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Research Article

Biderivations and Commuting Linear Maps on Topologically Simple $\mathscr{L}^*$-Algebras

Abdelaziz Ben Yahya

Department of Mathematics, Faculty of Science, University Moulay Ismail, B.P. 11201, Zitoune, Meknès, Morocco

Correspondence should be addressed to Abdelaziz Ben Yahya; yahya.abdelazizben@edu.umi.ac.ma

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Let $\mathscr{L}$ be a topologically simple $\mathscr{L}^*$-algebra of arbitrary dimension. In this paper, we introduce the notion of semi-inner biderivation in order to prove that every continuous commuting linear mapping on $\mathscr{L}$ is a scalar multiple of the identity mapping.

1. Introduction

Let $(\mathscr{L}, [.,.])$ be a Lie algebra over a field $\mathbb{F}$ of a characteristic different from two. A linear map $D: \mathscr{L} \longrightarrow \mathscr{L}$ is called derivation on $(\mathscr{L}, [.,.])$ if it satisfies the following identity:

$$D([x, y]) = [D(x), y] + [x, D(y)], \tag{1}$$

for all $x, y \in \mathscr{L}$.

A bilinear map $\delta: \mathscr{L} \times \mathscr{L} \longrightarrow \mathscr{L}$ is called biderivation on $(\mathscr{L}, [.,.])$ if it satisfies the following identities:

$$\begin{aligned} \delta([x, y], z) &= [x, \delta(y, z)] + [\delta(x, z), y], \\ \delta(x, [y, z]) &= [\delta(x, y), z] + [y, \delta(x, z)]. \end{aligned} \tag{2}$$

For all $x, y, z \in \mathscr{L}$, which means that it is a derivation with respect to both components. In addition, if $\delta(x, y) = -\delta(y, x)$, δ will be called skew-symmetric biderivation. Let $\lambda \in \mathbb{F}$ and f be the bilinear map $f: \mathscr{L} \times \mathscr{L} \longrightarrow \mathscr{L}$ sending (x, y) to $\lambda[x, y]$; it is straightforward to prove that f is a biderivation of $(\mathscr{L}, [.,.])$; and the biderivations of this type are called inner biderivations of $(\mathscr{L}, [.,.])$. A linear map $\phi: \mathscr{L} \longrightarrow \mathscr{L}$ is called a commuting linear map on $(\mathscr{L}, [.,.])$ if it satisfies the following identity: $[\phi(x), x] = 0$, for all $x \in \mathscr{L}$. It is easy to show that

$$[\phi(x), y] = [x, \phi(y)], \quad \forall x, y \in \mathscr{L}, \tag{3}$$

which implies that the bilinear map δ defined by

$$\delta(x, y) = [\phi(x), y] = [x, \phi(y)], \tag{4}$$

is a skew-symmetric biderivation on $(\mathscr{L}, [.,.])$.

Commuting maps and biderivations arose first in the associative ring theory [1, 2]. Since then, many authors have made considerable efforts to make their study very successful (see, for example, [3–10]). The way used in [8] requires the finiteness of the dimension of the simple Lie algebra. However, the purpose of this paper is to extend the results given in [8] concerning the commuting linear maps to topologically simple $\mathscr{L}^*$-algebras, which are of arbitrary dimension. To overcome the problem of the nonfiniteness of the dimension, we use some techniques related to these algebras. The $\mathscr{L}^*$-algebras are introduced by Schue in [11]. We recall that an $\mathscr{L}^*$-algebra over $\mathbb{C}$ (the complex field) is a Lie algebra $\mathscr{L}$, which is also a complex Hilbert space with the inner product $(.,.)$ endowed with a (conjugate-linear) algebra involution * such that $([x, y], z) = (y, [x^*, z])$, for all x, y, z in $\mathscr{L}$.

