Research Article

Real Root Polynomials and Real Root Preserving Transformations

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Polynomials can be used to represent real-world situations, and their roots have real-world meanings when they are real numbers. The fundamental theorem of algebra tells us that every nonconstant polynomial p with complex coefficients has a complex root. However, no analogous result holds for guaranteeing that a real root exists to p if we restrict the coefficients to be real. Let $n \geq 1$ and P_n be the vector space of all polynomials of degree n or less with real coefficients. In this article, we give explicit forms of polynomials in P_n such that all of their roots are real. Furthermore, we present explicit forms of linear transformations on P_n which preserve real roots of polynomials in a certain subset of P_n.

1. Introduction

Polynomials are the simplest and most commonly used in mathematics. We can approximate functions from real-world situation models by polynomials, and they give us results that are "close enough" to what we would get by using the actual functions and are a lot easier to use. The roots of a polynomial which are also the x-intercepts of the graph are key information when it comes to draw the polynomial graph, and they have real-world meaning when they are real numbers. Numerous approaches to compute the roots of polynomials have been developed (for example, [1–3]), and various efficient algorithms have been proposed (for example, [4, 5]). It is natural to ask which forms of polynomials guarantee all real roots and when we map a polynomial to another polynomial, how linear transformations that preserve real roots of polynomials look like. The obvious result is proved in [6] that only nonzero multiples of the identity transformation preserve roots of all polynomials in the vector space of all polynomials of degree n or less with real coefficients, P_n.

The aim of this article is to provide explicit forms of polynomials in P_n such that all of their roots are real as well as give explicit forms of linear transformations on P_n that preserve real roots of polynomials in a certain subset of P_n.

The organization of this article is as follows. In Section 2, we review some basic definitions and properties about q-factorial as well as polynomials and their applications. In Section 3, we present explicit forms of polynomials and show that they always have all real roots. In Section 4, we define a subset S of P_n and linear transformations on P_n. Then, we prove that they preserve real roots of all polynomials in S. In Section 5, we give a conclusion of our results.

2. Preliminary

In this section, we introduce q-factorial and recall basic theorems of polynomials and their applications.

2.1. q-Factorial

Definition 1. For nonnegative integers n and k, the number of combinations of n objects, taken k at a time, is given by

$$\frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}. \tag{1}$$

By convention, $\begin{pmatrix} n \\ 0 \end{pmatrix}$ is defined to be 1, and $\begin{pmatrix} n \\ k \end{pmatrix}$ is defined to be 0 for $n < k$.

Definition 2. For a nonnegative integer n, the q-analog of n is defined to be

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}. \tag{2}$$

By convention, $[0]_q$ is defined to be 0. Note that $\lim_{q \longrightarrow 1}[n]_q = n$.

Definition 3. For a nonnegative integer n, the q-factorial of n denoted by $[n]_q!$ is defined to be

$$[n]_q! = [n]_q[n-1]_q \ldots [2]_q[1]_q. \tag{3}$$

By convention, $[0]_q!$ is defined to be 1. Note that $\lim_{q \longrightarrow 1}[n]_q! = n!$, and

$$\begin{aligned}
[n]_q! &= \frac{1-q^n}{1-q}\frac{1-q^{n-1}}{1-q}\cdots\frac{1-q^2}{1-q}\frac{1-q}{1-q} \\
&= \left(1 + q + q^2 + \cdots + q^{n-1}\right) \\
&\quad \cdot \left(1 + q + q^2 + \cdots + q^{n-2}\right)\ldots(1+q)(1).
\end{aligned} \tag{4}$$

Example 1. Consider $n = 4$; we have

$$\begin{aligned}
[4]_1! &= \left(1 + 1 + 1^2 + 1^3\right)\left(1 + 1 + 1^2\right)(1+1) = 24 = 4!, \\
[4]_2! &= \left(1 + 2 + 2^2 + 2^3\right)\left(1 + 2 + 2^2\right)(1+2) = 315.
\end{aligned} \tag{5}$$

2.2. Polynomials

Definition 4. An element w in $\mathbb{C}$ is said to be a root of a given polynomial p over $\mathbb{C}$ if $p(w) = 0$. The set of all roots of p is denoted by $Z(p)$.

