_2'}|y(t)|^2 dt\\ &\quad\le M(\varepsilon_1',\varepsilon_2')\int_0^1 |y(t)|^2 dt + M(\varepsilon_1',\varepsilon_2')\\ &\qquad\cdot\int_0^1 t^{1+\varepsilon_1'}(1-t)^{1+\varepsilon_2'}|y'(t)|^2 dt,\quad \forall y\in \overset{0}{\mathrm{H}}, \varepsilon_1',\varepsilon_2'\neq 0. \end{aligned} \tag{66}$$

Since $|q(t)U'(t)| \le M$ by (3), we have the following inequality:

$$\begin{aligned} &\int_0^1 \left|(\rho^2(t))'\right|^2 |(B(\lambda)u(t))|_{\mathbb{C}^\ell}^2 dt\\ &\quad\le M_2\int_0^1 t^{\alpha-2+2\varepsilon_1'}(1-t)^{\beta-2+2\varepsilon_2'}|(B(\lambda)u(t))|_{\mathbb{C}^\ell}^2 dt\\ &\quad\le M_3\int_0^1 \left\|t^{\alpha}(1-t)^{\beta}\rho^{-2}(t)\right\|\left|\rho^2((B(\lambda)u)(t))_t'\right|_{\mathbb{C}^\ell}^2 dt\\ &\qquad+ M|B(\lambda)u|_{H_\ell}^2. \end{aligned} \tag{67}$$

Now by (3) and estimate (9), it follows

$$\begin{aligned} &\int_0^1 \left|(\rho^2(t))'\right|^2 |(B(\lambda)u(t))|_{\mathbb{C}^\ell}^2 dt\\ &\quad\le M\int_0^1 \rho^2|(B(\lambda)u(t))|_{\mathbb{C}^\ell}^2 dt + M|B(\lambda)u|_{H_\ell}^2\\ &\quad\le M'|\lambda|^{-1}|u|_{H_\ell}^2,\quad (\lambda\in\Phi, |\lambda|>\mathbb{C}). \end{aligned} \tag{68}$$

Then, $\|T_2^0\| \le M'|\lambda|^{-(1/2)}$ for sufficiently large in modulus of $\lambda\in\Phi$; consequently,

$$\begin{aligned} &(A-\lambda I)\Gamma(\lambda) = I + F(\lambda),\\ &\qquad\|F(\lambda)\| \le M|\lambda|^{-(1/2)},\quad (\lambda\in\Phi, |\lambda|>\mathbb{C}). \end{aligned} \tag{69}$$

Proceeding as at the end of Section 2 (e.g., see (43)) from $\|F(\lambda)\| \le M|\lambda|^{-(1/2)}$, it easily follows that $I+F(\lambda)$ is inversible and then that $(A-\lambda I)\Gamma(\lambda)$ is inversible, that is,

$$((A-\lambda I)\Gamma(\lambda))^{-1} = (I+F(\lambda))^{-1}. \tag{70}$$

Then by adding $\pm I$, the last relation we have is

$$(I+F(\lambda))^{-1} = (I+F(\lambda))^{-1} + I - I. \tag{71}$$

Since $\|F(\lambda)\| \le M|\lambda|^{-(1/2)}$, in a calculation as in Section 2, take $y(\lambda) = (I+F(\lambda))^{-1} - I$. Then, $y(\lambda)$satisfies

$$\|y(\lambda)\| \le M|\lambda|^{-1},\quad (\lambda\in\Phi, |\lambda|>\mathbb{C}). \tag{72}$$

Consequently, $(A-\lambda I)^{-1} = \Gamma(\lambda)(I+y(\lambda))$ since

$$\begin{aligned} &\Gamma(\lambda) = UB(\lambda)U^{-1},\\ &B(\lambda) = \operatorname{diag}\left\{(A_1-\lambda I)^{-1},\ldots,(A_\ell-\lambda I)^{-1}\right\}. \end{aligned} \tag{73}$$

Put $P_j = A_j$, $j=1,\ldots,\ell$ as in (39). By (72) and (73), we have $\|(A_j-\lambda I)^{1}\| \le M|\lambda|^{-1}$, $j=1,\ldots,\ell$ and it follows that $\|\Gamma(\lambda)\| \le |\lambda|^{-1}$, so

$$\begin{aligned} &\left\|(A-\lambda I)^{-1}\right\| \le \|\Gamma(\lambda)\|\|I+y(\lambda)\|\\ &\qquad\le M|\lambda|^{-1}\left(1+M|\lambda|^{-1}\right) \le M|\lambda|^{-1}. \end{aligned} \tag{74}$$

Now we prove estimate (39). Since $\|\rho(d/dt)(A_j-\lambda I)^{-1}\| \le M|\lambda|^{-1}$, $j=1,\ldots,\ell$for $\Gamma_1(\lambda)$, we can get the corresponding estimate $\|\Gamma_1(\lambda)\| \le M_1|\lambda|^{-(1/2)}$, and this implies

$$\left\|\rho\frac{d}{dx}(A-\lambda I)^{-1}\right\| \le \left\|\Gamma_1(\lambda)\right\|\left\|(I+y_1(\lambda))\right\|. \tag{75}$$

Since $\|y_1(\lambda)\| \le M_1'|\lambda|^{-1}$, we have

$$\left\|\rho\frac{d}{dx}\left(A_j-\lambda I\right)^{-1}\right\|\left\|\rho\frac{d}{dx}\left(P_j-\lambda I\right)^{-1}\right\| \le M|\lambda|^{-(1/2)}\left(1+M_1'|\lambda|^{-1}\right), \tag{76}$$

which implies $\|\rho(d/dx)(A_j - \lambda I)^{-1}\| \le M|\lambda|^{-(1/2)}$ $(\lambda \in \Phi, |\lambda| \ge \mathbb{C})$ so that the proof of the fundamental Theorem 3 in the general case $H_\ell = L^2(\Omega)^\ell$ is completed. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally to this work. All authors read and approved the final manuscript.

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