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Research Article

Fuzzy Prime Ideal Theorem in Residuated Lattices

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This paper mainly focuses on building the fuzzy prime ideal theorem of residuated lattices. Firstly, we introduce the notion of fuzzy ideal generated by a fuzzy subset of a residuated lattice and we give a characterization. Also, we introduce different types of fuzzy prime ideals and establish existing relationships between them. We prove that any fuzzy maximal ideal is a fuzzy prime ideal in residuated lattice. Finally, we give and prove the fuzzy prime ideal theorem in residuated lattice.

1. Introduction

Nonclassical logic is closely related to logic algebraic systems. Many researches have motivated to develop nonclassical logic and also enrich the content of algebra [1, 2]. In modern fuzzy logic theory, residuated lattices and some related algebraic systems play an extremely important role because they provide algebraic frameworks to fuzzy logic and fuzzy reasoning.

Ward and Dilworth [3] initiated the notion of residuated lattice and it interested other authors [4–8]. The notion of ideal has been introduced in several algebraic structures such as BL-algebras [9] and residuated lattice [5, 8]. Piciu [10] gives and proves the prime ideal theorem in residuated lattices. Dealing with certain and uncertain information is an important task of the artificial intelligence, in order to make computer simulate human being. To handle such information, Zadeh [11] introduced the notion of fuzzy subset of a nonempty set A as a function μ: $A \longrightarrow I$, where $I = [0; 1]$ is the unit interval of real numbers. Since then, a lot of works have been done on fuzzy mathematical structures and most authors used the above original definition of a fuzzy set. The notion of fuzzy ideal has been studied in several structures such as rings [12], lattices [13, 14], MV-algebras [15], BL-algebras [16], and residuated lattices [6, 8, 17]. However, recent work of Piciu [10] gives the prime ideal theorem in residuated lattices. But that theorem is not yet investigated in fuzzy logic. In this work, we give and demonstrate the fuzzy prime ideal theorem in residuated lattice.

The remainder of this paper is organized as follows: Section 2 is a review on residuated lattices and ideals, whereas Section 3 contains the characterization of a fuzzy ideal generated by a fuzzy subset in residuated lattice. In Section 4, we study the different types of fuzzy prime ideals in residuated lattice, and we give some relations between them. We also define the notion of fuzzy maximal ideal and we give the fuzzy prime ideal theorem of residuated lattice.

2. Review on Residuated Lattices and Ideals

Definition 1 (see [3, 17]). A residuated lattice is an algebraic structure $\mathscr{A} = (A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following axioms:

(R1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice

(R2) $(A, \odot, 1)$ is a commutative monoid

(R3) For all $a, b, c \in A$, $c \leq a \longrightarrow b$ if and only if $a \odot c \leq b$

Let us give the following notations in a residuated lattice $\mathscr{A}$:

(i) $\forall a \in A$, $a' := a \longrightarrow 0$ and $(a')' = a''$

(ii) $\forall a, b \in A$, $a \oslash b := a' \longrightarrow b$

Definition 2 (see [1, 18]). A residuated lattice $\mathscr{A} = (A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ is an MTL-algebra if it satisfies the prelinearity condition, that is, $(a \longrightarrow b) \vee (b \longrightarrow a) = 1$, for all $a, b \in A$.

Definition 3 (see [19]). A De Morgan residuated lattice is a residuated lattice $\mathscr{A} = (A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ such that, for all $a, b \in A$, $(a \wedge b)' = a' \vee b'$.

Example 1. Let A be a lattice defined by the Hass diagram of Figure 1.

Define $\longrightarrow$ and $\odot$ as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$(A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ is an MTL-algebra.

Example 2. Let A be a lattice defined by the Hass diagram of Figure 2.

Define $\longrightarrow$ and $\odot$ as follows:

$\rightarrow$	0	a	c	d	m	1
0	1	1	1	1	1	1
a	d	1	d	d	1	1
c	a	a	1	1	1	1
d	a	a	m	1	1	1
m	0	a	d	d	1	1
1	0	a	c	d	m	1

$\odot$	0	a	c	d	m	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
c	0	0	c	c	c	c
d	0	0	c	c	c	d
m	0	a	c	c	m	m
1	0	a	c	d	m	1

$(A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ is not a De Morgan residuated lattice. We have $(a \wedge d)' = 0' = 1$ and $a' \vee d' = d \vee a = m$; then $(a \wedge d)' \neq a' \vee d'$.

Any MTL-algebra is a De Morgan residuated lattice but the converse is not always true.

Example 3. Let A be a lattice defined by the Hass diagram of Figure 3.

Define $\longrightarrow$ and $\odot$ as follows:

$\rightarrow$	0	n	a	b	c	d	m	1
0	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1
a	d	d	1	d	1	d	1	1
b	c	c	c	1	1	1	1	1
c	b	b	c	d	1	d	1	1
d	a	a	a	c	c	1	1	1
m	n	n	a	b	c	d	1	1
1	0	n	a	b	c	d	m	1

$\odot$	0	n	a	b	c	d	m	1
0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	a
b	0	0	0	0	0	b	b	b
c	0	0	a	0	a	b	c	c
d	0	0	0	b	b	d	d	d
m	0	0	a	b	c	d	m	m
1	0	n	a	b	c	d	m	1

$(A, \leq, \odot, \longrightarrow, 0, 1)$ is a De Morgan residuated lattice. Since $(a \longrightarrow b) \vee (b \longrightarrow a) = f \vee e = m \neq 1$, it follows that $\mathscr{A}$ is not an MTL-algebra.