Let $\mathscr{L}$ be an $\mathscr{L}^*$-algebra; for subsets M and N of $\mathscr{L}$, we recall that $[M, N]$ denotes the closed subspace spanned by $\{[m, n]: m \in M, n \in N\}$. $\mathscr{L}$ is said to be semisimple as an $\mathscr{L}^*$-algebra if and only if $\mathscr{L} = [\mathscr{L}, \mathscr{L}]$. From [11], a finite-dimensional Lie algebra $\mathscr{L}$ is semisimple as an $\mathscr{L}^*$-algebra if and only if it is semisimple in the usual sense. The

$\mathscr{L}^*$-algebra $\mathscr{L}$ will be called topologically simple if and only if there are no nontrivial closed ideals. In [11], the author shows that the $\mathscr{L}^*$-algebras are reductive, the semisimple ones are the Hilbert space direct sum of its closed topologically simple ideals, and the author also gives the classification of the topologically simple $\mathscr{L}^*$-algebras in the separable case. The classification of the topologically simple $\mathscr{L}^*$-algebras in the arbitrary dimensional case can be found in [12]. In [13], Schue shows that every semisimple $\mathscr{L}^*$-algebra has a Cartan decomposition relative to a Cartan subalgebra. We recall that a Cartan subalgebra of a semisimple $\mathscr{L}^*$-algebra is defined as a maximal self-adjoint abelian subalgebra.

The paper is organized as follows. In the second section, we give some definitions and basic results related to $\mathscr{L}^*$-algebras. In Section 3, we introduce the notion of semi-inner biderivation in order to show that every semi-inner biderivation on an arbitrary dimensional topologically simple $\mathscr{L}^*$-algebra is inner; using this result, we determine all continuous commuting linear maps on an arbitrary dimensional topologically simple $\mathscr{L}^*$-algebras.

2. Preliminaries

In this section, we summarize some basic results related to $\mathscr{L}^*$-algebras, collected from [14, 15]. First of all, we point out that the notation concerning Lie algebras follows principally from [8, 14]. Let $\mathbb{C}$ be the complex numbers field, $\mathscr{L}$ a semisimple $\mathscr{L}^*$-algebra, H a fixed Cartan subalgebra of $\mathscr{L}$, and $^-$ denotes the conjugation operator on $\mathbb{C}$, a root of $\mathscr{L}$ relative to H is a linear form commuting with the involution:

$$\alpha\colon (H, *) \longrightarrow (\mathbb{C}, ^-). \tag{5}$$

That is, $\alpha(h^*) = \overline{\alpha(h)}$ for any $h \in H$, such that there exists $v_\alpha \in \mathscr{L}$, $v_\alpha \neq 0$ satisfying $[h, v_\alpha] = \alpha(h) v_\alpha$ for any $h \in H$. The subspace

$$\mathscr{V}_\alpha = \{v_\alpha \in \mathscr{L}\colon [h, v_\alpha] = \alpha(h) v_\alpha \text{ for all } h \in H\}, \tag{6}$$

is called the root space associated to α; it follows from this that if α is a root, then $-\alpha$ is also one and $(\mathscr{V}_\alpha)^* = \mathscr{V}_{-\alpha}$. The root space associated to the zero root is equal to the Cartan subalgebra H, using the Jacobi identity; one proves that if $\alpha + \beta$ is a root, then $[\mathscr{L}_\alpha, \mathscr{L}_\beta] \subseteq \mathscr{L}_{\alpha+\beta}$, and if $\alpha + \beta$ is not a root, then $[\mathscr{L}_\alpha, \mathscr{L}_\beta] = 0$. Let Φ denote the set of nonzero roots of $\mathscr{L}$ relative to H, then we have the following Cartan decomposition $\mathscr{L} = H \oplus (\sum_{\alpha\in\Phi} \oplus \mathscr{V}_\alpha)$, where $\oplus$ is the usual Hilbert space direct sum.

Let α be a root of $\mathscr{L}$ relative to H, then α is a linear functional on H; this implies that there exists a unique vector $h_\alpha \in H$ such that $\alpha(h) = (h, h_\alpha)$ where $(.,.)$ denotes the inner product of $\mathscr{L}$. Consequently, h_α is self-adjoint, which means that $h_\alpha^* = h_\alpha$ and $h_\alpha = [v_\alpha, v_\alpha^*]$ for any $v_\alpha \in \mathscr{V}_\alpha$ with $\|v_\alpha\| = 1$. Then, we have the following result.

Lemma 1 (see [15]). *The set $\{h_\alpha\colon \alpha \in \Phi\}$ is total in H, i.e., for any $h \in H$, $(h, h_\alpha) = 0$ for all h_α, implies $h = 0$.*

Let $\mathscr{H}$ be a Hilbert space, the orthogonal dimension of $\mathscr{H}$ is denoted by $\dim\mathscr{H}$, i.e., the cardinality of an orthonormal basis for $\mathscr{H}$. We will denote the cardinality of an arbitrary set E by $|E|$.