Theorem 1 (the fundamental theorem of algebra; see [7]). *Every nonconstant polynomial p with complex coefficients has a complex root.*

This is a remarkable statement; however, no analogous result holds for guaranteeing that a real root exists to p if we restrict the coefficients to be real numbers.

The following theorem which we use to prove throughout our main results provides a sufficient condition for the existence of all real roots.

Theorem 2 (see [8]). *Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial of degree $n \geq 2$ with positive real coefficients. If*

$$a_i^2 - 4a_{i-1}a_{i+1} > 0 \quad \text{for all } i = 1, 2, 3, \ldots, n-1, \tag{6}$$

then $Z(p) \subseteq \mathbb{R}$.

There are a variety of different applications of polynomials that we can look at. The following examples show real roots of polynomials apply to real-world situations.

Example 2. Gravity is roughly constant on the earth's surface with an acceleration of $g \approx 32$ feet/second^2, and the height of a rigid object in free fall at time t is modeled by Newton's equation of motion:

$$p(t) = \frac{1}{2}gt^2 + v_0 t + p_0, \tag{7}$$

where t is the elapsed time in seconds, v_0 is the initial velocity in feet/second, and p_0 is the initial height of the object above the ground level in feet. When v_0 and p_0 are known, we have $p(t_0) = 0$ if t_0 is a root of $p(t)$. This means that the object will take t_0 seconds to hit the ground. Note that since t represents time, only nonnegative real roots apply.

In [9], the special case $v_0 = 0$ is used to translate height measurements of the moving object in the image to metric units in 3D world coordinates and derive relations for the case of rigid objects and then for articulated motion to estimate a person's height from the video.

Example 3. Chemical equilibrium is a state in which the rate of the forward reaction equals the rate of the backward reaction. Consider chemical equilibrium in the gaseous system $Cl_2 + 2NO = 2NOCl$ described by

$$A + 2B = 2C. \tag{8}$$

In [10], if $P = 1$ bar and the equilibrium constant $K_p = 70.23$ bar at 500 K, the expression for K_p can be written in the equation

$$f(x) = x^3 - 3x^2 + 3.173x - 0.9422 = 0, \tag{9}$$

where x is the number of moles of A reacted at equilibrium from an initial state consisting of 1 mol of A, 2 mol of B, and 2 mol of C. Since $f(x)$ has two complex roots and one real root $x_0 \approx 0.479764$, only the real root is a chemical root of $f(x)$.

3. Real Root Polynomials

Fisk [11] observed that if the coefficients of a polynomial are decreasing sufficiently rapidly, then all of the roots of the polynomial are real numbers. Motivated by this observation, we construct polynomials in the following forms and use Theorem 2 to show that they have all real roots.

Proposition 1. *Let p be a polynomial in one of the following forms. Then, $Z(p) \subset \mathbb{R}$.*

(1) $p(x) = \sum_{i=0}^{n} \alpha^{-\binom{i}{2}} x^i$, $\alpha > 4$

(2) $p(x) = \sum_{i=0}^{n} \alpha^{-i^2} x^i$, $\alpha > 2$

(3) $p(x) = \sum_{i=0}^{n} x^i/[i]_q!$, $q \geq 4$

Proof. First, note that if $\alpha > 4$, then for $1 \leq i \leq n-1$,

$$\left(\alpha^{-\binom{i}{2}}\right)^2 - 4\alpha^{-\binom{i-1}{2}}\alpha^{-\binom{i+1}{2}} = \alpha^{-i^2+i} - 4\alpha^{\left(-2i^2+2i-2/2\right)} = \alpha^{-\left(i^2-i\right)}\left(1 - \frac{4}{\alpha}\right) > 0. \tag{10}$$