Theorem 1 (see [8]). *For any residuated lattice $\mathscr{A} = (A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$, the following properties hold for every $x, y, z \in A$:*

(P1) $(x \odot y) \longrightarrow z = x \longrightarrow (y \longrightarrow z)$

(P2) If $x \leq y$, *then* $y' \leq x'$

(P3) $y \longrightarrow z \leq (x \longrightarrow y) \longrightarrow (x \longrightarrow z)$

(P4) $(x \odot y)' = x \longrightarrow y'$

(P5) $1 \longrightarrow x = x$, $x \longrightarrow x = 1$

(P6) $x \longrightarrow (y \longrightarrow z) = y \longrightarrow (x \longrightarrow z)$;

(P7) $0' = 1, 1' = 0, x' = x''', x \leq x''$

(P8) $y \longrightarrow x \leq (x \longrightarrow z) \longrightarrow (y \longrightarrow z)$

(P9) $x \longrightarrow (y \wedge z) = (x \longrightarrow y) \wedge (x \longrightarrow z)$

(P10) $(x \vee y) \longrightarrow z = (x \longrightarrow z) \wedge (y \longrightarrow z)$

Definition 4 (see [6]). Let $\mathscr{A} = (A, \leq)$ be a lattice and let F be a nonempty subset of A. We say that F is a filter of $\mathscr{A}$, if it satisfies the following conditions:

(F1) For every $a, b \in F$, $a \wedge b \in F$

(F2) For every $a, b \in A$, if $a \leq b$ and $a \in F$, $b \in F$

Definition 5 (see [5, 8, 20]). Let $\mathscr{A} = (A, \wedge, \vee, \otimes, \longrightarrow, 0, 1)$ be a residuated lattice and let I be a nonempty subset of A. We say that I is an ideal of $\mathscr{A}$ if it satisfies the following conditions:

(I1) For every $a, b \in I$, $a \oslash b \in I$

(I2) For every $a, b \in A$, if $a \leq b$ and $b \in I$, then $a \in I$

We denote by $\mathscr{I}(A)$ the set of all ideals of A. An ideal I is called proper if $I \neq A$. If $I \in \mathscr{I}(A)$, then $0 \in I$ and $x \in I$ if $x'' \in I$.

There are two types of prime ideal in any residuated lattice.

Definition 6 (see [10, 20]). Let I be a proper ideal of a residuated lattice $\mathscr{A}$. I is said to be a prime ideal, if, for all $I_1, I_2 \in \mathscr{I}(A)$, if $I_1 \cap I_2 \subseteq I$, then $I_1 \subseteq I$ and $I_2 \subseteq I$.

Proposition 1 (see [10, 20]). *Let P be a subset of residuated lattice $\mathscr{A}$. P is a prime ideal if and only if, for all $a, b \in A$, if $a'' \wedge b'' \in P$, then $a \in P$ or $b \in P$.*

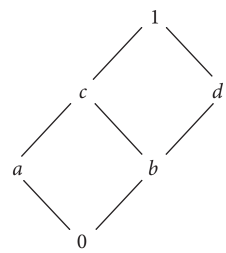

Figure 1: Hass diagram.

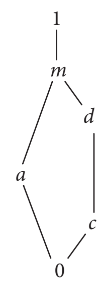

Figure 2: Example of lattice 2.

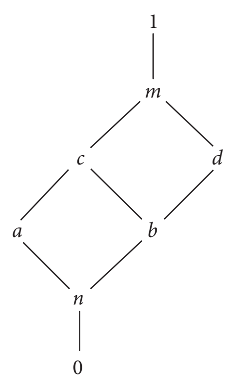

Figure 3: Hass diagram.

Remark 1 (see [19]). If $\mathscr{A}$ is a De Morgan residuated lattice, then $I \in \mathscr{I}(A)$ is a prime ideal if, for all $a, b \in A$, $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

Definition 7 (see [20]). Let I be a proper ideal of a residuated lattice $\mathscr{A}$. I is said to be a prime ideal of second kind, if and only if, for any $a, b \in A$, $(a \longrightarrow b)' \in I$ or $(b \longrightarrow a)' \in I$.

The next theorem gives the relation between these two types of prime ideal.

Theorem 2 (see [8]). *Let $\mathscr{A}$ be a residuated lattice. Every prime ideal of second kind of $\mathscr{A}$ is also a prime ideal. If $\mathscr{A}$ is an MTL-algebra, then prime ideal of $\mathscr{A}$ and prime ideal of second kind of $\mathscr{A}$ are equivalent.*

Let us recall the notion of maximal ideal.

Definition 8 (see [20]). Let I be a proper ideal of a residuated lattice $\mathscr{A}$. I is said to be a maximal ideal, if, for any ideal J of $\mathscr{A}$, $I \subseteq J$ implies that $J = I$ or $J = A$.

Proposition 2 (see [10]). *Let I be an ideal of A. If I is a maximal ideal of A, then it is a prime ideal of A.*

Theorem 3 (see [10]). *Let A be a residuated lattice. If I is an ideal of A and J is a filter of the lattice $(A, \wedge, \vee, 0, 1)$ such that $I \cap J = \varnothing$, then there is a prime ideal P of A such that $I \subseteq P$ and $P \cap J = \varnothing$.*

3. Fuzzy Ideal Generated by a Fuzzy Subset of Residuated Lattice

Let $\mathscr{A} = (A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ be a residuated lattice.