Now, we will define the root system relative to a Cartan subalgebra of the semisimple $\mathscr{L}^*$-algebra $\mathscr{L}$.

Definition 1. Let $\mathscr{L}$ be a semisimple $\mathscr{L}^*$-algebra, H a Cartan subalgebra of $\mathscr{L}$, and Φ the set of nonzero roots of $\mathscr{L}$ relative to H. A subset Φ_0 of Φ will be called a root system of Φ if the following conditions are satisfied:

(i) If $\alpha \in \Phi_0$, then $-\alpha \in \Phi_0$
(ii) If $\alpha, \beta \in \Phi_0$, such that $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi_0$

We need some further notations; $\mathbb{R}$ and $\mathbb{Q}$ will refer to the real and the rational fields, respectively. For a subset $\mathscr{S}$ of Φ, the set of all $\mathbb{C}$-linear combinations of elements of $\mathscr{S}$ will be denoted by $Sp_{\mathbb{C}}\mathscr{S} = Sp\mathscr{S}$ and the set of all $\mathbb{Q}$-linear combinations of elements of $\mathscr{S}$ by $Sp_{\mathbb{Q}}\mathscr{S}$. If we write $(Sp)\hat{\mathscr{S}} = Sp\mathscr{S} \cap \Phi$, then $(Sp)\hat{\mathscr{S}}$ is obviously a root system. The following results will be useful in our main proofs.

Lemma 2. *Let $\mathscr{L}$ be a topologically simple $\mathscr{L}^*$-algebra and Φ the set of its nonzero roots relative to some Cartan subalgebra H. For any subset S of Φ, there exists a topologically simple $\mathscr{L}^*$-subalgebra $\mathscr{L}_S$ of $\mathscr{L}$, with Cartan subalgebra H_S, such that*

(i) $S \subseteq \Phi_S$, where Φ_S is the set of roots of $\mathscr{L}_S$ (relative to H_S) and $\Phi_S = (Sp)\hat{\mathscr{S}} = Sp_{\mathbb{Q}}\mathscr{S} \cap \Phi$
(ii) $\mathscr{L}_S$ is finite-dimensional if S is finite
(iii) $\mathscr{L}_S$ is infinite-dimensional and $\dim\mathscr{L}_S = |S|$ if S is infinite

Proof. See Proposition 3 in [14]. □

Definition 2. Let $\mathscr{L}$ be a topologically simple $\mathscr{L}^*$-algebra and Φ the set of nonzero roots relative to a Cartan subalgebra H. Let $\alpha, \beta \in \Phi$; we say that α is connected to β if there are some $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Phi$ such that $\alpha + \gamma_1, \gamma_1 + \gamma_2, \ldots, \gamma_k + \beta \in \Phi \cup \{0\}$.

Obviously, the connected relation is an equivalence relation on Φ.

Lemma 3. *Any two roots of a topologically simple $\mathscr{L}^*$-algebra are connected.*

Proof. Let $S = \{\beta_1, \beta_l\} \subset \Phi$, then by Lemma 2 there exists a finite-dimensional simple $\mathscr{L}^*$-algebra $\mathscr{L}_S$ of $\mathscr{L}$, with Cartan subalgebra H_S, such that β_1 and β_l are roots of $\mathscr{L}_S$. Using Lemma 1.3 in [8], we obtain that β_1 and β_l are connected in $\mathscr{L}_S$. Since $\Phi_S \subset \Phi$, then β_1 and β_l are connected in $\mathscr{L}$. □

3. Semi-Inner Biderivations on $\mathscr{L}^*$-Algebras and Commuting Linear Maps on Topologically Simple L^*-Algebras

In this section, we introduce the notion of semi-inner biderivation in order to show that every continuous commuting linear map on a topologically simple $\mathscr{L}^*$-algebra $\mathscr{L}$

is a scalar multiple of the identity mapping. The aim of the first main theorem of this section is to prove that every semi-inner biderivation of a topologically simple $\mathscr{L}^*$-algebra is inner. To get this result, we have to show two lemmas.