Similarly, if $\alpha > 2$, then for $1 \le i \le n-1$,

$$\left(\alpha^{-i^2}\right)^2 - 4\alpha^{-(i-1)^2}\alpha^{-(i+1)^2} = \alpha^{-2i^2} - 4\alpha^{-2i^2-2}$$

$$= \alpha^{-2i^2}\left(1 - \frac{4}{\alpha^2}\right) > 0. \tag{11}$$

Also, if $q \ge 4$, then for $1 \le i \le n-1$, $[i+1]_q > q[i]_q$, and hence,

$$\left(\frac{1}{[i]_q!}\right)^2 - \frac{4}{[i-1]_q![i+1]_q!} = \frac{1}{\left([i-1]_q!\right)^2 [i+1]_q [i]_q}$$

$$\cdot \left(\frac{[i+1]_q}{[i]_q} - 4\right) > 0. \tag{12}$$

Therefore, by Theorem 2, all roots of p in the above forms are real. □

Example 4. By Proposition 1,

$$\sum_{i=0}^{4} 5^{-\binom{i}{2}} x^i = 1 + x + \frac{1}{5}x^2 + \frac{1}{125}x^3 + \frac{1}{15625}x^4,$$

$$\sum_{i=0}^{3} 3^{-i^2} x^i = 1 + \frac{1}{3}x + \frac{1}{81}x^2 + \frac{1}{19683}x^3, \tag{13}$$

$$\sum_{i=0}^{5} \frac{x^i}{[i]_4!} = 1 + x + \frac{x^2}{5} + \frac{x^3}{105} + \frac{x^4}{8925} + \frac{x^5}{3043425}$$

have all real roots.

4. Real Root Preserving Transformations

Let $n \ge 3$ and S be the set of polynomials satisfying conditions in Theorem 2, i.e., $S = \{p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \in P_n | a_i > 0$ for $0 \le i \le n$ and $a_i^2 - 4a_{i-1}a_{i+1} > 0$ for $1 \le i \le n-1\}$. Clearly, S is a subset of P_n, and all of the roots of polynomials in S are real numbers.

Proposition 2. *The following linear transformations on P_n preserve real roots of polynomials in S.*

(1) $x^i \mapsto \beta x^i$ for $\beta > 0$

(2) $x^i \mapsto \binom{n}{i} x^i$

(3) $x^i \mapsto i x^{i-1}$

(4) $x^i \mapsto x^i / i!$

(5) $x^i \mapsto x^i / (n-i)!$

(6) $x^i \mapsto x^i / (i!(n-i)!)$

(7) $x^i \mapsto x^i / [i]_q!$ for $q > 0$

Proof. Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is in S. Then, for $1 \le i \le n-1$, we have $a_i^2 - 4a_{i-1}a_{i+1} > 0$.

For each linear transformation, we show that it preserves real roots of p using Theorem 2.

(1) Let $\beta > 0$. Define $T_1(r(x)) = \beta r(x)$ for $r(x) \in P_n$. It is obvious that T_1 preserves real roots of all polynomials in P_n.

(2) Define

$$T_2(r(x)) = \binom{n}{n} k_n x^n + \binom{n}{n-1} k_{n-1} x^{n-1} + \cdots$$
$$+ \binom{n}{2} k_2 x^2 + \binom{n}{1} k_1 x + \binom{n}{0} k_0, \tag{14}$$

for $r(x) = k_n x^n + k_{n-1}x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0 \in P_n$. Then, for $1 \le i \le n-1$,

$$\left(\binom{n}{i} a_i\right)^2 - 4\left(\binom{n}{i-1} a_{i-1}\right)\left(\binom{n}{i+1} a_{i+1}\right)$$

$$= \frac{n!^2}{(n-i+1)!(i-1)!(n-i-1)!(i+1)!}$$

$$\cdot \left[\frac{(n-i+1)(i+1)}{(n-i)i} a_i^2 - 4a_{i-1}a_{i+1}\right]. \tag{15}$$

Since $(n-i+1)(i+1) > (n-i)i$, we have $((n-i+1)(i+1)/((n-i)i))a_i^2 - 4a_{i-1}a_{i+1} > 0$, and hence, $Z(T_2 p) \subseteq \mathbb{R}$.