We first recall some definitions and properties of fuzzy subset.

Definition 9 (see [11]). Let A be a nonempty set.

(i) A map $\eta\colon A \longrightarrow [0,1]$ is called a **fuzzy subset** of A

(ii) A fuzzy subset η of A is called proper if it is not a constant map

(iii) If $\eta\colon A \longrightarrow [0,1]$ is a fuzzy subset of A and $t \in [0,1]$, then $\eta_t = \{a \in A / \eta(a) \geq t\}$ is called the **t-cut set** of η

Definition 10 (see [8]). Let η be a fuzzy subset of A. η is a **fuzzy ideal** of $\mathscr{A}$, if it satisfies the following conditions:

(FI1) for any $a, b \in A$, if $a \leq b$, then $\eta(a) \geq \eta(b)$

(FI2) for any $a, b \in A$, $\eta(a \oslash b) \geq \min\{\eta(a), \eta(b)\}$

Example 4. Let A be a lattice defined by the Hass diagram of Figure 4.

Define $\longrightarrow$ and $\odot$ as follows (Table 1):

$\mathscr{A} = (A, \leq, \odot, \longrightarrow, 0, 1)$ is a residuated lattice. The fuzzy subset μ of A, defined by

$$\mu(x) = \begin{cases} \dfrac{1}{2}, & \text{if } x = 0, \\[2ex] \dfrac{1}{3}, & \text{if } x = b, \\[2ex] \dfrac{1}{5}, & \text{if } x \in \{1, c, a\}, \end{cases} \tag{1}$$

is a fuzzy ideal of A.

Definition 11 (see [6, 13]). Let η be a fuzzy subset of A.

(1) η is a **fuzzy filter of lattice** $(A, \wedge, \vee, 0, 1)$, if it satisfies the following conditions:

(i) for any $a, b \in A$, if $a \leq b$, then $\eta(a) \leq \eta(b)$

(ii) for any $a, b \in A$, $\eta(a \wedge b) = \min\{\eta(a), \eta(b)\}$

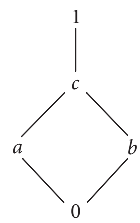

Figure 4: Example lattice 2.

Table 1: Table of operations ⊙ and ⟶ defined on A.

→	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

⊙	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

(2) η is a **fuzzy filter of residuated lattice** $\mathscr{A}$, if it satisfies the following conditions:

(i) for any $a, b \in A$, if $a \leq b$, then $\eta(a) \leq \eta(b)$

(ii) for any $a, b \in A$, $\eta(a \odot b) = \min\{\eta(a), \eta(b)\}$

Note that any fuzzy filter of residuated lattice $\mathscr{A}$ is a fuzzy filter of lattice $(A, \wedge, \vee, 0, 1)$; but the converse is not always true.

Example 5. Let us consider the residuated lattice $\mathscr{A}$ defined in Example 3. The fuzzy subset θ of A, defined by

$$\theta(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{4}, & \text{if } x \in \{a, n\}, \\ \frac{1}{2}, & \text{if } x \in \{b, d\}, \\ \frac{3}{4}, & \text{if } x \in \{c, m\}, \\ 1, & \text{if } x = 1, \end{cases} \quad (2)$$

is a fuzzy filter of lattice $(A, \leq)$ but it is not a filter of a residuated lattice $\mathscr{A}$ because $\theta(n \odot n) = \theta(0) \neq \theta(n)$.

Let η, θ be two fuzzy subsets of A. We define the order relation $\leq$ by $\eta \leq \theta \Longleftrightarrow \eta(a) \leq \theta(a)$, for all $a \in A$ and, for any family $\{\eta_i; i \in \Lambda\}$ of fuzzy ideals (or fuzzy filters) of A, $(\vee_{i\in\Lambda}\eta_i)(a) = \sup_{i\in\Lambda}\eta_i(a)$ and $(\wedge_{i\in\Lambda}\eta_i)(a) = \inf_{i\in\Lambda}\eta_i(a)$.

Let $FI(A)$ and $FF(A)$ denote, respectively, the set of fuzzy ideals of residuated lattice $\mathscr{A}$ and the set of fuzzy filters of lattice A.

Proposition 3. *Let $\{\eta_i; i \in \Lambda\}$ be a family of fuzzy ideals of a residuated lattice A. Then $\mu = \wedge_{i\in\Lambda}\eta_i$ is a fuzzy ideal of $\mathscr{A}$.*

Proof. Let $a, b \in A$ such that $a \leq b$. We have $\eta_i(a) \geq \eta_i(b)$; then $\mu(a) \geq \mu(b)$.

Moreover, for all $a, b \in A$, we have $\eta_i(a \oslash b) \geq \min\{\eta_i(a), \eta_i(b)\}$; then $\mu(a \oslash b) = \inf_{i\in\Lambda}\eta_i(a \oslash b) \geq \inf_{i\in\Lambda}(\min\{\eta_i(a), \eta_i(b)\}) = \min\{\mu(a), \mu(b)\}$. Thus, μ is a fuzzy ideal of A.