Definition 3. Let $\mathscr{L}$ be an $\mathscr{L}^*$-algebra and $f\colon \mathscr{L}\times\mathscr{L}\longrightarrow\mathscr{L}$ a biderivation of $\mathscr{L}$; f is said to be a semi-inner biderivation of $\mathscr{L}$ if there exists two continuous linear maps $\phi\colon \mathscr{L}\longrightarrow\mathscr{L}$ and $\psi\colon \mathscr{L}\longrightarrow\mathscr{L}$ such that

$$f(x,y)=[\phi(x),y]=[x,\psi(y)],\quad \forall x,y\in\mathscr{L}, \tag{7}$$

where f will be denoted by $f_{\phi,\psi}$.

Remark 1. By using Lemma 2.1 in [8], any biderivation of a finite-dimensional simple complex Lie algebra is semi-inner (a linear map between two finite-dimensional vector spaces is continuous).

From now on, $\mathscr{L}$ will represent an infinite-dimensional topologically simple $\mathscr{L}^*$-algebra of arbitrary dimension and $\mathscr{L}=H\oplus(\sum_{\alpha\in\Phi}\oplus\mathscr{V}_\alpha)$ where its Cartan decomposition is relative to a Cartan subalgebra H (Φ is the set of nonzero roots relative to H).

Lemma 4. *Let $f_{\phi,\psi}\colon \mathscr{L}\times\mathscr{L}\longrightarrow\mathscr{L}$ be a semi-inner biderivation of $\mathscr{L}$, then for any $h\in H$, we have $\phi(h),\psi(h)\in H$.*

Proof. For any $\alpha\in\Phi_+$, we select $x_\alpha\in\mathscr{V}_\alpha$, such that $\|x_\alpha\|=1$, $x_\alpha^*=x_{-\alpha}\in\mathscr{V}_{-\alpha}$, and $h\in H$. Then, we have

$$\begin{aligned}[x_\alpha,x_{-\alpha}]&=h_\alpha,\\ [h_\alpha,x_\alpha]&=\alpha(h_\alpha)x_\alpha,\\ [h_\alpha,x_{-\alpha}]&=-\alpha(h_\alpha)x_{-\alpha}.\end{aligned} \tag{8}$$

Let $\alpha\in\Phi$, we denote by $\widetilde{\Phi}_\alpha$ the set $\Phi\backslash\{\alpha,-\alpha\}$. Let

$$\phi(h_\alpha)=a_1h_1+a_2x_\alpha+a_3x_{-\alpha}+\sum_{\beta\in\widetilde{\Phi}_\alpha}k_\beta x_\beta, \tag{9}$$

$$\phi(x_\alpha)=b_1h_2+b_2x_\alpha+b_3x_{-\alpha}+\sum_{\beta\in\widetilde{\Phi}_\alpha}t_\beta x_\beta, \tag{10}$$

$$\phi(x_{-\alpha})=c_1h_3+c_2x_\alpha+c_3x_{-\alpha}+\sum_{\beta\in\widetilde{\Phi}_\alpha}t_\beta x_\beta, \tag{11}$$

$$\psi(h_\alpha)=s_1h_4+s_2x_\alpha+s_3x_{-\alpha}+\sum_{\beta\in\widetilde{\Phi}_\alpha}m_\beta x_\beta, \tag{12}$$

$$\psi(x_\alpha)=p_1h_5+p_2x_\alpha+p_3x_{-\alpha}+\sum_{\beta\in\widetilde{\Phi}_\alpha}n_\beta x_\beta, \tag{13}$$

$$\psi(x_{-\alpha})=q_1h_6+q_2x_\alpha+q_3x_{-\alpha}+\sum_{\beta\in\widetilde{\Phi}_\alpha}r_\beta x_\beta. \tag{14}$$

For some $a_i,b_i,c_i,s_i,p_i,q_i,k_\beta,t_\beta,l_\beta,m_\beta,n_\beta,r_\beta\in\mathbb{C}$, the sums are orthogonal, $i=1,2,3,\beta\in\widetilde{\Phi}_\alpha$, and $h_j\in H$, $j=1,2,\dots,6$.