(3) Define

$$T_3(r(x)) = n k_n x^{n-1} + (n-1)k_{n-1}x^{n-2} + \cdots + 2k_2 x + k_1, \tag{16}$$

for $\quad r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0$
$\in P_n$. Then, for $2 \leq i \leq n-2$,

$$(ia_i)^2 - 4(i-1)a_{i-1}(i+1)a_{i+1} = (i^2-1)\left[\frac{i^2}{i^2-1}a_i^2 - 4a_{i-1}a_{i+1}\right] > 0, \tag{17}$$

and hence, $Z(T_3 p) \subseteq \mathbb{R}$.

(4) Define

$$T_4(r(x)) = \frac{k_n x^n}{n!} + \frac{k_{n-1} x^{n-1}}{(n-1)!} + \cdots + \frac{k_2 x^2}{2!} + \frac{k_1 x}{1!} + \frac{k_0}{0!}, \tag{18}$$

for
$r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0 \in P_n$.
Then, for $1 \leq i \leq n-1$,

$$\left(\frac{a_i}{i!}\right)^2 - \frac{4a_{i-1}}{(i-1)!}\frac{a_{i+1}}{(i+1)!} = \frac{1}{(i-1)!^2(i+1)i}\left[\frac{i+1}{i}a_i^2 - 4a_{i-1}a_{i+1}\right] > 0, \tag{19}$$

and hence, $Z(T_4 p) \subseteq \mathbb{R}$.

(5) Define

$$T_5(r(x)) = \frac{k_n x^n}{0!} + \frac{k_{n-1} x^{n-1}}{1!} + \cdots + \frac{k_2 x^2}{(n-2)!}$$
$$+ \frac{k_1 x}{(n-1)!} + \frac{k_0}{n!}, \tag{20}$$

for $\quad r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0$
$\in P_n$. Then, for $1 \leq i \leq n-1$, we have

$$\left(\frac{a_i}{(n-i)!}\right)^2 - \frac{4a_{i-1}}{(n-(i-1))!}\frac{a_{i+1}}{(n-(i+1))!}$$
$$= \frac{1}{(n-i-1)!^2(n-i+1)(n-i)} \tag{21}$$
$$\cdot \left[\frac{n-i+1}{n-i}a_i^2 - 4a_{i-1}a_{i+1}\right] > 0.$$

Thus, $Z(T_5 p) \subseteq \mathbb{R}$.

(6) Define

$$T_6(r(x)) = \frac{k_n x^n}{n!(n-n)!} + \frac{k_{n-1} x^{n-1}}{(n-1)!(n-(n-1))!} + \cdots$$
$$+ \frac{k_2 x^2}{2!(n-2)!} + \frac{k_1 x}{1!(n-1)!} + \frac{k_0}{0!(n-0)!}, \tag{22}$$

for $\quad r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0$
$\in P_n$. Then, for $1 \leq i \leq n-1$,

$$\left(\frac{a_i}{i!(n-i)!}\right)^2 - \frac{4a_{i-1}}{(i-1)!(n-(i-1))!}\frac{a_{i+1}}{(i+1)!(n-(i+1))!} = \frac{1}{(i+1)i(i-1)!^2(n-i+1)(n-i)(n-i-1)!^2} \tag{23}$$
$$\left[\frac{(i+1)(n-i+1)}{i(n-i)}a_i^2 - 4a_{i-1}a_{i+1}\right].$$

Since $(i+1)(n-i+1) > i(n-i) > 1$, we have that $(i+1)(n-i+1)/(i(n-i)) > 1$, and hence, $((i+1)(n-i+1)/(i(n-i)))a_i^2 - 4a_{i-1}a_{i+1} > 0$. Therefore, $Z(T_6 p) \subseteq \mathbb{R}$.