In general, $\vee_{i\in\Lambda}\eta_i$ of fuzzy ideals of A is not always a fuzzy ideal of A. Indeed, this following example shows it. □

Example 6. Let us consider the fuzzy ideals,

$$\eta_1(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{9}{10}, & \text{if } x = b, \\ \frac{3}{10}, & \text{if } x \in \{1, c, a\}, \end{cases}$$

$$\eta_2(x) = \begin{cases} 1, & \text{if } x = 0 \\ \frac{4}{5}, & \text{if } x = a, \\ \frac{3}{5}, & \text{if } x \in \{1, c, b\}, \end{cases} \quad (3)$$

of residuated lattice $\mathscr{A}$ defined in Example 4. For $\eta = \eta_1 \vee \eta_2$, the fuzzy subset of $\mathscr{A}$, we have

$$\eta(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{4}{5}, & \text{if } x = a, \\ \frac{9}{10}, & \text{if } x = b, \\ \frac{3}{5}, & \text{if } x \in \{1, c\}, \end{cases} \quad (4)$$

which is not a fuzzy ideal of $\mathscr{A}$ because $\eta(a \oslash b) = \eta(1) = (3/5) \ngeq \eta(a) \wedge \eta(b) = (4/5)$.

Let $\alpha \in [0; 1]$; we define $C_\alpha: A \longrightarrow [0; 1]$ by $C_\alpha(a) = \alpha$, for all $a \in A$.

Proposition 4 (see [7]). *Let η be a fuzzy ideal of $\mathscr{A}$ and $\alpha \in [0; 1]$ such that $\alpha < \eta(0)$. Then the fuzzy subset $\eta \vee C_\alpha$ of A is a fuzzy ideal of $\mathscr{A}$.*

Theorem 4 (see [8]). *Let η be a fuzzy subset of A. η is a fuzzy ideal of $\mathscr{A}$ if and only if, for each $t \in L$, $\eta_t \neq \varnothing$ implies that η_t is an ideal of $\mathscr{A}$.*

Notation 1. In the remainder of this paper, the following map will be very useful. Let I be a nonempty subset of A and

$\alpha, \beta \in [0;1]$ such that $\alpha > \beta$. Define the map $\eta_{I_{\alpha,\beta}}$ as follows: $\eta_{I_{\alpha,\beta}}(a) = \begin{cases} \alpha, & \text{if } a \in I \\ \beta, & \text{otherwise} \end{cases}$.

Proposition 5 (see [8]). *$\eta_{I_{\alpha,\beta}}$ is a proper fuzzy ideal of A if and only if I is a proper ideal of $\mathscr{A}$.*

Definition 12. Let $\eta: A \longrightarrow [0;1]$ be a fuzzy subset of A. A fuzzy ideal θ of $\mathscr{A}$ is said to be generated by η if $\eta \le \theta$ and, for any fuzzy ideal μ of $\mathscr{A}$, if $\eta \le \mu$, then $\theta \le \mu$. The fuzzy ideal generated by η will be denoted by $\widehat{\eta}$.

Let η be a fuzzy subset of A. For all $a \in A, a \in \eta_{\eta(a)} \subseteq \widehat{\eta}_{\eta(a)}$ and $\eta(a) \in [0;1]$; then $\{t \in [0;1]/a \in \widehat{\eta}_t\} \neq \varnothing$.

Theorem 5. *Let η be a fuzzy subset of A. Then the fuzzy subset η^* of A defined by $\eta^*(a) = \sup\{t \in [0;1]/a \in \widehat{\eta}_t\}$, for all $a \in A$, is the fuzzy ideal generated by η.*

Proof. Let $\alpha \in [0;1]$. Suppose that $\alpha \in \mathrm{Im}(\eta^*)$; take $\alpha_n = \alpha - (1/n)$, for all $n \in \mathbb{N}^*$ and $a \in \eta^*_\alpha$. We have $\eta^*(a) \ge \alpha \ge \alpha_n$; then there exists $t_0 \in [0;1]$ such that $a \in \widehat{\eta}_{t_0}$ and $t_0 \ge \alpha_n$, for all $n \in \mathbb{N}^*$. That is, $a \in \widehat{\eta}_{t_0}$ and $\eta_{t_0} \subseteq \eta_{\alpha_n}$, for all $n \in \mathbb{N}^*$. Then $a \in \widehat{\eta}_{t_0} \subseteq \widehat{\eta}_{\alpha_n}$, for all $n \in \mathbb{N}^*$, which implies that $a \in \cap_{n\in\mathbb{N}^*} \widehat{\eta}_{\alpha_n}$. Thus, $\eta^*_\alpha \subseteq \cap_{n\in\mathbb{N}^*} \widehat{\eta}_{\alpha_n}$. If $a \in \cap_{n\in\mathbb{N}^*} \widehat{\eta}_{\alpha_n}$, then, for all $n \in \mathbb{N}^*$, $\alpha_n \in \{t \in [0;1]/a \in \widehat{\eta}_t\}$; therefore, for all $n \in \mathbb{N}^*$, $\eta^*(a) \ge \alpha_n = \alpha - (1/n)$, which implies that $\eta^*(a) \ge \alpha$; that is, $x \in \eta^*_\alpha$. Thus, $\eta^*_\alpha = \cap_{n\in\mathbb{N}^*} \widehat{\eta}_{\alpha_n}$. By Proposition 3, η^*_α is an ideal of A. If $\alpha \notin \mathrm{Im}(\eta^*)$, then $\eta^*_\alpha = \eta^*_\beta$, where $\beta = \sup\{t \in \mathrm{Im}(\eta^*)/t > \alpha\} \in \mathrm{Im}(\eta^*)$ or $\eta^*_\alpha = \varnothing$. Therefore, by Theorem 4, η^* is a fuzzy ideal of $\mathscr{A}$.