By equations (10) and (14), we have

$$\begin{aligned}f(h_\alpha,x_\alpha)&=[\phi(h_\alpha),x_\alpha]=a_1\alpha(h_1)x_\alpha-a_3h_\alpha\\&\quad+\sum_{\beta\in\widetilde{\Phi}_\alpha}k_\beta[x_\beta,x_\alpha],\\ f(h_\alpha,x_\alpha)&=[h_\alpha,\psi(x_\alpha)]=\alpha(h_\alpha)p_2x_\alpha-\alpha(h_\alpha)p_3x_{-\alpha}\\&\quad+\sum_{\beta\in\widetilde{\Phi}_\alpha}m_\beta\beta(h_\alpha)x_\beta.\end{aligned} \tag{15}$$

If we compare the two equations above and $[x_\beta,x_\alpha]\in L_{\alpha+\beta}\neq H$ since $\beta\in\widetilde{\Phi}_\alpha$, we obtain $a_3=0$. In the same way, by considering $f(h_\alpha,x_{-\alpha})$ with equations (10) and (14), $a_2=0$. Similarly, considering the images $f(x_\alpha,h_\alpha)$ and $f(x_{-\alpha},h_\alpha)$, we obtain $s_2=s_3=0$, by equations (11)–(13).

If we put $a_1h_1=\widehat{h}^\alpha\in\eta$ and $s_1h_4=\breve{h}^\alpha\in\eta$, then equations (9) and (12) can be written as follows:

$$\phi(h_\alpha)=\widehat{h}^\alpha+\sum_{\beta\in\widetilde{\Phi}_\alpha}k_\beta x_\beta, \tag{16}$$

$$\psi(h_\alpha)=\breve{h}^\alpha+\sum_{\beta\in\widetilde{\Phi}_\alpha}m_\beta x_\beta. \tag{17}$$

Now, for any root $\gamma\in\widetilde{\Phi}_\alpha$, the set $\Phi\backslash\{\alpha,-\alpha,\gamma,-\gamma\}$ is denoted by $\widetilde{\Phi}_{\alpha,\gamma}$. By equations (16) and (17), we can write

$$\begin{aligned}\phi(h_\alpha)&=\widehat{h}^\alpha+k_\gamma x_\gamma+k_{-\gamma}x_{-\gamma}+\sum_{\beta\in\widetilde{\Phi}_{\alpha,\gamma}}k_\beta x_\beta,\\ \psi(h_\gamma)&=\breve{h}^\gamma+\mu_\alpha x_\alpha+\mu_{-\alpha}x_{-\alpha}+\sum_{\beta\in\widetilde{\Phi}_{\alpha,\gamma}}\mu_\beta x_\beta.\end{aligned} \tag{18}$$

The two equations above imply that

$$\begin{aligned}f(h_\alpha,h_\gamma)&=[\phi(h_\alpha),h_\gamma]=-\gamma(h_\gamma)k_\gamma x_\gamma+\gamma(h_\gamma)k_{-\gamma}x_{-\gamma}\\&\quad-\sum_{\beta\in\widetilde{\Phi}_{\alpha,\gamma}}k_\beta\beta(h_\gamma)x_\beta,\\ f(h_\alpha,h_\gamma)&=[h_\alpha,\psi(h_\gamma)]=\alpha(h_\alpha)\mu_\alpha x_\alpha-\alpha(h_\alpha)\mu_{-\alpha}x_{-\alpha}\\&\quad+\sum_{\beta\in\widetilde{\Phi}_{\alpha,\gamma}}\mu_\beta\beta(h_\alpha)x_\beta.\end{aligned} \tag{19}$$

By comparing, one has $k_\gamma=k_{-\gamma}=\mu_\alpha=\mu_{-\alpha}=0$. Due to the arbitrariness of γ, we obtain $\varphi(h_\alpha)=\widehat{h}^\alpha\in H$ and $\psi(h_\alpha)=\breve{h}^\alpha\in H$. By Lemma 1, the set $\{h_\alpha,\alpha\in\Phi\}$ is total in H and ϕ and ψ are continuous, and we have $\phi(H)\subset H$ and $\psi(H)\subset H$. □

Lemma 5. *Let $f_{\phi,\psi}\colon \mathscr{L}\times\mathscr{L}\longrightarrow\mathscr{L}$ be a semi-inner biderivation of $\mathscr{L}$, then there is a complex number λ such that*

$$\phi(x) = \psi(x) = \lambda x, \quad \forall x \in \mathscr{L}_\alpha, \alpha \in \Phi. \tag{20}$$

Proof. Let $h \in H$ and $\alpha, \gamma \in \Phi$ such that $\alpha \neq \gamma$, then we have

$$f(x_\alpha, h) = [\phi(x_\alpha), h] = -[h, \phi(x_\alpha)], \tag{21}$$

$$f(x_\alpha, h) = [x_\alpha, \psi(h)] = -[\psi(h), x_\alpha] = -\alpha(\psi(h))x_\alpha. \tag{22}$$