(7) Let $q > 0$. Define

$$T_7(r(x)) = \frac{k_n x^n}{[n]_q!} + \frac{k_{n-1} x^{n-1}}{[n-1]_q!} + \cdots + \frac{k_2 x^2}{[2]_q!} + \frac{k_1 x}{[1]_q!} + \frac{k_0}{[0]_q!}, \tag{24}$$

for $\quad r(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_2 x^2 + k_1 x + k_0 \in P_n$. Then, for $1 \leq i \leq n-1$,

$$\left(\frac{a_i}{[i]_q!}\right)^2 - \frac{4a_{i-1}}{[i-1]_q!}\frac{a_{i+1}}{[i+1]_q!} = \frac{1}{[i-1]_q!^2[i]_q[i+1]_q}$$

$$\cdot\left[\frac{[i+1]_q}{[i]_q}a_i^2 - 4a_{i-1}a_{i+1}\right]. \tag{25}$$

Since $[i+1]_q/[i]_q = (1 + q + q^2 + \cdots + q^{i-1} + q^i)/(1 + q + q^2 + \cdots + q^{i-1}) > 1$, we have $([i+1]_q/[i]_q)a_i^2 - 4a_{i-1}a_{i+1} > 0$. Hence, $Z(T_7p) \subseteq \mathbb{R}$. $\square$

5. Conclusion

Real-world situations that cannot be modeled using a linear function can be approximated using polynomials, and this article gives three explicit forms of real root polynomials and seven explicit forms of real root preserving linear transformations of polynomials that guarantee real-world interpretations and understanding.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] J. M. McNamee, *Numerical Methods for Roots of Polynomials - Part I, Studies in Computational Mathematics*, Elsevier Science, Amsterdam, Netherlands, 2007.

[2] J. M. McNamee and V. Y. Pan, *Numerical Methods for Roots of Polynomials - Part II, Studies in Computational Mathematics*, Elsevier Science, Amsterdam, Netherlands, 2013.

[3] V. Y. Pan, "Solving a polynomial equation: some history and recent progress," *SIAM Review*, vol. 39, no. 2, pp. 187–220, 1997.

[4] V. Y. Pan, "Univariate polynomials: nearly optimal algorithms for numerical factorization and root-finding," *Journal of Symbolic Computation*, vol. 33, no. 5, pp. 701–733, 2002.

[5] D. A. Bini and G. Fiorentino, "Design, analysis, and implementation of a multiprecision polynomial rootfinder," *Numerical Algorithms*, vol. 23, no. 2/3, pp. 127–173, 2000.

[6] B. Curgus and V. Mascioni, "Root preserving transformations of polynomials," *Mathematics Magazine*, vol. 80, pp. 136–138, 2007.

[7] M. Artin, *Algebra*, Prentice-Hall, Massachusetts, MA, USA, 2011.

[8] D. C. Kurtz, "A sufficient condition for all the roots of a polynomial to be real," *The American Mathematical Monthly*, vol. 99, no. 3, pp. 259–263, 1992.

[9] D. Bieler, S. G. Günel, P. Fua, and H. Rhodin, "Gravity as a reference for estimating a person's height from video," in *Proceedings of the 2019 IEEE/CVF International Conference on Computer Vision (ICCV)*, pp. 8568–8576, Seoul, South Korea, November 2019.

[10] W. R. Smith and R. W. Missen, "Chemical equilibrium and polynomial equations: beware of roots," *Journal of Chemical Education*, vol. 66, no. 6, pp. 489-490, 1989.

[11] S. Fisk, "Polynomials, roots, and interlacing," 2006, https://arxiv.org/abs/math/0612833.