Let $a \in A$; we have $a \in \eta_\alpha$, where $\alpha = \eta(a)$; thus, $a \in \widehat{\eta}_\alpha$; that is, $\alpha \in \{t \in [0;1]/a \in \widehat{\eta}_t\}$. Therefore, $\eta(a) = \alpha \le \eta^*(a)$; that is, $\eta \le \eta^*$.

Let θ be a fuzzy ideal of $\mathscr{A}$ such that $\eta \le \theta$ and $a \in A$. If $\eta^*(a) = 0$, then $\eta^*(a) \le \theta(a)$. Suppose that $\eta^*(a) = \alpha \neq 0$. Then $a \in \eta^*_\alpha = \cap_{n\in\mathbb{N}^*} \widehat{\eta}_{\alpha_n}$; that is, for all $n \in \mathbb{N}^*$, $\eta(a) \ge \alpha_n = \alpha - (1/n)$. Since $\theta(a) \ge \eta(a) \ge \alpha = \eta^*(a)$, $\theta \ge \eta^*$. Hence, $\eta^* = \widehat{\eta}$. □

Example 7. Let us consider the residuated lattice $\mathscr{A}$ defined in Example 4. Let us consider the fuzzy subset η of A defined by

$$\eta(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0, \\ \frac{1}{3}, & \text{if } x = b, \\ \frac{1}{5}, & \text{if } x = a, \\ \frac{1}{10}, & \text{if } x \in \{1, c\}. \end{cases} \quad (5)$$

Let $t \in [0;1]$; if $t \in [0;(1/10)]$, then $\eta_t = A = \widehat{\eta}_t$. If $t \in (1/10);(1/5)$, then $\eta_t = \{0, a, b\}$ and $\widehat{\eta}_t = A$. If $t \in (1/5);(1/3)$, then $\eta_t = \{0, b\} = \widehat{\eta}_t$. If $t \in (1/3);(1/2)$, then $\eta_t = \{0\} = \widehat{\eta}_t$. If $t \in (1/2);1$, then $\eta_t = \varnothing$. Therefore, we have

$$\eta^*(0) = \sup\{t \in [0;1]/0 \in \widehat{\eta}_t\} = (1/2)$$
$$\eta^*(a) = \sup\{t \in [0;1]/a \in \widehat{\eta}_t\} = (1/5)$$
$$\eta^*(b) = \sup\{t \in [0;1]/b \in \widehat{\eta}_t\} = (1/3)$$
$$\eta^*(c) = \sup\{t \in [0;1]/c \in \widehat{\eta}_t\} = (1/5)$$
$$\eta^*(1) = \sup\{t \in [0;1]/1 \in \widehat{\eta}_t\} = (1/5)$$

Thus,

$$\eta^*(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0, \\ \frac{1}{3}, & \text{if } x = b, \\ \frac{1}{5}, & \text{if } x \in \{1, c, a\}, \end{cases} \quad (6)$$

4. Fuzzy Prime Ideal Theorem

Definition 13. A proper fuzzy ideal η of a residuated lattice $\mathscr{A}$ is said to be fuzzy prime if $\eta(x'' \wedge y'') = \eta(x)$ or $\eta(x'' \wedge y'') = \eta(y)$, for any $x, y \in A$.

Proposition 6. *Let η be a proper fuzzy subset of a residuated lattice $\mathscr{A}$. η is a fuzzy prime ideal of $\mathscr{A}$ if and only if, for any $t \in [0;1]$, if $\varnothing \neq \eta_t \neq A$, then η_t is a prime ideal of $\mathscr{A}$.*

Proof. Let η be a proper fuzzy subset of A. Suppose that η is a fuzzy prime ideal. Let $t \in [0;1]$ such that $\varnothing \neq \eta_t \neq A$. By Theorem 4, η_t is an ideal of A. Let $x, y \in A$ such that $x'' \wedge y'' \in \eta_t$; that is, $\eta(x'' \wedge y'') \ge t$; then $\eta(x) \ge t$ or $\eta(y) \ge t$; that is, $x \in \eta_t$ or $y \in \eta_t$. Therefore, η_t is a prime ideal of $\mathscr{A}$.

Conversely, suppose that, for all $t \in [0;1]$, if η_t is nontrivial, then η_t is a prime ideal. By Theorem 4, η is a fuzzy ideal of $\mathscr{A}$. Let $x, y \in A$. Take $\alpha = \eta(x'' \wedge y'')$; we have $\eta(x) = \eta(x'') \neq \eta(x'' \wedge y'')$ or $\eta(y) = \eta(y'') \neq \eta(x'' \wedge y'')$ because if it is not, then $\eta(x) = \eta(y)$, which is absurd (η is proper); that is, $\eta(x) < \eta(x'' \wedge y'') = \alpha$ or $\eta(y) < \eta(x'' \wedge y'') = \alpha$; that is, $x \notin \eta_\alpha$ or $y \notin \eta_\alpha$. Thus, $\eta_\alpha \neq A$. Moreover, $\eta_\alpha \neq \varnothing$ because $x'' \wedge y'' \in \eta_\alpha$; then by hypothesis η_α is a prime ideal and $x \in \eta_\alpha$ or $y \in \eta_\alpha$; that is, $\eta(x) \ge \eta(x'' \wedge y'')$ or $\eta(y) \ge \eta(x'' \wedge y'')$. In addition, $\eta(x'' \wedge y'') \ge \eta(x'') = \eta(x)$ and $\eta(x'' \wedge y'') \ge \eta(y'') = \eta(y)$ because η is a fuzzy ideal. Therefore, $\eta(x'' \wedge y'') = \eta(x)$ or $\eta(x'' \wedge y'') = \eta(y)$.