Let

$$\phi(x_\alpha) = \widehat{h}^\alpha + \sum_{\beta\in\Phi} t_\beta x_\beta, \tag{23}$$

where $t_\beta \in \mathbb{C}$ and $\widehat{h}^\alpha \in H$. Using equations (21) and (23), we obtain

$$f(x_\alpha, h) = -\sum_{\beta\in\Phi} t_\beta \beta(h) x_\beta. \tag{24}$$

Combining equations (22) and (24), we obtain $t_\alpha \alpha(h) = \alpha(\psi(h))$ and $t_\beta \beta(h) = 0$ for every $\beta \in \Phi\backslash\{\beta\}$.

For any $\beta \in \Phi\backslash\{\alpha\}$, if we replace h by h_β in equation (24), we get $t_\beta = 0$; this implies that

$$\begin{aligned} \phi(x_\alpha) &= t_\alpha x_\alpha + \widehat{h}^\alpha, \\ \alpha(h)t_\alpha &= \alpha(\psi(h)). \end{aligned} \tag{25}$$

The image of $f(h, x_\alpha)$ is computed. Similarly, we get

$$\begin{aligned} \psi(x_\alpha) &= k_\alpha x_\alpha + \breve{h}^\alpha, \\ \alpha(h)k_\alpha &= \alpha(\phi(h)), \end{aligned} \tag{26}$$

where $k_\beta \in \mathbb{C}$ and $\breve{h}^\alpha \in H$.

For any $\alpha, \beta \in \Phi$, using equations (25) and (26), we have

$$f(x_\alpha, x_\beta) = [\phi(x_\alpha), x_\beta] = t_\alpha [x_\alpha, x_\beta] + \beta(\widehat{h}^\alpha) x_\beta, \tag{27}$$

$$f(x_\alpha, x_\beta) = [x_\alpha, \psi(x_\beta)] = k_\beta [x_\alpha, x_\beta] - \alpha(\breve{h}^\beta) x_\alpha. \tag{28}$$

By combining equations (27) with (28), we first see that $\beta(\widehat{h}^\alpha) = \alpha(\breve{h}^\beta) = 0$ if $\alpha \neq \beta$. However, $-\alpha(\widehat{h}^\alpha) = \alpha(\breve{h}^{-\alpha}) = 0$ by taking $\beta = -\alpha$. This means that $\beta(\widehat{h}^\alpha) = 0$ for all $\beta \in \Phi$, i.e., $\widehat{h}^\alpha \in \cap_{\beta\in\Phi} \ker \beta$, which gives $\widehat{h}^\alpha = 0$. Similarly, by taking $\alpha = -\beta$, we have $\breve{h}^\beta = 0$. Therefore, by equations (25) and (26), one can obtain

$$\begin{aligned} \phi(x_\alpha) &= t_\alpha x_\alpha, \\ \psi(x_\alpha) &= k_\alpha x_\alpha. \end{aligned} \tag{29}$$

Using equation (29) and $f(x_\alpha, x_{-\alpha}) = [\phi(x_\alpha), x_{-\alpha}] = [x_\alpha, \psi(x_{-\alpha})]$, it follows that

$$t_\alpha = k_{-\alpha}, \quad \forall \alpha \in \Phi. \tag{30}$$

Then, comparing equations (27) with (28), we obtain

$$t_\alpha = k_\beta, \quad \text{for } \alpha + \beta \in \Phi. \tag{31}$$

Let $S = \{\alpha\} \subseteq \Phi$, then by Lemma 2 there exists a finite-dimensional simple $\mathscr{L}^*$-algebra $\mathscr{L}_S$ of $\mathscr{L}$, with Cartan subalgebra H_S, such that α is a root of $\mathscr{L}_S$. Now, let us prove that $f_{\phi,\psi}(\mathscr{L}_S \times \mathscr{L}_S) \subseteq \mathscr{L}_S$ indeed; let $x, y \in \mathscr{L}_S$, then $x = h_x + \sum_{\beta\in\Phi_S} x_\beta$ and $y = h_y + \sum_{\gamma\in\Phi_S} y_\gamma$.

$$\begin{aligned} f_{\phi,\psi}(x, y) = [\phi(x), y] &= \sum_{\gamma\in\Phi_S} [\phi(h_x), y_\gamma] + \sum_{\beta\in\Phi_S} [t_\beta x_\beta, h_y] \\ &\quad + \sum_{\beta\in\Phi_S} \sum_{\gamma\in\Phi_S} [t_\beta x_\beta, y_\gamma]. \end{aligned} \tag{32}$$

Then, $f_{\phi,\psi} / \mathscr{L}_S \times \mathscr{L}_S$ is a biderivation of $\mathscr{L}_S$; by Theorem 2.4 in [8], there exists $\mu \in \mathbb{C}$ such that $f_{\phi,\psi}(x, y) = \mu[x, y]$ for any $x, y \in \mathscr{L}_S$. Then, $\mu[x_\beta, y] = t_\beta [x_\beta, y] = -k_\beta [y, x_\beta]$ for any $\beta \in \Phi_S$ and $y \in \mathscr{L}_S$; this implies that

$$t_\alpha = t_{-\alpha} = k_\alpha = k_{-\alpha} = \mu, \quad \text{for arbitrary } \alpha \in \Phi. \tag{33}$$

Equations (30)–(33) imply that if $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi \cup \{0\}$, then $t_\alpha = k_\beta = t_\beta = k_\alpha$. Additionally, for arbitrary connected roots $\alpha, \beta \in \Phi, t_\alpha = k_\beta = t_\beta = k_\alpha$, from Lemma 3, we conclude that

$$t_\alpha = t_{\alpha'} = k_\beta = k_{\beta'}, \quad \forall \alpha, \alpha', \beta, \beta' \in \Phi. \tag{34}$$

If we pose $t_\alpha = \lambda$ in equation (34), we get our result. □

Remark 2. In the above proof, there is another method to show that $t_\gamma = 0$ if $\gamma \neq \alpha$: Indeed, using equation (23), we have

$$\begin{aligned} (f(x_\alpha, h_\gamma), x_\gamma) &= -([h_\gamma, \phi(x_\alpha)], x_\gamma) = -(\phi(x_\alpha), [h_\gamma^*, x_\gamma]) \\ &= \left(\widehat{h}^\alpha + \sum_{\beta\in\Phi} t_\beta x_\beta, \gamma(h_\gamma^*) x_\gamma\right) \\ &= t_\gamma \gamma(h_\gamma^*)(x_\gamma, x_\gamma). \end{aligned} \tag{35}$$

We also have $(f(x_\alpha, h_\gamma), x_\gamma) = (-\alpha(\psi(h_\gamma))x_\alpha, x_\gamma) = 0$. Then, $t_\gamma = 0$ if $\gamma \neq \alpha$.

Remark 3. In the case where $\mathscr{L}$ is separable, Theorem 2 in [16] will facilitate some difficulties encountered in the above proof (the above proof will look like the proof of Lemma 2.2 in [8]).

Thanks to the above Lemmas, we can state our first main theorem.

Theorem 1. *Let $\mathscr{L}$ be a topologically simple $\mathscr{L}^*$-algebra. Then, f is a semi-inner biderivation of $\mathscr{L}$ if and only if it is inner, i.e., there is a complex number λ such that $f(x, y) = \lambda[x, y]$.*

Proof. Let $\mathscr{L}$ be a topologically simple $\mathscr{L}^*$-algebra, any inner biderivation f of $\mathscr{L}$ such that $f(x, y) = \lambda[x, y]$ is a semi-inner biderivation $f_{\phi,\psi}$ with

$$\phi = \psi = \lambda \mathrm{id}_{\mathscr{L}}. \tag{36}$$

Now, let us prove the "only if direction."

The first case: if $\mathscr{L}$ is finite-dimensional, the result follows from Theorem 2.4 in [8]. The second case: if $\mathscr{L}$ is infinite-dimensional, then, for $f_{\phi,\psi}$ a semi-inner biderivation of $\mathscr{L}$, by using Lemma 5, there is $\lambda \in \mathbb{C}$ such that $\phi(x) = \psi(x) = \lambda x, \forall x \in L_\alpha, \alpha \in \Phi$. For any $h \in H$ and $\alpha \in \Phi$, we have $f(h, x_\alpha) = [\phi(h), x_\alpha] = [h, \psi(x_\alpha)]$. This implies that $\alpha(\phi(h))x_\alpha = \lambda\alpha(h)x_\alpha$ which means that