If $\mathscr{A}$ is a De Morgan residuated lattice, then η is a fuzzy prime ideal of $\mathscr{A}$ if $\eta(x \wedge y) = \eta(x)$ or $\eta(x \wedge y) = \eta(y)$, for any $x, y \in A$. □

Definition 14. A fuzzy ideal η of a residuated lattice $\mathscr{A}$ is said to be fuzzy prime of the second kind if it is nonconstant and $\eta((x \longrightarrow y)') = \eta(0)$ or $\eta((y \longrightarrow x)') = \eta(0)$ for any $x, y \in A$.

Example 8. Let $\mathscr{A}$ be the residuated lattice defined in Example 4. Then, η, defined by

$$\eta(x)=\begin{cases}\frac{1}{5}, & \text{if } x\in\{0,a\},\\ \frac{1}{10}, & \text{if } x\in\{1,c,b\},\end{cases} \tag{7}$$

is a fuzzy prime ideal of the second kind of $\mathscr{A}$.

Lemma 1. *Let $\mathscr{A}$ be a residuated lattice. For all $x, y \in A$, we have $(x''\wedge y'')\oslash(x\longrightarrow y)'\geq x$ and $(x''\wedge y'')\oslash(y\longrightarrow x)'\geq y$.*

Proof. Let $x, y\in A$. We have $(x''\wedge y'')\oslash(x\longrightarrow y)' = (x\longrightarrow y)\longrightarrow (x''\wedge y'')''$ by property (P4) of Theorem 1. Since $(x''\wedge y'')''\geq x''\wedge y''\geq x\wedge y$, then $(x''\wedge y'')\oslash(x\longrightarrow y)'\geq (x\longrightarrow y)\longrightarrow (x\wedge y) = ((x\longrightarrow y)\longrightarrow x)\wedge((x\longrightarrow y)\longrightarrow y)\geq x\wedge x = x$. Thus, $(x''\wedge y'')\oslash(x\longrightarrow y)'\geq x$. In the same way, we show that $(x''\wedge y'')\oslash(y\longrightarrow x)'\geq y$. □

Theorem 6. *Any fuzzy prime ideal of the second kind of residuated lattice $\mathscr{A}$ is a fuzzy prime ideal of $\mathscr{A}$. If $\mathscr{A}$ is an MTL-algebra, then the notions of fuzzy prime ideal of $\mathscr{A}$ and fuzzy prime ideal of the second kind of $\mathscr{A}$ are equivalent.*

Proof. Let η be a fuzzy prime ideal of the second kind of $\mathscr{A}$. Let $x, y\in A$; we have $\eta((x\longrightarrow y)')=\eta(0)$ or $\eta((y\longrightarrow x)')=\eta(0)$. By Lemma 1, $(x''\wedge y'')\oslash(x\longrightarrow y)'\geq x$ and $(x''\wedge y'')\oslash(y\longrightarrow x)'\geq y$; then $\eta(x)\geq\eta((x''\wedge y'')\oslash(x\longrightarrow y)')\geq\min\{\eta(x''\wedge y''); \eta((x\longrightarrow y)')\}$ and $\eta(y)\geq\eta((x''\wedge y'')\oslash(y\longrightarrow x)')\geq\min\{\eta(x''\wedge y''); \eta((y\longrightarrow x)')\}$. If $\eta((x\longrightarrow y)')=\eta(0)$, then $\eta(x)\geq\min\{\eta(x''\wedge y''); \eta(0)\}=\eta(x''\wedge y'')$. From the fact that η is a fuzzy ideal, we have $\eta(x''\wedge y'')\geq\eta(x)$; then $\eta(x)=\eta(x''\wedge y'')$. Identically, if $\eta((y\longrightarrow x)')=\eta(0)$, then $\eta(y)=\eta(x''\wedge y'')$. In conclusion, η is a fuzzy prime ideal of $\mathscr{A}$.

Assume that $\mathscr{A}$ is an MTL-algebra and let η be a fuzzy prime ideal of $\mathscr{A}$. Then, $\mathscr{A}$ satisfies the prelinearity condition; that is, for any $x, y\in A$, we have $(x\longrightarrow y)\vee(y\longrightarrow x)=1$; that is, $(x\longrightarrow y)'\wedge(y\longrightarrow x)'=0$; then $\eta(0)=\eta((x\longrightarrow y)'''\wedge(y\longrightarrow x)''')$. Therefore, $\eta(0)=\eta((x\longrightarrow y)')$ or $\eta(0)=\eta((y\longrightarrow x)')$, by hypothesis. Thus, η is a fuzzy prime ideal of the second kind of $\mathscr{A}$.