$$\alpha(\lambda h - \phi(h)) = 0, \quad \forall \alpha \in \Phi, \tag{37}$$

this implies $\phi(h) = \lambda h$ for all $h \in H$. For any $x, y \in L$, with $x = h + \sum_{\alpha\in\Phi} l_\alpha x_\alpha$, where $l_\alpha \in \mathbb{C}$, with $l_\alpha = 0$ except for a countable number. Therefore,

$$\begin{aligned} f(x, y) = [\phi(x), y] &= \left[\sum_{\alpha\in\Phi} l_\alpha \phi(x_\alpha) + \phi(h), y\right] \\ &= \left[\sum_{\alpha\in\Phi} l_\alpha \lambda x_\alpha + \lambda h, y\right] = [\lambda x, y]. \end{aligned} \tag{38}$$

In recent years, many authors have studied commuting linear maps of certain algebra structures; for some of these achievements, refer to [3, 4, 6, 8, 9, 17, 18]. It should be noted that this subject is not new since it was studied in 1957, exactly in Posner's works [19]. As we said in the introduction, if ϕ is a commuting linear map on $\mathscr{L}$, then $[\phi(x), y] = [x, \phi(y)]$ for any $x, y \in \mathscr{L}$, and $f(x, y) = [\phi(x), y] = [x, \phi(y)]$ is a biderivation of $\mathscr{L}$. Using Theorem 1, the present theorem aims to determine all continuous commuting linear maps on topologically simple L^*-algebras. □

Theorem 2. *Let $\mathscr{L}$ be a topologically simple $\mathscr{L}^*$-algebra. Then, every continuous linear map ϕ on $\mathscr{L}$ is a commuting linear map if and only if it is a scalar multiplication map on $\mathscr{L}$.*

Proof. Let ϕ be a continuous commuting linear map on a topologically simple $\mathscr{L}^*$-algebra $\mathscr{L}$. Then, f defined by $f(x, y) = [\phi(x), y] = [x, \phi(y)]$ is a semi-inner biderivation of $\mathscr{L}$. By Theorem 1, we have $f(x, y) = [\phi(x), y] = \lambda[x, y]$ for some $\lambda \in \mathbb{C}$. Since $\mathscr{L}$ is topologically simple $\mathscr{L}^*$-algebra and y is arbitrary, therefore, we have $\phi(x) = \lambda x$.

The following Lemma is one of the interesting results given in (see p.7 in [3]). □

Lemma 6. *Let $\mathscr{L}$ be a simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic different from 2 such that $card(\mathbb{F}) > \dim(\mathscr{L})$ ($card(\mathbb{F})$ is the cardinality of $\mathbb{F}$ and $\dim(\mathscr{L})$ is the dimension of $\mathscr{L}$). For any skew-symmetric biderivation f of $\mathscr{L}$, there exists $\lambda \in \mathbb{F}$ such that*

$$f(x, y) = \lambda[x, y], \quad \text{for all } x, y \in \mathscr{L}. \tag{39}$$

Remark 4. Let $\mathscr{L}$ be a simple complex Lie algebra of countable dimension and ϕ a commuting linear map on $\mathscr{L}$. Then, ϕ is of the form $\phi(x) = \lambda x$ for all $x \in \mathscr{L}$ where $\lambda \in \mathbb{C}$. Indeed, let ϕ be a commuting linear map on $\mathscr{L}$, then $f(x, y) = [\phi(x), y] = [x, \phi(y)]$ for all $x, y \in \mathscr{L}$ which is a skew-symmetric biderivation of $\mathscr{L}$. By Lemma 6, there exists $\lambda \in \mathbb{C}$, such that f is of the form $f(x, y) = \lambda[x, y]$ for all $x, y \in \mathscr{L}$. Then, ϕ is of the form $\phi(x) = \lambda x$ for all $x \in \mathscr{L}$.

We mention here that this proof does not seem to work in our case when the underlying Hilbert space of the $\mathscr{L}^*$-algebra is infinite-dimensional. However, Corollary 2.4 in [3] may simplify some of the proofs in our paper.

4. Conclusions

This paper aimed to show that every continuous commuting linear map on a topologically simple $\mathscr{L}^*$-algebra $\mathscr{L}$ is a scalar multiplication map on $\mathscr{L}$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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