In general, a fuzzy prime ideal of a residuated lattice is not always a fuzzy prime ideal of the second kind, unless the residuated lattice is an MTL-algebra. The proof of this statement is given by the following counterexample. □

Example 9. Let us consider the residuated lattice $\mathscr{A}$ defined in Example 3 and let η be the fuzzy ideal of $\mathscr{A}$ defined by

$$\eta(x)=\begin{cases}\frac{1}{2}, & \text{if } x\in\{1,m,c,d,b\},\\ \frac{8}{5}, & \text{if } x\in\{a,n\},\\ 1, & \text{if } x=0.\end{cases} \tag{8}$$

η is a fuzzy prime ideal of $\mathscr{A}$ which is not a fuzzy prime ideal of the second kind of $\mathscr{A}$, because $\eta((a\longrightarrow b)')=\eta(a)\neq\eta(0)$ and $\eta((b\longrightarrow a)')=\eta(b)\neq\eta(0)$.

Definition 15. A proper fuzzy ideal η of a residuated lattice $\mathscr{A}$ is prime fuzzy if, for fuzzy ideals μ and θ of A, $\theta\wedge\mu\leq\eta$ implies that $\theta\leq\eta$ or $\mu\leq\eta$.

Example 10. Let us consider the residuated lattice defined in Example 3.

Consider that the fuzzy subset η, defined by

$$\eta(x)=\begin{cases}\frac{1}{2}, & \text{if } x\in\{1,m,c,d,b\},\\ 1, & \text{if } x\in\{0,a,n\},\end{cases} \tag{9}$$

is a prime fuzzy ideal of $\mathscr{A}$.

Remark 2. A fuzzy prime ideal of $\mathscr{A}$ is not necessarily a prime fuzzy ideal of $\mathscr{A}$ as the following example shows.

Example 11. Let us consider the residuated lattice defined in Example 3. Let η be a fuzzy subset of $\mathscr{A}$ defined by

$$\eta(x)=\begin{cases}\frac{1}{2}, & \text{if } x\in\{1,m,c,d,b\},\\ \frac{8}{5}, & \text{if } x\in\{0,a,n\}.\end{cases} \tag{10}$$

η is a fuzzy prime ideal of $\mathscr{A}$. Let μ and θ be two fuzzy ideals of $\mathscr{A}$ defined by

$$\mu(x)=\begin{cases}\frac{1}{5}, & \text{if } x\in\{1,m,c,d,b\},\\ 1, & \text{if } x\in\{0,a,n\},\end{cases}$$
$$\theta(x)=\begin{cases}\frac{3}{5}, & \text{if } x\in\{1,m,c,d,b\},\\ \frac{7}{10}, & \text{if } x\in\{0,a,n\}.\end{cases} \tag{11}$$

We have

$$(\mu\wedge\theta)(x)=\begin{cases}\dfrac{1}{5}, & \text{if } x\in\{1,m,c,d,b\},\\[2ex] \dfrac{7}{10}, & \text{if } x\in\{0,a,n\}.\end{cases} \tag{12}$$

Then $\mu\wedge\theta\le\eta$, $\mu\nleq\eta$, and $\theta\nleq\eta$; therefore, η is not a prime fuzzy ideal of $\mathscr{A}$.

Definition 16. Let η: $A\longrightarrow[0,1]$ be a proper fuzzy ideal of $\mathscr{A}$. η is called fuzzy maximal ideal if, for all $t\in[0,1]$, $\varnothing\neq\eta_t\neq A$ implies that η_t is a maximal ideal of $\mathscr{A}$.

Example 12. Let us consider the residuated lattice defined in Example 3 and let η be a fuzzy subset of A defined by

$$\eta(x)=\begin{cases}0, & \text{if } x\in\{1,m,c,d,b\},\\ 1, & \text{if } x\in\{0,a,n\},\end{cases} \tag{13}$$

η is a fuzzy maximal ideal of $\mathscr{A}$.

Proposition 7. *Let η be a fuzzy ideal of $\mathscr{A}$. If η is a fuzzy maximal ideal of $\mathscr{A}$, then η is a fuzzy prime ideal of $\mathscr{A}$.*

Proof. Let η be a fuzzy maximal ideal of $\mathscr{A}$. Then, for all $t\in[0,1]$ such that $\varnothing\neq\eta_t\neq A$, η_t is a maximal ideal of $\mathscr{A}$. By Proposition 2, η_t is a prime ideal of $\mathscr{A}$ for all $t\in[0,1]$ such that $\varnothing\neq\eta_t\neq A$. Thus, according to Proposition 6, η is a fuzzy prime ideal of $\mathscr{A}$.

The converse of the above proposition is not true. Let us consider the residuated lattice defined in Example 3 and let η be a fuzzy subset of $\mathscr{A}$ defined by

$$\eta(x)=\begin{cases}\dfrac{1}{5}, & \text{if } x\in A,\{0\},\\[2ex] \dfrac{7}{10}, & \text{if } x=0.\end{cases} \tag{14}$$

η is a fuzzy prime ideal of $\mathscr{A}$, which is not a fuzzy maximal ideal of $\mathscr{A}$ because $\eta_{3/5}=\{0\}$ is not maximal. □

Lemma 2. *Let $t\in[0;1]$, let η be a fuzzy ideal of a residuated lattice $\mathscr{A}$, and let μ be a fuzzy filter of the lattice $(A,\wedge,\vee,0,1)$. Then $I=\{x\in A|\eta(x)>t\}$ and $J=\{x\in A\,|\,\mu(x)>t\}$ are, respectively, the ideal of the residuated lattice $\mathscr{A}$ and the filter of the lattice $(A,\wedge,\vee,0,1)$.*

Proof. Let $x\in A$ and $y\in I$ such that $x\le y$. Then $\eta(x)\ge\eta(y)>t$; that is, $x\in I$. Let $x,y\in I$. We have $\eta((x'\odot y')')=\eta(x'\longrightarrow y'')\ge\eta(x)\wedge\eta(y'')$. Since η is a fuzzy ideal of $\mathscr{A}$, $\eta(y'')=\eta(y)$. Therefore, $\eta((x'\odot y')')\ge\eta(x)\wedge\eta(y)>t$ because $x,y\in I$. Thus, $(x'\odot y')'\in I$. Hence, I is an ideal of $\mathscr{A}$.

Let $x\in J$ and $y\in A$ such that $x\le y$. Then $\mu(y)\ge\mu(x)>t$; that is, $y\in J$. Let $x,y\in J$; we have $\mu(x\wedge y)\ge\mu(x)\wedge\mu(y)>t$. Therefore, $x\wedge y\in J$. Thus, J is a filter of the lattice $(A,\wedge,\vee,0,1)$. □

Theorem 7. *Let $t\in[0;1]$, let η be a fuzzy ideal of a residuated lattice $\mathscr{A}$, and let μ be a fuzzy filter of the lattice $(A,\wedge,\vee,0,1)$ such that $\eta\wedge\mu\le t$. Then there exists a fuzzy prime ideal θ of $\mathscr{A}$ such that $\eta\le\theta$ and $\mu\wedge\theta\le t$.*

Proof. Let $I=\{x\in A|\eta(x)>t\}$ and $J=\{x\in A\,|\,\mu(x)>t\}$. By Lemma 2, I is an ideal of the residuated lattice $\mathscr{A}$ and J is a filter of the lattice $(A,\wedge,\vee,0,1)$.

Let us show that $I\cap J=\varnothing$. Suppose that there exists $x\in I\cap J$. Then $\eta(x)>t$ and $\mu(x)>t$. Therefore, $(\eta\wedge\mu)(x)>t$; this is a contradiction because, by hypothesis, $\eta\wedge\mu\le t$. Thus, $I\cap J=\varnothing$.

Using the fact that I is an ideal of the residuated lattice $\mathscr{A}$, J is a filter of lattice $(A,\wedge,\vee,0,1)$ and $I\cap J=\varnothing$ and, by Theorem 4, there exists a prime ideal P of the residuated lattice $\mathscr{A}$ such that $I\subseteq P$ and $P\cap J=\varnothing$. Let

$$\theta(x)=\begin{cases}1, & \text{if } x\in P,\\ t, & \text{if } x\notin P.\end{cases} \tag{15}$$

We have $\theta=\eta_{P_{1,t}}$; then, by Proposition 5, θ is a fuzzy ideal of $\mathscr{A}$. If $x''\wedge y''\in P$, then $x\in P$ or $y\in P$; that is, $\theta(x)=1$ or $\theta(y)=1$; therefore, $\theta(x)=\theta(x''\wedge y'')$ or $\theta(y)=\theta(x''\wedge y'')$. If $x''\wedge y''\notin P$, then $\theta(x''\wedge y'')=t\le\ \theta(x),\theta(y)$ because $\theta(x),\theta(y)\in\{t,1\}$. In addition, θ is a fuzzy ideal of $\mathscr{A}$ and, for all $x,y\in A$, $x''\wedge y''\le x'',y''$; then, $\theta(x''\wedge y'')\ge\theta(x''),\theta(y'')$. Therefore, $\theta(x''\wedge y'')=\theta(x)$ or $\theta(x''\wedge y'')=\theta(y)$, since $\theta(x'')=\theta(x)$ and $\theta(y'')=\theta(y)$. Thus, θ is a fuzzy prime ideal of $\mathscr{A}$.

Let $x\in A$; if $x\in P$, then $\theta(x)=1\ge\eta(x)$. Else, $x\notin P$; then $x\notin I$ and $\theta(x)=t$, because $I\subseteq P$; in this case, $\eta(x)<t=\theta(x)$. Thus, $\eta\le\theta$.

Let $x\in A$; if $x\in P$, then $\theta(x)=1$ and $x\notin J$ (because $P\cap J=\varnothing$). Therefore, $(\mu\wedge\theta)(x)=\mu(x)<t$. If $x\notin P$, then $(\mu\wedge\theta)(x)=\mu(x)\wedge t<t$. Thus, $\mu\wedge\theta\le t$. □

5. Conclusion

In this paper, we have investigated the notion of fuzzy ideal generated by a fuzzy subset of residuated lattice and established the fuzzy prime ideal theorem of this structure. Besides, we have studied different types of fuzzy prime ideals (fuzzy prime ideal, fuzzy prime ideal of second kind, and prime fuzzy ideal) and fuzzy maximal ideal of residuated lattice. We have proved that any fuzzy maximal ideal is a fuzzy prime ideal, but the converse is not always true. Moreover, any prime fuzzy ideal and any fuzzy prime ideal of the second kind are fuzzy prime ideals and the converse is not also true. We have also given a characterization of fuzzy ideal generated by a fuzzy subset of residuated lattice. The last result of this paper is the fuzzy prime ideal theorem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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